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POSITIVE SOLUTIONS OF A FRACTIONAL INTEGRO-DIFFERENTIAL EQUATION WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract. We study the existence and uniqueness of positive solutions of a fractional integro-differential equation with integral boundary conditions. We convert the given fractional integro-differential equation into an equivalent integral equation. Then we construct appropriate mappings and employ the Schauder fixed point theorem and the method of upper and lower solutions to show the existence of a positive solution. We also use the Banach fixed point theorem to show the existence of a unique positive solution. Finally, an example is given to illustrate our results.

Keywords. Fractional integro-differential equations; Positive solutions; Upper and lower solutions; Fixed point theorem.

1. INTRODUCTION

Fractional differential equations with and without delay arise from a variety of applications including in various fields of science and engineering such as applied sciences, practical problems concerning mechanics, the engineering technique fields, economy, control systems, physics, chemistry, biology, medicine, atomic energy, information theory, harmonic oscillator, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, etc. In particular, problems concerning qualitative analysis of linear and nonlinear fractional differential equations with and without delay have received the attention of many authors, see [1]- [17] and the references therein.

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Zhang [17] investigated the existence and uniqueness of positive solutions for the nonlinear fractional differential equation

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f(t, u(t)),\ 0 < t \leq 1, \\ u(0) = 0, \end{cases}$$

where $D_{0^+}^{\alpha}$ is the standard Riemann Liouville fractional derivative of order $0 < \alpha < 1$ and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a given continuous function. By using the method of the upper and lower solution and cone fixed-point theorem, the author obtained the existence and uniqueness of a positive solution.

The nonlinear fractional integro-differential equation with nonlinear conditions

$$\begin{cases} {}^cD_{0^+}^{\alpha} u(t) = f(t, u(t)) + \int_0^t k(t,s,u(s))ds,\ t \in (0,T], \\ u(0) = u_0 - g(u), \end{cases}$$

was investigated in [5], where ${}^cD_{0^+}^{\alpha}$ is the standard Caputo fractional derivative of order $1 < \alpha < 1$, $u_0 \in \mathbb{R}$, g, f and k are given continuous functions. By employing the Krasnoselskii and Banach fixed point theorems, Ahmad and Sivasundaram obtained the existence and uniqueness results.

Wang, Wang and Zeng [16] discussed the existence of solutions of the following fractional differential equation with integral boundary conditions

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f(t, u(t)),\ t \in (0,T], \\ u(0) = \lambda \int_0^T u(s)\,ds + d, \end{cases}$$

where $0 < \alpha < 1$, $\lambda \geq 0$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function. The authors applied the upper and lower solutions combined with a monotone iterative technique and obtained their main results.

In [2], Abdo, Wahash and Panchat discussed the existence and uniqueness of positive solutions of the following nonlinear fractional differential equation with integral boundary conditions

$$\begin{cases} {}^cD_{0^+}^{\alpha} x(t) = f(t, x(t)),\ t \in (0,1], \\ x(0) = \lambda \int_0^1 x(s)\,ds + d, \end{cases}$$

where $0 < \alpha < 1$, $\lambda \geq 0$, $d > 0$ and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a given continuous function. By using the method of the upper and lower solutions and Schauder and Banach fixed point theorems, the existence and uniqueness of solutions has been established.

In this paper, we are interested in the analysis of qualitative theory of the problems of the positive solutions to fractional integro-differential equations with integral boundary conditions. Inspired and motivated by the works mentioned above, we concentrate on the positivity of the solutions for the nonlinear fractional integro-differential equation with integral boundary conditions

$$\begin{cases} {}^cD_{0^+}^{\alpha} u(t) = f(t, u(t)) + \int_0^t k(t,s,u(s))ds,\ t \in (0,T], \\ u(0) = \lambda \int_0^T u(s)\,ds + d, \end{cases} \tag{1.1}$$

where $0 < \alpha < 1$, $\lambda \geq 0$, $d > 0$, $f : [0,T] \times [0,\infty) \to [0,\infty)$ and $k : [0,T] \times [0,T] \times [0,\infty) \to [0,\infty)$ are given continuous functions, k is non-decreasing on u.

To show the existence and uniqueness of positive solutions, we transform (1.1) into an integral equation and then by the method of upper and lower solutions and use the Schauder and Banach fixed point theorems.

This paper is organized as follows. In section 2, we introduce some notations and lemmas, and state some preliminaries results needed in later sections. Also, we present the inversion of (1.1) and the Banach and Schauder fixed point theorems. For details on the Banach and Schauder theorems we refer the reader to [18]. In Sections 3, we give and prove our main results on positivity and provide an example to illustrate our results.

2. Preliminaries

In this section, we recall some basic definitions and necessary lemmas related to fractional calculus and fixed point theorems that will be used throughout this paper.

Let $C([0,T])$ be the Banach space endowed with the infinity norm and A a nonempty closed subset of $C([0,T])$ defined as

$$A = \{u \in C([0,T]) : u(t) \geq 0,\ t \in [0,T]\}.$$

$C^n([0,T])$ denotes the class of all real valued functions defined on $[0,T]$ which have a continuous n th order derivative.

Definition 2.1. [11] The left sided Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u \in C([0,T])$ is given by

$$I_{0^+}^{\alpha} u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)\, ds,\ t > 0,$$

where Γ denotes the Gamma function.

Definition 2.2. [11] Let $n-1 < \alpha < n$. The left sided Riemann-Liouville fractional derivative of order α of a function $u : [0,T] \to \mathbb{R}$ is defined by

$$D_{0^+}^{\alpha} u(t) = \frac{d^n}{dt^n} I_{0^+}^{n-\alpha} u(t),\ t > 0,$$

provided the right side integral is pointwise defined on $[0,T]$ and $n = [\alpha]+1$, $[\alpha]$ denotes the integer part of the real number α. In particular, if $0 < \alpha < 1$, then

$$D_{0^+}^{\alpha} u(t) = \frac{d}{dt} I_{0^+}^{1-\alpha} u(t),\ t > 0.$$

Definition 2.3. [11] Let $n-1 < \alpha < n$. The left sided Caputo derivative of order $\alpha > 0$ of a function $u \in C^n([0,T])$ is given by

$${}^cD_{0^+}^{\alpha} u(t) = I_{0^+}^{n-\alpha} \frac{d^n}{dt^n} u(t),\ t > 0.$$

In particular, if $0 < \alpha < 1$, then

$${}^cD_{0^+}^{\alpha} u(t) = I_{0^+}^{1-\alpha} \frac{d}{dt} u(t),\ t > 0.$$

Moreover, the Caputo derivative of a constant is equal to zero.

Lemma 2.4. *[11] Let $\alpha > 0$ and $u \in C^n([0,T])$. Then*

1) ${}^cD^{\alpha}_{0^+} I^{\alpha}_{0^+} u(t) = u(t)$.

2) $I^{\alpha}_{0^+} {}^cD^{\alpha}_{0^+} u(t) = u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(0^+)}{k!} t^k$.

In particular, when $0 < \alpha < 1$, $I^{\alpha}_{0^+} {}^cD^{\alpha}_{0^+} u(t) = u(t) - u(0)$.

Theorem 2.5 (Banach's fixed point theorem [18]). *Let Ω be a non-empty closed subset of a Banach space $(S, \|.\|)$, then any contraction mapping Φ of Ω into itself has a unique fixed point.*

Theorem 2.6 (Schauder's fixed point theorem [18]). *Let Ω be a nonempty closed bounded convex subset of a Banach space S and $\Phi : \Omega \to \Omega$ be a continuous compact operator. Then Φ has a fixed point in Ω.*

Definition 2.7. A function $u \in C^1([0,T])$ is said to be a solution of (1.1) if u satisfies the equation

$$ {}^cD^{\alpha}_{0^+} u(t) = f(t, u(t)) + \int_0^t k(t,s,u(s))ds,\ t \in (0,T], $$

with integral boundary conditions

$$ u(0) = \lambda \int_0^T u(s)\, ds + d. $$

Definition 2.8. A function $u \in C^1([0,T])$ is called a positive solution of (1.1) if $u(t) \geq 0$ for all $t \in [0,T]$ and u satisfies the problem (1.1).

Definition 2.9. Let $a, b \in \mathbb{R}^+$ and $b > a$. For any $u \in [a,b]$, we define the upper-control function $U(t,u) = \sup_{a \leq \rho \leq u} f(t,\rho)$ and lower-control function $L(t,u) = \inf_{u \leq \rho \leq b} f(t,\rho)$.

Obviously, $U(t,u)$ and $L(t,u)$ are monotonous non-decreasing on u and

$$ L(t,u) \leq f(t,u) \leq U(t,u). $$

3. Main results

In this section, we shall give the existence and uniqueness results of (1.1) and prove it. Before starting and proving the main results, we introduce the following lemma.

Lemma 3.1. *$u \in C^1([0,T])$ is a solution of the boundary value problem (1.1) if and only if u is a solution of the integral equation*

$$ \begin{aligned} u(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[f(s,x(s)) + \int_0^s k(s,\tau,u(\tau))d\tau \right] ds \\ & + \lambda \int_0^T u(s)\, ds + d,\ t \in [0,T]. \end{aligned} $$

Proof. Suppose u satisfies the problem (1.1). Then applying $I^{\alpha}_{0^+}$ to both sides of (1.1), we have

$$ I^{\alpha}_{0^+} {}^cD^{\alpha}_{0^+} u(t) = I^{\alpha}_{0^+} \left(f(t,u(t)) + \int_0^t k(t,s,u(s))ds \right). $$

In view of Lemma 2.4 and the integral boundary condition, we get

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[f(s,x(s)) + \int_0^s k(s,\tau,u(\tau))d\tau \right] ds + \lambda \int_0^T u(s)\,ds + d,\ t \in [0,T]. \tag{3.1}$$

Conversely, suppose u satisfies (3.1). By Definition 2.3 and Lemma 2.4, for $t \in [0,T]$, we observe that

$$\begin{aligned}
{}^cD_{0^+}^{\alpha} u(t) ={}& {}^c D_{0^+}^{\alpha} \left(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[f(s,x(s)) + \int_0^s k(s,\tau,u(\tau))d\tau \right] ds \right. \\
& \left. + \lambda \int_0^T u(s)\,ds + d \right) \\
={}& {}^c D_{0^+}^{\alpha} I_{0^+}^{\alpha} \left(f(t,u(t)) \right) + {}^c D_{0^+}^{\alpha} I_{0^+}^{\alpha} \left(\int_0^s k(s,\tau,u(\tau))d\tau \right) \\
& + {}^c D_{0^+}^{\alpha} \left(\lambda \int_0^T u(s)\,ds + d \right) \\
={}& f(t,u(t)) + \int_0^t k(t,s,u(s))ds.
\end{aligned}$$

Moreover, the integral boundary condition $u(0) = \lambda \int_0^T u(s)\,ds + d$ holds. □

To transform (3.1) to be applicable to Schauder's fixed point, we define the operator $\Phi : A \to A$ by

$$(\Phi u)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[f(s,x(s)) + \int_0^s k(s,\tau,u(\tau))d\tau \right] ds + \lambda \int_0^T u(s)\,ds + d,\ t \in [0,T], \tag{3.2}$$

where figured fixed point must satisfy the identity operator equation $\Phi u = u$.

We introduce the following assumptions

(H_1) Let $u^*, u_* \in A$ such that $a \le u_*(t) \le u^*(t) \le b$ and

$$\begin{cases} {}^cD_{0^+}^{\alpha} u^*(t) - \int_0^t k(t,s,u^*(s))ds \ge U(t,u^*(t)), \\ {}^cD_{0^+}^{\alpha} u_*(t) - \int_0^t k(t,s,u_*(s))ds \le L(t,u_*(t)), \end{cases}$$

for any $t \in [0,T]$.

(H_2) For $t,s \in [0,T]$ and $u,v \in \mathbb{R}$, there exist two positive constants l_f and l_k such that

$$\begin{aligned} |f(t,u) - f(t,v)| &\le l_f |u-v|, \\ |k(t,s,u) - k(t,s,v)| &\le l_k |u-y|. \end{aligned}$$

The functions u^* and u_* are respectively called the pair of upper and lower solutions for problem (1.1).

The first result is based on the Schauder fixed point theorem.

Theorem 3.2. *Assume that (H_1) is satisfied. Then, problem (1.1) has at least one positive solution.*

Proof. Let

$$\Omega=\{u\in A: u_*(t)\leq u(t)\leq u^*(t),\ t\in[0,T]\}$$

endowed with the norm $\|u\|=\max_{t\in[0,T]}|u(t)|$. Then, $\|u\|\leq b$. Hence, Ω is convex bounded and closed subset of the Banach space $C([0,T])$. Moreover, the continuity of f and k imply the continuity of the operator Φ on Ω defined by (3.2). Now, if $u\in\Omega$, there exist two positive constants c_f and c_k such that

$$\max\{f(t,u(t)): t\in[0,T],\ u(t)\leq b\}\leq c_f,$$

and

$$\max\{k(t,s,u(s)): t,s\in[0,T],\ u(s)\leq b\}\leq c_k.$$

Then

$$\begin{aligned}|(\Phi u)(t)|&\leq\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left[|f(s,x(s))|+\int_0^s|k(s,\tau,u(\tau))|\,d\tau\right]ds\\&\quad+\lambda\int_0^T|u(s)|\,ds+d\\&\leq\frac{c_fT^\alpha}{\Gamma(\alpha+1)}+\frac{c_kT^{\alpha+1}}{\Gamma(\alpha+2)}+\lambda bT+d.\end{aligned}$$

Thus

$$\|\Phi u\|\leq\frac{c_fT^\alpha}{\Gamma(\alpha+1)}+\frac{c_kT^{\alpha+1}}{\Gamma(\alpha+2)}+\lambda bT+d.$$

Hence, $\Phi(\Omega)$ is uniformly bounded.

Next, we prove the equicontinuity of $\Phi(\Omega)$. For each $u\in\Omega$. Then for $t_1,t_2\in[0,T]$ with $t_1<t_2$, we have

$$\begin{aligned}|(\Phi u)(t_2)-(\Phi u)(t_1)|&\leq\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\right]|f(s,u(s))|\,ds\\&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|f(s,u(s))|\,ds\\&\quad+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\right]\left(\int_0^s|k(s,\tau,u(\tau))|\,d\tau\right)ds\\&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\left(\int_0^s|k(s,\tau,u(\tau))|\,d\tau\right)ds\\&\leq\frac{c_f}{\Gamma(\alpha+1)}\left[2(t_2-t_1)^\alpha+t_1^\alpha-t_2^\alpha\right]\\&\quad+\frac{c_k}{\Gamma(\alpha+2)}\left[2(t_2-t_1)^{\alpha+1}+t_1^{\alpha+1}-t_2^{\alpha+1}\right]\\&\leq\frac{2c_f}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha+\frac{2c_k}{\Gamma(\alpha+2)}(t_2-t_1)^{\alpha+1}.\end{aligned}\tag{3.3}$$

As $t_1\to t_2$, the right-hand side of inequality (3.3) tends to zero and the convergence is independent of u in Ω, which means that $\Phi(\Omega)$ is equicontinuous. The Arzela-Ascoli theorem implies that $\Phi:\Omega\to A$ is compact. The only thing to apply the Schauder fixed point is to prove that $\Phi(\Omega)\subset\Omega$. For any $u\in\Omega$, then $u_*(t)\le u(t)\le u^*(t)$ and by (H_1), we have

$$\begin{aligned}
(\Phi u)(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[f(s,u(s))+\int_0^s k(s,\tau,u(\tau))d\tau\right]ds+\lambda\int_0^T u(s)\,ds+d\\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[U(s,u(s))+\int_0^s k(s,\tau,u(\tau))d\tau\right]ds+\lambda\int_0^T u(s)\,ds+d\\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[U(s,u^*(s))+\int_0^s k(s,\tau,u^*(\tau))d\tau\right]ds+\lambda\int_0^T u^*(s)\,ds+d\\
&\le u^*(t),
\end{aligned}$$

and

$$\begin{aligned}
(\Phi u)(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[f(s,u(s))+\int_0^s k(s,\tau,u(\tau))d\tau\right]ds+\lambda\int_0^T u(s)\,ds+d\\
&\ge \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[L(s,u(s))+\int_0^s k(s,\tau,u(\tau))d\tau\right]ds+\lambda\int_0^T u(s)\,ds+d\\
&\ge \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[L(s,u_*(t))+\int_0^s k(s,\tau,u_*(\tau))d\tau\right]ds+\lambda\int_0^T u_*(s)\,ds+d\\
&\ge u_*(t).
\end{aligned}$$

Hence, $u_*(t)\le(\Phi u)(t)\le u^*(t)$, $t\in[0,T]$, that is, $\Phi(\Omega)\subset\Omega$. According to the Schauder fixed point theorem, the operator Φ has at least one fixed point $u\in\Omega$. Therefore, problem (1.1) has at least one positive solution. □

The second result is based on the Banach fixed point theorem.

Theorem 3.3. *Assume that (H_1) and (H_2) are satisfied and*

$$\frac{l_f T^{\alpha}}{\Gamma(\alpha+1)}+\frac{l_k T^{\alpha+1}}{\Gamma(\alpha+2)}+\lambda T<1. \tag{3.4}$$

Then problem (1.1) has a unique positive solution.

Proof. From Theorem 3.2, it follows that problem (1.1) has at least one positive solution. Hence, we need only to prove that the operator defined in (3.2) is a contraction in Ω. In fact, for any $u,v\in\Omega$, we have

$$\begin{aligned}
|(\Phi u)(t)-(\Phi v)(t)| &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,u(s))-f(s,v(s))|\,ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\int_0^s |k(s,\tau,u(\tau))-k(s,\tau,v(\tau))|\,d\tau\right)ds\\
&\quad+\lambda\int_0^T |u(s)-v(s)|\,ds\\
&\le\left(\frac{l_f T^{\alpha}}{\Gamma(\alpha+1)}+\frac{l_k T^{\alpha+1}}{\Gamma(\alpha+2)}+\lambda T\right)\|u-v\|.
\end{aligned}$$

Hence, the operator Φ is a contraction mapping by (3.4). Therefore, by the Banach fixed point theorem, we conclude that problem (1.1) has a unique positive solution. □

Finally, we give an example to illustrate our results.

Example 3.4. Consider the fractional integro-differential equation with integral boundary conditions

$$\begin{cases} {}^{c}D_{0^+}^{\frac{1}{2}}u(t) = \frac{1}{3}(2+\sin(u(t))) + \frac{1}{4}\int_0^t u(s)\exp\left(-\left(t^2+s^2\right)\right)ds,\ t\in(0,1], \\ u(0) = \frac{1}{6}\int_0^1 u(s)\,ds + \frac{1}{2}, \end{cases} \tag{3.5}$$

where $\alpha = 1/2$, $\lambda = 1/3$, $d = 1/2$, $f(t,u(t)) = \frac{1}{3}(2+\sin(u(t)))$ and

$$k(t,s,u(s)) = \frac{1}{4}u(s)\exp\left(-\left(t^2+s^2\right)\right).$$

Since f and g are continuous positive functions, k is non-decreasing on u and

$$\frac{l_f T^{\alpha}}{\Gamma(\alpha+1)} + \frac{l_k T^{\alpha+1}}{\Gamma(\alpha+2)} + \lambda T \simeq 0.73 < 1.$$

By using Theorem 3.3, we find that (3.5) has a unique positive solution.

References

[1] S. Abbas, Existence of solutions to fractional order ordinary and delay differential equations and applications, Electron. J. Differential Equations 2011 (2011), 1-11.

[2] M.A. Abdo, H.A. Wahash, S. K. Panchat, Positive solutions of a fractional differential equation with integral boundary conditions, J. Appl. Math. Mech. 17 (2018), 5-15.

[3] R.P. Agarwal, Y. Zhou, Y. He, Existence of fractional functional differential equations, Comput. Math. Appl. 59 (2010), 1095-1100.

[4] B. Ahmad, S.K. Ntouyas, Existence and uniqueness of solutions for Caputo-Hadamard sequential fractional order neutral functional differential equations, Electron. J. Differential Equations 2017 (2017), 1-11.

[5] B. Ahmad, S. Sivasundaram, Some existence results for fractional integro-differential equations with nonlinear conditions, Commun. Appl. Anal. 12 (2008), 107-112

[6] H. Boulares, A. Ardjouni, Y. Laskri, Positive solutions for nonlinear fractional differential equations, Positivity 21 (2017), 1201-1212.

[7] H. Boulares, A. Ardjouni, Y. Laskri, Stability in delay nonlinear fractional differential equations, Rend. Circ. Mat. Palermo 65 (2016), 243-253.

[8] A. Chidouh, A. Guezane-Lakoud, R. Bebbouchi, Positive solutions of the fractional relaxation equation using lower and upper solutions, Vietnam J. Math. 44 (2016), 739-748.

[9] F. Ge, C. Kou, Stability analysis by Krasnoselskii's fixed point theorem for nonlinear fractional differential equations, Appl. Math. Comput. 257 (2015), 308-316.

[10] F. Ge, C. Kou, Asymptotic stability of solutions of nonlinear fractional differential equations of order $1 < \alpha < 2$, J. Shanghai Normal Univ. 44 (2015), 284-290.

[11] A.A. Kilbas, H.H. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier Science B. V., Amsterdam, 2006.

[12] C. Kou, H. Zhou, Y. Yan, Existence of solutions of initial value problems for nonlinear fractional differential equations on the half-axis, Nonlinear Anal. 74 (2011), 5975-5986.

[13] V. Lakshmikantham, A.S. Vatsala, Basic theory of fractional differential equations, Nonlinear Anal. 69 (2008), 2677-2682.

[14] N. Li, C. Wang, New existence results of positive solution for a class of nonlinear fractional differential equations, Acta Math. Sci. 33 (2013), 847–854.

[15] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

[16] X. Wang, L. Wang, Q. Zeng, Fractional differential equations with integral boundary conditions, J. of Nonlinear Sci. Appl. 8 (2015), 309-314.

[17] S. Zhang, The existence of a positive solution for a nonlinear fractional differential equation, J. Math. Anal. Appl. 252 (2000), 804-812.

[18] D.R. Smart, Fixed Point Theorems, Cambridge Tracts in Mathematics, no. 66, Cambridge University Press, London–New York, 1974.