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DYNAMICS OF A TWO SPECIES RATIO-DEPENDENT LOTKA-VOLTERRA COMPETITIVE SYSTEM WITH DELAYS

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Abstract. This paper studies a class of nonautonomous two-species Lotka-Volterra competitive system with delays. Some sufficient conditions on the boundedness, permanence, existence of periodic solutions and global attractivity of the system are established by means of the comparison method and the Lyapunov function method.

Keywords. Ratio-dependent competitive system; Permanence; Periodic solution; Delay; Global attractivity.

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1. INTRODUCTION

Recently, the ratio-dependent population dynamical systems are extensively studied [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. Especially, the population ratio-dependent predator-prey dynamical systems [1, 2, 3, 4, 5, 6, 7], the population ratio-dependent competitive dynamical systems [8] and the population ratio-dependent cooperative dynamical systems [9, 10, 11, 12]. Till now, most results only involve persistence, permanence, extinction, global stability, global attractivity of systems [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. Fore example, Muhammadhaji and Teng considered the following ratio-dependent predator-prey

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system stage structure

$$
\begin{aligned}
\dot{x_1}(t) =& \quad r(t)x_2(t) - B(t)x_1(t) - d_1(t)x_1^2(t),\\
\dot{x_2}(t) =& \quad B(t)x_1(t) - d_2(t)x_2^2(t) - \frac{a_1(t)x_2(t)y_1(t)}{k(t)x_2^2(t)+\beta_1(t)x_2(t)+\alpha_1(t)}\\
&-\frac{a_2(t)x_2(t)y_2(t)}{\alpha_2(t)+\beta_2(t)x_2(t)+\gamma(t)y_2(t)},\\
\dot{y_1}(t) =& \quad y_1(t)\big(-d_3(t)+\frac{e_1(t)x_2(t)}{k(t)x_2^2(t)+\beta_1(t)x_2(t)+\alpha_1(t)}-D_1(t)y_1(t)-\frac{c_1(t)y_2(t)}{b_1(t)+y_2(t)}\big),\\
\dot{y_2}(t) =& \quad y_2(t)\big(-d_4(t)+\frac{e_2(t)x_2(t)}{\alpha_2(t)+\beta_2(t)x_2(t)+\gamma(t)y_2(t)}-D_2(t)y_2(t)-\frac{c_2(t)y_1(t)}{b_2(t)+y_1(t)}\big).
\end{aligned}
\tag{1.1}
$$

By means of the comparison method, they obtained the sufficient and necessary conditions for the permanence of system (1.1). In addition, sufficient conditions are derived for the existence of positive periodic solutions to the system (1.1). In [8], Sun et al. considered the following Lotka-Volterra ratio-dependent competitive system with diffusion and delays

$$
\begin{aligned}
&\dot{x_1}(t) = x_1(t)\big[a_1(t)-a_{11}(t)x_1(t)-\frac{a_{13}(t)y(t-\tau)}{x_1(t-\tau)+\alpha(t)y(t-\tau)}\big]+D_1(t)(x_2(t)-x_1(t))\\
&\dot{x_2}(t) = x_2(t)\big[a_2(t)-a_{22}(t)x_2(t)\big]+D_2(t)(x_1(t)-x_2(t))\\
&\dot{y}(t) = \ y(t)\big[a_3(t)-a_{33}(t)y(t)-\frac{a_{31}(t)x_1(t-\tau)}{x_1(t-\tau)+\alpha(t)y(t-\tau)}\big].
\end{aligned}
\tag{1.2}
$$

By using the Mawhin's continuation theorem, the sufficient conditions on the existence of positive periodic solutions were established for system (1.2). In [9], Muhammadhaji, Teng and Abdurahman considered the following ratio-dependent cooperative system with stage structure and delays

$$
\begin{aligned}
&\dot{x}_1(t) = r_1(t)x_2(t) - d_1(t)x_1(t) - r_1(t-\tau)e^{-\int_{t-\tau}^{t} d_1(s)ds}x_2(t-\tau),\\
&\dot{x}_2(t) = r_1(t-\tau)e^{-\int_{t-\tau}^{t} d_1(s)ds}x_2(t-\tau) - d_2(t)x_2^2(t) - c_1(t)x_2(t)\big(a_1(t)+b_1(t)y(t)\big)^{-1},\\
&\dot{y}(t) = y(t)\big[r_2(t)-d(t)y(t)-c_2(t)\big(a_2(t)+b_2(t)x_2(t)\big)^{-1}\big].
\end{aligned}
\tag{1.3}
$$

By using the comparison method, they obtained some sufficient conditions on the permanence and extinction of system (1.3). To the best of our knowledge, results on the ratio-dependent competitive system with delays are fairly rare. Based on the above observations, we propose and investigate the following nonautonomous Lotka-Volterra competitive system with delays

$$
\begin{aligned}
\dot{x}(t) =& \quad x(t)\big[r_1(t)-a_{11}(t)x(t)-\frac{a_{12}(t)y(t-\tau)}{x(t-\tau)+\alpha(t)y(t-\tau)}\big]\\
\dot{y}(t) =& \quad y(t)\big[r_2(t)-a_{22}(t)y(t)-\frac{a_{21}(t)x(t-\tau)}{x(t-\tau)+\alpha(t)y(t-\tau)}\big].
\end{aligned}
\tag{1.4}
$$

Our main purpose is to establish some sufficient conditions on the boundedness, permanence and attractivity of system (1.4) by using the comparison method and Lyapunov function method.

2. Preliminaries

In system (1.4), we have that $x(t), y(t)$ are represent the population density of species x and species y at time t respectively, $r_i(t)(i=1,2)$ denote the intrinsic growth rate of species x and species y at time t, respectively; $a_{ii}(t)(i=1,2)$ denote the intrapatch restriction density of species x and species y at time t, respectively; $a_{12}(t)$ and $a_{21}(t)$ represent the the competitive coefficients of species x and species y at time t,respectively; $\alpha(t)$ is ratio coefficient and $\tau > 0$ is a constant.

In this paper, we always assume that

($\mathbf{H_1}$) $r_i(t), \alpha(t), a_{ij}(t)$ $(i, j = 1, 2)$ are all strictly positive continuous functions.

($\mathbf{H_2}$) $r_i(t), \alpha(t), a_{ij}(t)$ $(i, j = 1, 2)$ are all strictly positive ω-periodic continuous functions with $\omega > 0$.

Throughout this paper for system (1.4) we consider the solution with the following initial condition

$$x(s) = \phi(s) \quad \text{and} \quad y(s) = \psi(s), \quad \text{for all} \quad s \in [-\tau, 0], \tag{2.1}$$

where $\phi(t),\ \psi(t)$ are nonnegative continuous functions defined on $[-\tau, 0]$ satisfying $\phi(0) > 0,\ \psi(0) > 0$.

In this paper, for any ω-periodic continuous function $f(t)$ we denote

$$f^L = \min_{t\in[0,\omega]} f(t), \quad f^M = \max_{t\in[0,\omega]} f(t).$$

Now, we present some useful definition and lemmas.

Definition 1 System (1.4) is said to be permanent if there exist positive constants $m_i, M_i (i = 1, 2)$ and T, such that each positive solution $(x(t), y(t))$ of system (1.4) with any positive initial value φ, fulfill $m_1 \le x(t) \le M_1, \quad m_2 \le y(t) \le M_2$ for all $t \ge T$, where T may depend on φ.

Definition 2. System (1.4) is said to be global attractive, if for any two positive solutions $(x(t), y(t))$ and $(u(t), v(t))$ of system (1.4), one has

$$\lim_{t\to\infty}(x(t) - u(t)) = 0, \quad \lim_{t\to\infty}(y(t) - v(t)) = 0.$$

Lemma 1.[4] Consider the following equation: $\dot{u}(t) = u(t)(d_1 - d_2u(t))$, where $d_2 > 0$. Then

(1) If $d_1 > 0, then \lim_{t\to+\infty} u(t) = d_1/d_2$.

(2) If $d_1 < 0, then \lim_{t\to+\infty} u(t) = 0$.

Lemma 2. [13] If there exist positive constants m and M for any $\Phi \in C^n_+[-\tau, 0]$ such that

$$m < \liminf_{t\to\infty} x_i(t, 0, \Phi) \le \limsup_{t\to\infty} x_i(t, 0, \Phi) < M, \qquad i = 1, 2, \cdots, n,$$

then the following general functional differential equation

$$\frac{dx}{dt} = F(t, x_t)$$

admits at least one positive ω-periodic solution, where $x(t) \in R^n$ and $F(t, x_t)$ is a n-dimensional continuous functional, $x(t, 0, \Phi) = (x_1(t, 0, \Phi), x_2(t, 0, \Phi), \cdots, x_n(t, 0, \Phi))$ is a solution of the functional differential equation with initial condition $x_0 = \Phi$.

3. Boundedness, permanence and periodic solution

In this section, we obtain some sufficient conditions for the ultimately boundedness, permanence and positive $\omega-$periodic solution of system (1.4).

Theorem 1 Assume that (H_1) holds and $a_{12}^M < \alpha^L r_1^L,\ \ a_{21}^M < r_2^L$. Then system (1.4) is permanent.

We need the following two lemmas to prove Theorem 1.

Lemma 3 Assume that (H_1) holds. Then solutions of system (1.4) with initial condition (2.1) are ultimately bounded from above.

Proof. Suppose that $z(t) = (x(t), y(t))$ is any solution of system (1.4) with initial conditions (2.1). First, it follows from the first equation of system (1.4) that, for $t > \tau$,

$$\dot{x}(t) \leq x(t)\left[r_1^M - a_{11}^L x(t)\right].$$

We consider the following auxiliary equation

$$\dot{u}(t) = u(t)\left[r_1^M - a_{11}^L u(t)\right]$$

By Lemma 1, we derive

$$\lim_{t\to+\infty} u(t) = \frac{r_1^M}{a_{11}^L} \triangleq M_1.$$

By comparison, there exists a $T_0 > \tau$ such that $x(t) \leq M_1$ for $t \geq T_0$. Next from the second equation of system (1.4) for $t > \tau$, we have

$$\dot{y}(t) \leq y(t)\left[r_2^M - a_{22}^L y(t)\right].$$

We consider the following auxiliary equation

$$\dot{u}(t) = u(t)\left[r_2^M - a_{22}^L u(t)\right].$$

By Lemma 1, we derive

$$\lim_{t\to+\infty} u(t) = \frac{r_2^M}{a_{22}^L} \triangleq M_2.$$

By comparison, there exists a $T_1 > \tau$ such that $y(t) \leq M_2$ for $t \geq T_1$.

Lemma 4 Assume that (H_1) holds and $a_{12}^M < \alpha^L r_1^L,\ \ a_{21}^M < r_2^L$, then solutions of system (1.4) with initial condition (2.1) are ultimately bounded from below.

Proof. Suppose $z(t) = (x(t), y(t))$ is any solution of system (1.4) with initial conditions (2.1). Firstly, it follows from the first equation of system (1.4) that for $t > \tau$, we have

$$\dot{x}(t) \geq x(t)\left[r_1^L - \frac{a_{12}^M}{\alpha^L} - a_{11}^M x(t)\right].$$

We consider the following auxiliary equation

$$\dot{u}(t) = u(t)\left[r_1^L - \frac{a_{12}^M}{\alpha^L} - a_{11}^M u(t)\right].$$

By Lemma 1, we derive

$$\lim_{t\to+\infty} u(t) = \frac{r_1^L - \frac{a_{12}^M}{\alpha^L}}{a_{11}^M} \triangleq m_1.$$

By comparison, there exists a $T_2 > \tau$ such that $x(t) \geq m_1$ for $t \geq T_2$. Next from the second equation of system (1.4) for $t > \tau$, we have

$$\dot{y}(t) \geq y(t)\left[r_2^L - a_{21}^M - a_{22}^M y(t)\right].$$

We consider the following auxiliary equation

$$\dot{u}(t) = u(t)\left[r_2^L - a_{21}^M - a_{22}^M u(t)\right].$$

By Lemma 1, we derive

$$\lim_{t\to+\infty} u(t) = \frac{r_2^L - a_{21}^M}{a_{22}^M} \triangleq m_2.$$

By comparison, there exists a $T_3 > \tau$ such that $y(t) \geq m_2$ for $t \geq T_3$. This completes the proof.

From Lemma 3 and Lemma 4 we can see the boundedness and permanence of system (1.4).

Corollary 1. Assume that (H_2) holds and $a_{12}^M < \alpha^L r_1^L,\ \ a_{21}^M < r_2^L$. Then system (1.4) is permanent.

As a direct result of Lemma 2, from Corollary 1, we have

Corollary 2. If (H_2) holds and $a_{12}^M < \alpha^L r_1^L,\ \ a_{21}^M < r_2^L$, then system (1.4) admits at least one positive $\omega-$periodic solution.

4. GLOBAL ATTRACTIVITY

In this section, we obtain the sufficient conditions for the global attractivity of system (1.4). First, for convenience we denote the following functions

$$A_1(t) = \frac{a_{12}(t)v(t-\tau)}{\gamma(t)}, \quad A_2(t) = \frac{a_{21}(t)u(t-\tau)}{\gamma(t)}$$

$$B_1(t) = \frac{a_{12}(t)u(t-\tau)}{\gamma(t)}, \quad B_2(t) = \frac{a_{21}(t)v(t-\tau)}{\gamma(t)}$$

$$\gamma(t) = \big(x(t-\tau)+\alpha(t)y(t-\tau)\big)\big(u(t-\tau)+\alpha(t)v(t-\tau)\big)$$

$$A_1 = \frac{a_{12}^M}{\alpha^L m_1} \geq \frac{a_{12}(t)}{\alpha(t)x(t-\tau)} > A_1(t),\ \ A_2 = \frac{a_{21}^M}{\alpha^L m_2} \geq \frac{a_{21}(t)}{\alpha(t)y(t-\tau)} > A_2(t)$$

$$B_1 = \frac{a_{12}^M}{\alpha^L m_2} \geq \frac{a_{12}(t)}{\alpha(t)y(t-\tau)} > B_1(t),\ \ B_2 = \frac{a_{21}^M}{\alpha^L m_1} \geq \frac{a_{21}(t)}{\alpha(t)x(t-\tau)} > B_2(t)$$

$$A = A_1 + A_2, \quad B = B_1 + B_2,$$

where

$$\gamma(t) = \big(x(t-\tau)+\alpha(t)y(t-\tau)\big)\big(u(t-\tau)+\alpha(t)v(t-\tau)\big).$$

On the global attractivity of system (1.4), we have the following result.

Theorem 4. Suppose that the conditions of Theorem 1 hold and $a_{11}^L - A > 0,\ \ a_{22}^L - B > 0$. Then system (1.4) is globally attractive.

Proof. Suppose that $(x(t), y(t))$ and (u(t),v(t)) are any two positive solutions of system (1.4). From Theorem 1, there exist $T > 0$ and positive constants M_i, m_i $(i = 1, 2)$ such that $m_1 < x(t) < M_1$, $m_2 < y(t) < M_2$ for all $t \geq T$. Now, we define a Lyapunov function as follows

$$V(t) = |\ln x(t) - \ln u(t)| + |\ln y(t) - \ln v(t)| + A\int_{t-\tau}^{t} |x(s) - u(s)|ds + B\int_{t-\tau}^{t} |y(s) - v(s)|ds.$$

Calculating the upper right derivative of $V(t)$ along system (1.4), we have

$$\begin{aligned}
D^+V(t) = \; & \mathrm{sign}(x(t) - u(t))\Big[-a_{11}(t)(x(t) - u(t)) + \\
& + a_{12}(t)\Big(\frac{y(t-\tau)}{x(t-\tau) + \alpha(t)y(t-\tau)} - \frac{v(t-\tau)}{u(t-\tau) + \alpha(t)v(t-\tau)}\Big)\Big] \\
& + \mathrm{sign}(y(t) - v(t))\Big[-a_{22}(t)(y(t) - v(t)) + \\
& + a_{21}(t)\Big(\frac{x(t-\tau)}{x(t-\tau) + \alpha(t)y(t-\tau)} - \frac{u(t-\tau)}{u(t-\tau) + \alpha(t)v(t-\tau)}\Big)\Big] \\
& + A|x(t) - u(t)| - A|x(t-\tau) - u(t-\tau)| \\
& + B|y(t) - v(t)| - B|y(t-\tau) - v(t-\tau)| \\
= \; & \mathrm{sign}(x(t) - u(t))\big[-a_{11}(t)(x(t) - u(t)) + \\
& + A_1(t)(x(t-\tau) - u(t-\tau)) - B_1(t)(y(t-\tau) - v(t-\tau))\big] \\
& + \mathrm{sign}(y(t) - v(t))\Big[-a_{22}(t)(y(t) - v(t)) + \\
& - A_2(t)(x(t-\tau) - u(t-\tau)) + B_2(t)(y(t-\tau) - v(t-\tau))\big] \\
& + A|x(t) - u(t)| - A|x(t-\tau) - u(t-\tau)| \\
& + B|y(t) - v(t)| - B|y(t-\tau) - v(t-\tau)| \\
\leq \; & -a_{11}(t)|x(t) - u(t)| + A_1(t)|x(t-\tau) - u(t-\tau)| + B_1(t)|y(t-\tau) - v(t-\tau)| \\
& - a_{22}(t)|y(t) - v(t)| + A_2(t)|x(t-\tau) - u(t-\tau)| + B_2(t)|y(t-\tau) - v(t-\tau)| \\
& + A|x(t) - u(t)| - A|x(t-\tau) - u(t-\tau)| \\
& + B|y(t) - v(t)| - B|y(t-\tau) - v(t-\tau)| \\
\leq \; & -a_{11}^L|x(t) - u(t)| + A_1|x(t-\tau) - u(t-\tau)| + B_1|y(t-\tau) - v(t-\tau)| \\
& - a_{22}^L|y(t) - v(t)| + A_2|x(t-\tau) - u(t-\tau)| + B_2|y(t-\tau) - v(t-\tau)| \\
& + A|x(t) - u(t)| - A|x(t-\tau) - u(t-\tau)| \\
& + B|y(t) - v(t)| - B|y(t-\tau) - v(t-\tau)| \\
= \; & -(a_{11}^L - A)|x(t) - u(t)| - (a_{22}^L - B)|y(t) - v(t)|.
\end{aligned}$$

Let $C = \min\{a_{11}^L - A, a_{22}^L - B\}$. Then we have

$$D^+V(t) \le -C\big(|x(t)-u(t)| + |y(t)-v(t)|\big). \tag{4.1}$$

Integrating from T to t on both sides of (4.1) produces

$$V(t) + C\int_T^t \big(|x(s)-u(s)| + |y(s)-v(s)|\big)ds. \le V(T).$$

Hence, $V(t)$ bounded on $[T,\infty)$ and we have

$$\int_T^t C\big(|x(s)-u(s)| + |y(s)-v(s)|\big)ds < \infty. \tag{4.2}$$

From (4.2) and permanence of system (1.4), we can obtain that $|x(s)-u(s)| + |y(s)-v(s)|$ and their derivatives remain bounded on $[T,\infty)$. As a consequence, $|x(s)-u(s)| + |y(s)-v(s)|$ is uniformly continuous on $[T,\infty)$. By Barbalat's lemma, it follows that

$$\lim_{t\to\infty}(|x(s)-u(s)| + |y(s)-v(s)|) = 0.$$

Hence,

$$\lim_{t\to\infty}(x(s)-u(s)) = 0 \quad \text{and} \quad \lim_{t\to\infty}(y(s)-v(s)) = 0.$$

This completes the proof.

On the existence and global attractivity of positive periodic solutions of system (1.4), we have the following result.

Corollary 2. If the conditions of Corollary 2 hold and $a_{11}^L - A > 0$, $a_{22}^L - B > 0$, then system (1.4) has a positive ω-periodic solution which is globally attractive.

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