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# GLOBAL ATTRACTORS FOR A NONLINEAR BRESSE SYSTEM WITH DELAY IN THE INTERNAL FEEDBACKS

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**Abstract.** The aim of this paper is to investigate a nonlinear Bresse system with delay in the internal feedbacks. The existence of global attractors with finite fractal dimension is established and the exponential stability of the system is obtained.

**Keywords.** Bresse system; Global attractor; Finite fractal dimension; Long-time dynamics.

**2010 Mathematics Subject Classification.** 37L05, 37L30.

## 1. Introduction

In this paper, we are concerned with the long-time dynamics of solutions of the following nonlinear Bresse system with time delay term in the internal feedbacks

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x+\psi+lw)_x - k_0 l(w_x - l\varphi) = g_1, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x+\psi+lw) + \gamma_1\psi_t + \mu_1\psi_t(x,t-\tau_1) + f_1(\psi) = g_2, \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x+\psi+lw) + \gamma_2 w_t + \mu_2 w_t(x,t-\tau_2) + f_2(w) = g_3, \end{cases} \tag{1.1}$$

where $\varphi$, $\psi$ and $w$ represent the vertical displacement, the shear angle and the longitudinal displacement, respectively, $\rho_1, \rho_2, b, k$ and $k_0$ are positive coefficients, $l$ is the curvature of the beam, $\gamma_1\psi_t$ and $\gamma_2 w_t$ represent frictional dampings, the functions $\mu_1\psi_t(x,\, t-\tau_1)$ and $\mu_2 w_t(x,\, t-\tau_2)$ are the time delay effects on the beam, $f_1(\psi), f_2(w), g_1, g_2$ and $g_3$ are the forcing terms, where $\gamma_i, \mu_i, \tau_i, i = 1,2$ are positive constants.

The system is subject to the Dirichlet boundary conditions

$$\varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = w(0,t) = w(1,t), \quad t > 0,$$

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and the initial conditions

$$\begin{cases} \varphi(x,0)=\varphi_0(x), \varphi_t(x,0)=\varphi_1(x), \\ \psi(x,0)=\psi_0(x), \psi_t(x,0)=\psi_1(x), \\ \psi_t(x,t-\tau_1)=f_0(x,t-\tau_1), \quad (x,t)\in(0,1)\times(0,\tau_1), \\ w(x,0)=w_0(x), w_t(x,0)=w_1(x), \\ w_t(x,t-\tau_2)=\tilde{f}_0(x,t-\tau_2), \quad (x,t)\in(0,1)\times(0,\tau_2). \end{cases}$$

Problems without delay $(i.e., \mu_1=\mu_2=0)$ have been considered by many authors during the past decades and many results have been obtained; see [1, 3, 5, 7, 10]. Recently, Feng and Yang [4] investigated the following nonlinear Timoshenko system

$$\begin{cases} \rho_1\varphi_{tt}-k(\varphi_x+\psi)_x=h, \\ \rho_2\psi_{tt}-b\psi_{xx}+k(\varphi_x+\psi)+\mu_1\psi_t+\mu_2\psi_t(x,t-\tau_)+f(\psi)=g. \end{cases}$$

They obtained the existence of a global attractor with finite fractal dimension for the case of equal speed wave propagation. The existence of exponential attractors was also derived.

For the Bresse system, Ma and Monteiro [11] studied the semilinear system with nonlinear damping and forcing terms

$$\begin{cases} \rho_1\varphi_{tt}-k(\varphi_x+\psi+lw)_x-k_0l(w_x-l\varphi)+g_1(\varphi_t)+f_1(\varphi,\psi,w)=0, \\ \rho_2\psi_{tt}-b\psi_{xx}+k(\varphi_x+\psi+lw)+g_2(\psi_t)+f_2(\varphi,\psi,w)=0, \\ \rho_1 w_{tt}-k_0(w_x-l\varphi)_x+kl(\varphi_x+\psi+lw)+g_3(w_t)+f_3(\varphi,\psi,w)=0. \end{cases}$$

They established the existence of a smooth global attractor and proved the regularity and finite dimension of attractors. Furthermore, they also proved the existence of exponential attractors.

Motivated by the above results, our purpose in this paper is to establish the well-posedness of system (1.1) by using semi-group methods and prove the existence of exponential attractors. The paper is organized as follows. In Section 2, we present lemmas. In Section 3, the well-posedness and the long-time dynamics of system (1.1) are discussed. Finally, in Section 4, we establish the existence of exponential attractors.

## 2. LEMMAS

We use the standard notations of Lebesgue integral spaces and Sobolev spaces as

$$\mathbb{L}^q(0,1)\ (1\le q\le\infty) \quad \text{and} \quad H_0^1(0,1).$$

For simplify, we write $\|u\|$ instead of $\|u\|_2$ when $q=2$. We impose the following assumptions on the forcing term $f$

$$|f(\psi^2)-f(\psi^1)|\le k_1\left(|\psi^1|^\theta+|\psi^2|^\theta\right)|\psi^1-\psi^2|, \quad \text{for all} \quad \psi^1,\psi^2\in\mathbb{R}, \tag{2.1}$$

with $k_1>0$ and $\theta>0$. In addition, we assume that, for some constant $k_2\ge 0$,

$$-k_2\le\hat{f}(\psi)\le f(\psi)\psi, \quad \text{for all} \quad \psi\in\mathbb{R}, \tag{2.2}$$

where $\hat{f}(z)=\int_0^z f(s)ds$.

**Lemma 2.1.** *There exists a positive constant* $\boldsymbol{C}$ *such that the following inequality holds, for every* $(\varphi,\psi,w)\in[H_0^1(0,1)]^3$,

$$\int_0^1\left(\varphi_x^2+\psi_x^2+w_x^2\right)dx\le \boldsymbol{C}\int_0^1\left(b\psi_x^2+k(\varphi_x+\psi+lw)^2+k_0(w_x-l\varphi)^2\right)dx.$$

In order to prove our well posedness result, we make the following operations, as in [8], the new variables

$$\begin{aligned}z_1(x,\rho,t)&=\psi_t(x,t-\tau_1\rho),\quad x\in(0,1),\quad \rho\in(0,1),\quad t>0,\\ z_2(x,\rho,t)&=w_t(x,t-\tau_2\rho),\quad x\in(0,1),\quad \rho\in(0,1),\quad t>0.\end{aligned}$$

Hence, we have

$$\begin{aligned}\tau_1z_{1t}(x,\rho,t)+z_{1\rho}(x,\rho,t)&=0,\quad in\quad (0,1)\times(0,1)\times(0,\infty),\\ \tau_2z_{2t}(x,\rho,t)+z_{2\rho}(x,\rho,t)&=0,\quad in\quad (0,1)\times(0,1)\times(0,\infty).\end{aligned}$$

Then, system (1.1) is transformed into

$$\begin{cases}\rho_1\varphi_{tt}-k(\varphi_x+\psi+lw)_x-k_0l(w_x-l\varphi)=g_1,\\ \rho_2\psi_{tt}-b\psi_{xx}+k(\varphi_x+\psi+lw)+\gamma_1\psi_t+\mu_1z_1(x,1,t)+f_1(\psi)=g_2,\\ \tau_1z_{1t}(x,\rho,t)+z_{1\rho}(x,\rho,t)=0,\\ \rho_1w_{tt}-k_0(w_x-l\varphi)_x+kl(\varphi_x+\psi+lw)+\gamma_2w_t+\mu_2z_2(x,1,t)+f_2(w)=g_3,\\ \tau_2z_{2t}(x,\rho,t)+z_{2\rho}(x,\rho,t)=0,\end{cases}\tag{2.3}$$

with $x\in(0,1),\rho\in(0,1)$. The above system subjected to the following initial and boundary conditions

$$\begin{cases}\varphi(x,0)=\varphi_0,\varphi_t(x,0)=\varphi_1,\psi(x,0)=\psi_0,\psi_t(x,0)=\psi_1,\quad x\in(0,1),\\ w(x,0)=w_0,w_t(x,0)=w_1,\quad x\in(0,1),\\ z_1(x,\rho,0)=f_0(x,-\rho\tau_1),\quad (x,t)\in(0,1)\times(0,\tau_1),\\ z_2(x,\rho,0)=\tilde{f}_0(x,-\rho\tau_2),\quad (x,t)\in(0,1)\times(0,\tau_2),\\ \varphi(0,t)=\varphi(1,t)=\psi(0,t)=\psi(1,t)=w(0,t)=w(1,t)=0\quad t>0,\\ z_1(x,0,t)=\psi_t,z_2(x,0,t)=w_t,\quad x\in(0,1),\end{cases}\tag{2.4}$$

where $\xi_1$ and $\xi_2$ are two positive constants such that

$$\begin{cases}\tau_1\mu_1\le\xi_1\le\tau_1(2\gamma_1-\mu_1),\\ \\ \tau_2\mu_2\le\xi_2\le\tau_2(2\gamma_2-\mu_2).\end{cases}\tag{2.5}$$

We define in $\mathscr{H}=(H_0^1(0,1))^3\times(\mathbb{L}^2(0,1))^3\times(\mathbb{L}^2(0,1)\times(0,1))^2$ the energy associated to the solution of problem (2.3)-(2.4) by

$$\begin{aligned}E(t)\ &=\frac{1}{2}\int_0^1[\rho_1\varphi_t^2+\rho_2\psi_t^2+\rho_1w_t^2+b\psi_x^2+k|\varphi_x+\psi+lw|^2\\ &\quad+k_0|w_x-l\varphi|^2]dx+\frac{\xi_1}{2}\int_0^1\int_0^1z_1^2(x,\rho,t)d\rho dx\\ &\quad+\frac{\xi_2}{2}\int_0^1\int_0^1z_2^2(x,\rho,t)d\rho dx+\int_0^1\widehat{f}_1(\psi)dx\\ &\quad+\int_0^1\widehat{f}_2(w)dx-\int_0^1(g_1\varphi+g_2\psi+g_3w)\,dx.\end{aligned}\tag{2.6}$$

**Lemma 2.2.** *If $(\varphi,\psi,w,z_1,z_2)$ is the solution of problem (2.3)-(2.4), then, for any $t\geq 0$, there exists a constant $C>0$ such that energy $E(t)$ satisfies*

$$E'(t)\leq -C\int_0^1\psi_t^2dx-C\int_0^1 w_t^2dx-C\int_0^1 z_1^2(x,1,t)-C\int_0^1 z_2^2(x,1,t)dx, \tag{2.7}$$

*and there exist tow positive constants $\delta_0$ and $C_1$ which are independent of initial data in $\mathscr{H}$, such that, for any $t\geq 0$,*

$$\begin{aligned} E(t) \quad &\geq \delta_0\Big(\int_0^1\varphi_t^2+\psi_t^2+\int_0^1\psi_x^2dx+\int_0^1|\varphi_x+\psi+lw|^2\\ &+\int_0^1(w_x-l\varphi)^2dx+\int_0^1\int_0^1 z_1^2(x,\rho,t)d\rho dx\\ &+\int_0^1\int_0^1 z_2^2(x,\rho,t)d\rho dx\Big)-C_1. \end{aligned}\tag{2.8}$$

*Proof.* We multiply the first equation in (2.3) by $\varphi_t$, the second equation by $\psi_t$ and the fourth by $w_t$, integrate the result over $(0,1)$ with respect to $x$ and use Young's inequality to get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\rho_1\varphi_t^2+\rho_2\psi_t^2+\rho_1 w_t^2+b\psi_x^2+k|\varphi_x+\psi+lw|^2+k_0|w_x-l\varphi|^2\Big)\\ &=-\gamma_1\int_0^1\psi_t^2dx-\mu_1\int_0^1 z_1(x,1,t)\psi_t dx\\ &-\gamma_2\int_0^1 w_t^2dx-\mu_2\int_0^1 z_2(x,1,t)\psi_t dx\\ &\leq\Big(-\gamma_1+\frac{\mu_1}{2}\Big)\int_0^1\psi_t^2dx+\Big(-\gamma_2+\frac{\mu_2}{2}\Big)\int_0^1 w_t^2dx\\ &+\frac{\mu_1}{2}\int_0^1 z_1^2(x,1,t)dx+\frac{\mu_2}{2}\int_0^1 z_2^2(x,1,t)dx. \end{aligned}\tag{2.9}$$

Multiplying the third and the fifth equation in (2.3) by $\frac{\xi_1}{\tau_1}z_1$ ,$\frac{\xi_2}{\tau_2}z_2$ and integrating over $(0,1)\times(0,1)$ with respect to $\rho$ and $x$, we obtain

$$\frac{\xi_1}{2}\frac{d}{dt}\int_0^1\int_0^1 z_1(x,\rho,t)d\rho dx=-\frac{\xi_1}{2\tau_1}\int_0^1\int_0^1\frac{\partial}{\partial\rho}z_1^2(x,\rho,t)d\rho dx=\frac{\xi_1}{2\tau_1}\int_0^1(z_1^2(x,0,t)-z_1^2(x,1,t))dx.$$

Similarly, we have

$$\frac{\xi_2}{2}\frac{d}{dt}\int_0^1\int_0^1 z_2(x,\rho,t)d\rho dx \quad =\frac{\xi_2}{2\tau_2}\int_0^1(z_2^2(x,0,t)-z_2^2(x,1,t))dx.$$

Using (2.9), (2.5) and invoking that $\frac{d}{dt}\widehat{f_1}(\psi)=f_1(\psi)\psi_t$ yield (2.7). Therefore, for any $\delta_1$,

$$\int_0^1(g_1\varphi+g_2\psi+g_3w)\,dx\leq\delta_1\int_0^1\left(\varphi_x^2+\psi_x^2+w_x^2\right)dx+C_{\delta_1}\int_0^1\left(g_1^2+g_2^2+g_3^2\right)dx,$$

which, together with (2.6) and (2.2), gives us (2.8). The proof is complete. □

## 3. WELL POSEDNESS

Now, we give well posedness results for problem (2.3)-(2.4) by using the semigroup theory. We introduce three new dependent variables $u=\phi_t, v=\psi_t$ and $\omega=w_t$. (2.3)-(2.4) is reduced to the following Cauchy problem of abstract first order evolutionary operator equation

$$\begin{cases}\dfrac{dU}{dt}(t)=\mathscr{A}U+F, \quad t>0,\\ U(0)=U_0=\left(\varphi_0,\varphi_1,\psi_0,\psi_1,w_0,w_1,f_0(.,-\tau_1),\tilde{f}_0(.,\tau_2)\right)^T,\end{cases}\tag{3.1}$$

where $U=(\varphi,u,\psi,v,w,\omega,z_1,z_2)^T$ and

$$\mathscr{A}U=\begin{pmatrix} u \\ \frac{k}{\rho_1}(\varphi_x+\psi+lw)_x+\frac{k_0 l}{\rho_1}(w_x-l\varphi) \\ v \\ \frac{b}{\rho_2}\psi_{xx}-\frac{k}{\rho_2}(\varphi_x+\psi+lw)-\frac{\gamma_1}{\rho_2}\psi_t-\frac{\mu_1}{\rho_2}z_1(x,1,t) \\ \omega \\ \frac{k_0}{\rho_1}(w_x-l\varphi)_x-\frac{kl}{\rho_1}(\varphi_x+\psi+lw)-\frac{\gamma_2}{\rho_1}w_t-\frac{\mu_2}{\rho_1}z_2(x,1,t) \\ -\frac{1}{\tau_1}z_{1\rho} \\ -\frac{1}{\tau_2}z_{2\rho} \end{pmatrix},\ F=\begin{pmatrix} 0 \\ g_1 \\ 0 \\ -\frac{1}{\rho_2}f_1(\psi)+g_2 \\ 0 \\ -\frac{1}{\rho_1}f_2(w)+g_3 \\ 0 \\ 0 \end{pmatrix}$$

with the domain $H$ given by

$$H=(H^2(0,1)\cap H_0^1(0,1))^3\times(H_0^1(0,1))^3\times(\mathbb{L}^2(0,1;H_0^1(0,1))^2,$$

and the energy space

$$\mathscr{H}=(H_0^1(0,1))^3\times(\mathbb{L}^2(0,1))^3\times(\mathbb{L}^2(0,1)\times(0,1))^2.$$

The domain $D(\mathscr{A})$ of $\mathscr{A}$ is defined by

$$D(\mathscr{A})=\Big\{(\varphi,u,\psi,v,w,\omega,z_1,z_2)^T\in H: v=z_1(.,0),\ \omega=z_2(.,0),\quad \text{in}\quad (0,1)\Big\}.$$

For $U=(\varphi,u,\psi,v,w,\omega,z_1,z_2)^T,\overline{U}=(\overline{\varphi},\overline{u},\overline{\psi},\overline{v},\overline{w},\overline{\omega},\overline{z_1},\overline{z_2})^T$, the energy space $\mathscr{H}$ is equipped with the inner product

$$\begin{aligned}<U,\overline{U}>_{\mathscr{H}} &=\int_0^1\big[\rho_1 u\overline{u}+\rho_2 v\overline{v}+k(\varphi_x+\psi+lw)(\overline{\varphi_x}+\overline{\psi}+\overline{lw})\\ &+k_0(w_x-l\varphi)(\overline{w_x}-l\overline{\varphi})\big]dx+\xi_i\int_0^1\int_0^1 z_i(x,\rho)\overline{z}(x,\rho)d\rho dx,\end{aligned}$$

with $i=1,2$.

**Theorem 3.1.** *(Local existence). Assume that (2.1) and (2.7) and $\mu_1\le\gamma_1,\mu_2\le\gamma_2$ hold. Then*

*$(i)$ Given $U_0\in\mathscr{H}$, problem (3.1) has a unique mild solution $U\in C([0,\infty),\mathscr{H})$ with $U(0)=U_0$.*

*$(ii)$ If $U_1$ and $U_2$ are two mild solutions of problem (3.1), then there exists a positive constant $C_0=C_0(U_1(0),U_2(0))$ such that*

$$\|U_1(t)-U_2(t)\|_{\mathscr{H}}\le e^{C_0T}\|U_1(0)-U_2(0)\|_{\mathscr{H}},\quad for\quad all\quad 0\le t\le T. \tag{3.2}$$

*$(iii)$ If $U_0\in D(\mathscr{A})$, then the above mild solution can be improved as a strong solution.*

**Lemma 3.2.** *The operator $\mathscr{A}$ defined in (3.1) is the infinitesimal generator of a $C_0$- semigroup in $\mathscr{H}$ .*

*Proof.* First, we show that $\mathscr{A}$ is dissipative. For $U=(\varphi,u,\psi,v,w,\omega,z_1,z_2)^T\in D(\mathscr{A})$, we have

$$\begin{aligned}<\mathscr{A}U,U>_{\mathscr{H}} &=-\gamma_1\int_0^1 v^2dx-\gamma_2\int_0^1\omega^2dx-\mu_1\int_0^1 z_1(x,1)v(x)dx\\ &-\mu_2\int_0^1 z_2(x,1)\omega(x)dx-\frac{\xi_1}{\tau_1}\int_0^1 z_1(x,\rho)z_{1\rho}(x,\rho)d\rho dx\\ &-\frac{\xi_2}{\tau_2}\int_0^1 z_2(x,\rho)z_{2\rho}(x,\rho)d\rho dx,\end{aligned} \tag{3.3}$$

since

$$\int_0^1\int_0^1 z_i(x,\rho)z_{i\rho}(x,\rho)d\rho dx = \frac{1}{2}\int_0^1\int_0^1 \frac{\partial}{\partial\rho}z_i^2(x,\rho)d\rho = \frac{1}{2}\int_0^1 \left(z_i^2(x,1)-z_i^2(x,0)\right)dx.$$

Consequently, (3.3) becomes

$$\begin{aligned} <\mathscr{A}U,U>_{\mathscr{H}} &= -\gamma_1\int_0^1 v^2(x)dx - \gamma_2\int_0^1 \omega^2 dx \\ &-\mu_1\int_0^1 z_1(x,\rho)v(x)dx - \mu_2\int_0^1 z_2(x,\rho)\omega(x)dx \\ &-\frac{\xi_1}{2\tau_1}\int_0^1 v^2(x)dx - \frac{\xi_2}{2\tau_2}\int_0^1 \omega^2(x)dx. \end{aligned} \tag{3.4}$$

By using the Young's inequality, we obtain from (3.4) that

$$\begin{aligned} <\mathscr{A}U,U>_{\mathscr{H}} &\leq \left(-\gamma_1+\frac{\mu_1}{2}+\frac{\xi_1}{2\tau_1}\right)\int_0^1 v^2(x)dx + \left(-\gamma_2+\frac{\mu_2}{2}+\frac{\xi_2}{2\tau_2}\right)\int_0^1 \omega^2(x)dx \\ &+\left(\frac{\mu_1}{2}-\frac{\xi_1}{2\tau_1}\right)\int_0^1 z_1^2(x,1)dx + \left(\frac{\mu_2}{2}-\frac{\xi_2}{2\tau_2}\right)\int_0^1 z_2^2(x,1)dx. \end{aligned}$$

It follows from (2.5) that

$$\begin{aligned} &-\gamma_1+\frac{\mu_1}{2}+\frac{\xi_1}{2\tau_1}<0, \quad \frac{\mu_1}{2}-\frac{\xi_1}{2\tau_1}<0, \\ &-\gamma_2+\frac{\mu_2}{2}+\frac{\xi_2}{2\tau_2}<0, \quad \frac{\mu_2}{2}-\frac{\xi_2}{2\tau_2}<0. \end{aligned}$$

Hence, operator $\mathscr{A}$ is dissipative. Now, we are in a position to prove that $\lambda I-\mathscr{A}$ is surjective for $\lambda>0$. For this purpose, we let $(f_1,f_2,f_3,f_4,f_5,f_6,f_7,f_8)^T\in\mathscr{H}$ and find $U=(\varphi,u,\psi,v,w,\omega,z_1,z_2)^T\in D(\mathscr{A})$, solution of the following system of equation:

$$\begin{cases} \lambda\varphi-u=f_1, \\ \lambda u-\frac{k}{\rho_1}(\varphi_x+\psi+lw)_x-\frac{k_0 l}{\rho_1}(w_x-l\varphi)=f_2, \\ \lambda\psi-v=f_3, \\ \lambda v-\frac{b}{\rho_2}\psi_{xx}+\frac{k}{\rho_2}(\varphi_x+\psi+lw)+\frac{\gamma_1}{\rho_2}v+\frac{\mu_1}{2}z_1(.,1)=f_4, \\ \lambda w-\omega=f_5, \\ \lambda\omega-\frac{k_0}{\rho_1}(w_x-l\varphi)_x+\frac{kl}{\rho_1}(\varphi_x+\psi+lw)+\frac{\gamma_2}{\rho_1}\omega+\frac{\mu_2}{\rho_2}z_2(.,1)=f_6, \\ \lambda z_1+\frac{1}{\tau_1}z_{1\rho}=f_7, \\ \lambda z_2+\frac{1}{\tau_2}z_{2\rho}=f_8. \end{cases} \tag{3.5}$$

From the first and the third and the five equations in (3.5), we have

$$\begin{cases} u=\lambda\varphi-f_1, \\ v=\lambda\psi-f_3, \\ \omega=\lambda w-f_5. \end{cases} \tag{3.6}$$

Then, it is clear that $u\in H_0^1(0,1), v\in H_0^1(0,1)$ and $\omega\in H_0^1(0,1)$. Furthermore, we can find from (3.5) that

$$z_1(x,\rho)=v(x), \quad z_2(x,\rho)=\omega(x), \quad \text{for} \quad x\in(0,1).$$

Following the same approach as in [8], we can obtain

$$z_1(x,\rho) = v(x)e^{-\lambda\rho\tau_1} + \tau_1 e^{-\lambda\rho\tau_1}\int_0^1 f_7(x,\sigma)e^{\lambda\sigma\tau_1}d\sigma,$$
$$z_2(x,\rho) = \omega(x)e^{-\lambda\rho\tau_2} + \tau_2 e^{-\lambda\rho\tau_2}\int_0^1 f_8(x,\sigma)e^{\lambda\sigma\tau_2}d\sigma.$$

Exploiting (3.6), we find that

$$\begin{aligned} z_1(x,\rho) &= \lambda\psi(x)e^{-\lambda\rho\tau_1} - f_3 e^{-\lambda\rho\tau_1} + \tau_1 e^{-\lambda\rho\tau_1}\int_0^1 f_7(x,\sigma)e^{\lambda\sigma\tau_1}d\sigma,\\ z_2(x,\rho) &= \lambda w(x)e^{-\lambda\rho\tau_2} - f_5 e^{-\lambda\rho\tau_2} + \tau_2 e^{-\lambda\rho\tau_2}\int_0^1 f_8(x,\sigma)e^{\lambda\sigma\tau_2}d\sigma. \end{aligned} \tag{3.7}$$

By using (3.5) and (3.6), $\varphi,\psi$ and $w$ satisfy the following system:

$$\begin{cases} \lambda^2\varphi - \dfrac{k}{\rho_1}(\varphi_x+\psi+lw)_x - \dfrac{k_0 l}{\rho_1}(w_x - l\varphi) = f_2 + \lambda f_1,\\ \lambda^2\psi - \dfrac{b}{\rho_2}\psi_{xx} + \dfrac{k}{\rho_2}(\varphi_x+\psi+lw) + \dfrac{\gamma_1}{\rho_2}v + \dfrac{\mu_1}{2}z_1(.,1) = f_4 + \lambda f_3,\\ \lambda^2 w - \dfrac{k}{\rho_1}(w_x - l\varphi)_x + \dfrac{kl}{\rho_1}(\varphi_x+\psi+lw) + \dfrac{\gamma_2}{\rho_1}\omega + \dfrac{\mu_2}{\rho_2}z_2(.,1) = f_6 + \lambda f_5. \end{cases} \tag{3.8}$$

Solving system (3.8) is equivalent to finding $(\varphi,\psi,w) \in H^2(0,1)\cap H_0^1(0,1) \times H^2(0,1)\cap H_0^1(0,1) \times H^2(0,1)\cap H_0^1(0,1)$ such that

$$\begin{cases} \displaystyle\int_0^1 \left(\rho_1\lambda^2\varphi\tilde{\varphi} + k(\varphi_x+\psi+lw)\tilde{\varphi}_x - k_0 l(w_x - l\varphi)\tilde{\varphi}\right)dx = \int_0^1 \rho_1(f_2+\lambda f_1)\tilde{\varphi}dx,\\ \displaystyle\int_0^1 \left(\rho_2\lambda^2\psi\tilde{\psi} + b\psi_x\tilde{\psi}_x + k(\varphi_x+\psi+lw)\tilde{\psi} + \gamma_1 v\tilde{\psi} + \mu_1 z_1(.,1)\tilde{\psi}\right)dx = \int_0^1 \rho_2(f_4+\lambda f_3)\tilde{\psi}dx,\\ \displaystyle\int_0^1 \left(\rho_1\lambda^2 w\tilde{w} + k(w_x - l\varphi)\tilde{w}_x + kl(\varphi_x+\psi+lw)\tilde{w} + \gamma_2\omega\tilde{w} + \mu_2 z_2(.,1)\tilde{w}\right)dx = \int \rho_1(f_6+\lambda f_5)\tilde{w}dx. \end{cases} \tag{3.9}$$

For all $(\tilde{\varphi},\tilde{\psi},\tilde{w}) \in H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)$, we obtain by using (3.7) and $x \in (0,1)$ that

$$z_1(x,1) = \lambda\psi(x)e^{-\lambda\tau_1} + z_0(x); \qquad z_2(x,1) = \lambda w(x)e^{-\lambda\tau_2} + \tilde{z}_0(x),$$

where

$$z_0(x) = -f_3 e^{-\lambda\tau_1} + \tau_1 e^{-\lambda\tau_1}\int_0^1 f_7(x,\sigma)e^{\lambda\sigma\tau_1}d\sigma,$$
$$\tilde{z}_0(x) = -f_5 e^{-\lambda\tau_2} + \tau_2 e^{-\lambda\tau_2}\int_0^1 f_8(x,\sigma)e^{\lambda\sigma\tau_2}d\sigma.$$

It is clear from above formula that $z_0$ and $\tilde{z}_0$ depend only on $f_3, f_5, f_7, f_8$. Consequently, problem (3.9) is equivalent to the problem

$$a((\varphi,\psi,w)(\tilde{\varphi},\tilde{\psi},\tilde{w})) = l(\tilde{\varphi},\tilde{\psi},\tilde{w}), \tag{3.10}$$

where the bilinear form $a : [H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)]^2 \to \mathbb{R}$ and the linear form $l : H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1) \to \mathbb{R}$ are defined by

$$\begin{aligned} a((\varphi,\psi,w)(\tilde{\varphi},\tilde{\psi},\tilde{w})) &= \int_0^1 \left(\rho_1\lambda^2\varphi\tilde{\varphi} + \rho_2\lambda^2\psi\tilde{\psi}) + k(\varphi_x+\psi+lw)(\tilde{\varphi}_x+\tilde{\psi}+\tilde{w})\right)dx\\ &+ \int_0^1 \left(b\psi_x\tilde{\psi}_x - k_0(\varphi_x - lw)(\tilde{\varphi}_x - l\tilde{w})\right)dx\\ &+ \int_0^1 \left(\gamma_1 + \mu_1 e^{-\lambda\tau_1}\right)\lambda\psi\tilde{\psi}dx + \int_0^1 \left(\gamma_2 + \mu_2 e^{-\lambda\tau_2}\right)\lambda w\tilde{w}dx, \end{aligned}$$

and

$$\begin{aligned} l(\tilde{\varphi},\tilde{\psi},\tilde{w}) &= \int_0^1 (\gamma_1 f_3\tilde{\psi} - \mu_1 z_0(x)\tilde{\psi})\,dx + \int_0^1 (\gamma_2 f_5\tilde{w} - \mu_2\tilde{z}_0(x)\tilde{w})\,dx \\ &+ \int_0^1 \rho_1\,((f_2+\lambda f_1)+(f_6+\lambda f_5))\,dx + \int_0^1 \rho_2\,(f_4+\lambda f_3)\,dx. \end{aligned}$$

It is easy to verify that $a$ is continuous and coercive, and $l$ is continuous. So applying the Lax-Milgram theorem, we deduce that, for all $(\tilde{\varphi},\tilde{\psi},\tilde{w}) \in H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)$, problem (3.10) admits a unique solution $(\varphi,\psi,w)\in H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)$. Applying the classical elliptic regularity, one obtains from (3.9) that $(\varphi,\psi,w)\in H^2(0,1)\times H^2(0,1)\times H^2(0,1)$. Therefore, $\lambda I-\mathscr{A}$ is surjective for any $\lambda>0$. Consequently, we can infer that $\mathscr{A}$ is maximal dissipative in $\mathscr{H}$. Since $D(\mathscr{A})$ is dense in $\mathscr{H}$, we can conclude that $\mathscr{A}$ is the infinitesimal generator of a $C_0$-semigroup in $\mathscr{H}$ by the Lummer-Phillips theorem. The proof is now complete. □

**Lemma 3.3.** *The function $F$ is locally Lipschitz.*

*Proof.* Letting $U_1=(\varphi^1,u^1,\psi^1,v^1,w^1,\omega^1,z_1^1,z_2^1)$ and $U_2=(\varphi^2,u^2,\psi^2,v^2,w^2,\omega^2,z_1^2,z_2^2)$, we have

$$\begin{aligned} \|F(U_1)-F(U_2)\|^2_{\mathscr{H}} &= \left\| \begin{pmatrix} 0\\0\\0\\ -\frac{1}{\rho_2}f_1(\psi^1)\\ 0\\ -\frac{1}{\rho_1}f_2(w^1)\\ 0\\0 \end{pmatrix} - \begin{pmatrix} 0\\0\\0\\ -\frac{1}{\rho_2}f_1(\psi^2)\\ 0\\ -\frac{1}{\rho_1}f_2(w^2)\\ 0\\0 \end{pmatrix} \right\|^2_{\mathscr{H}} \\ &\le \|f_1(\psi^1)-f_1(\psi^2)\|_{L^2} + \|f_2(w^1)-f_2(w^2)\|_{L^2} \\ &\le \left(\|\psi^1\|^{\theta}_{2\theta}+\|\psi^2\|^{\theta}_{2\theta}\right)\|\psi^1-\psi^2\| + \left(\|w^1\|^{\theta}_{2\theta}+\|w^2\|^{\theta}_{2\theta}\right)\|w^1-w^2\| \\ &\le \kappa_1\|\psi_x^1-\psi_x^1\| + \kappa_2\|w_x^1-w_x^2\|, \end{aligned}$$

which leads to

$$\|F(U_1)-F(U_2)\|_{\mathscr{H}} \le \|U_1-U_2\|_{\mathscr{H}}.$$

Hence, operator $F$ is locally Lipschitz in $\mathscr{H}$. The proof is complete. □

*Proof of Theorem 3.1.* We deduce from Lemma 3.2 and Lemma 3.3 that the Cauchy problem has a unique local mild solution

$$U(t)=e^{\mathscr{A}t}U_0+\int_0^t e^{\mathscr{A}(t-s)}F(U(s))ds, \tag{3.11}$$

defined in a maximal interval $(0,t_{\max})$. If $t_{\max}<0$, then

$$\lim_{t\to\infty}\|U(t)\|_{\mathscr{H}}=+\infty. \tag{3.12}$$

Let $U(t)$ be a mild solution with $U_0\in D(\mathscr{A})$. By using Theorem 6.1.5 in Pazy [12], we conclude that it is a strong solution. It follows from (2.8) that, for all $t\ge 0$,

$$\|U(t)\|^2_{\mathscr{H}} \le \frac{1}{\delta_0}(E(0)+C_1),$$

which, by density, holds for mild solutions. Then it is a contradiction to (3.12). Therefore, $t_{\max} = \infty$, that is, the solution is global. The proof of (i) of Theorem 3.1 is complete. By using (3.11), we obtain inequality (3.2), the local Lipschitz behavior of $F$ and the Gronwall's inequality. Then we can obtain the continuous dependence on the initial data for mild solutions. This proves item (ii) of Theorem 3.1. By using Theorem 6.1.5 in Pazy [12], we know that any mild solution with initial data in $D(\mathscr{A})$ is strong. This completes the proof of Theorem 3.1. □

**Theorem 3.4.** *(Global attractor)* [2] *Let $(\varphi, \psi, w, z_1, z_2)$ be the solution of problem (2.3)-(2.4). Assume that $\mu_1 \leq \gamma_1, \mu_2 \leq \gamma_2$. Let the functions $g_1, g_2, g_3 \in L^2(0,1)$. Then, for any $U_0 \in \mathscr{H}$, the dynamical system $(\mathscr{H}, S(t))$ corresponding to problem (3.1) possesses a compact connected global attractor $\mathfrak{A}$ in $\mathscr{H}$.*

**Theorem 3.5.** [2] *The attractor obtained in Theorem 3.4 has finite fractal dimension.*

**Remark 3.6.** Let $U(t)$ be the unique solution of problem (3.1) in Theorem 3.1. We can derive one parameter family of operators

$$\left\{ S(t) : \mathscr{H} \to \mathscr{H} | S(t)U_0 = U(t), \quad \text{for all} \quad t \geq 0 \right\}, \tag{3.13}$$

satisfying

$$S(0) = I \quad and \quad S(t+s) = S(t) \circ S(s), \quad for \quad all \quad s,t \geq 0.$$

Moreover, $S(t)$ is a nonlinear $C_0$-semigroup, which is locally Lipschitz continuous on $\mathscr{H}$. Hence, we can study the long time dynamics of problem (3.1) as a dynamical system $(\mathscr{H}, S(t))$.

## 4. Long-time dynamics

In this section, we establish the existence of global attractors of system (2.3)-(2.4) to complete the proof of Theorem 3.4.

4.1. **Generation of the dynamical system.** Let $X$ be a Banach space. Recall that the one-parameter operator $S(t) : X \to X \quad (t \geq 0)$ is said to be a semigroup if

$$S(t_1 + t_2) = S(t_1)S(t_2) \quad \text{and} \quad S(0) = \mathbb{I}_d,$$

hold for all $t_1, t_2 \geq 0$, where $\mathbb{I}_d$ is the identity operator. The existence of global attractors relies on two properties, namely, the dissipativeness and the compactness.

A dynamical system is said to be dissipative if it has a set $B_0 \subset X$, which is called an absorbing set for the semigroup $S(t)(t \geq 0)$ that attracts any bounded set $B \subset X$ in a finite time $t_1 = t_1(B) > 0$ such that, for all $t > t_1$, $S(t)B \subseteq B_0$.

For compactness, a dynamical system $(\mathscr{H}, S(t))$ is called asymptotically compact if for any bounded $B \subset \mathscr{H}$ and sequence $x_n \subset B$, the sequence $S(t_n)x_n$ has convergent subsequence whenever $t_n \to \infty$.

A compact set $\mathscr{A} \subset X$ is said to be a global attractor of semigroup $S(t)$ if

(i) $\mathscr{A}$ is strictly invariant with respect to $S(t)$, i.e., for all $t \geq 0$,

$$S(t)\mathscr{A} = \mathscr{A},$$

(ii) $\mathscr{A}$ attracts any bounded set $B \subset X$, i.e., for any $\varepsilon > 0$, there exists a time $t_1 = t_1(\varepsilon, B) > 0$ such that, for all $t \geq t_1(\varepsilon, B)$,

$$S(t)B \subseteq \mathscr{O}_\varepsilon(\mathscr{A}),$$

where $\mathscr{O}_\varepsilon(Y)$ is an $\varepsilon$-neighborhood of a set $Y$ in $X$.

The fractal dimension of compact set $M$ in a metric space $X$ is a number defined by

$$dim_f^X M = \lim_{\varepsilon\to 0}\sup \frac{lnN(M,\varepsilon)}{ln(1/\varepsilon)},$$

where $N_\varepsilon(M)$ is the minimal number of closed balls with radius $\varepsilon > 0$, which covers $M$.

We can find the following theorems in Hale [6], and Chueshov and Lasiecka [2, Chapter 7].

*A compactness criterion*

**Theorem 4.1.** *Let $X$ be a Banach space and $S(t)(t \geq 0)$ be a continuous semigroup on the Banach space $X$. Then, $S(t)$ has a global attractor $A \subset X$, if and only if*

*(i) $S(t)$ has a bounded absorbing set $B_0$,*

*(ii) $S(t)$ is asymptotically smooth.*

**Theorem 4.2.** *Suppose that, for any bounded positively invariant set $B \subset \mathscr{H}$ and for any $\varepsilon > 0$, there exists $T = T(\varepsilon, B)$ such that*

$$\|S(T)x - S(T)y\|_{\mathscr{H}} \leq \varepsilon + \phi_T(x,y), \quad \forall x,y \in B,$$

*where $\phi_T : B\times B \to \mathbb{R}$ satisfies*

$$\lim_{n\to\infty}\inf \lim_{n\to\infty} \phi_T(z_n, z_m) = 0,$$

*for any sequence $z_n$ in $B$. Then $S(t)$ is asymptotically smooth in $\mathscr{H}$.*

*Quasi-stable systems*

A semi-norm $n_X(.)$ defined on a Banach space $X$ is compact if there exists a sequence $x_j \to 0$ weakly in $X$ such that $n_X(x_j) \to 0$. Let $X,Y,Z$ be three reflexive Banach spaces with $X$ compactly embedded in $Y$ and put $\mathscr{H} = X\times Y\times X\times Y\times Z\times Z$. Consider the dynamical system $(\mathscr{H}, S(t))$ given by an evolution operator

$$S(t)U_0 = (\varphi,\varphi_t,\psi,\psi_t,w,w_t,z_1,z_2), U_0 = (\varphi_0,\varphi_1,\psi_0,\psi_1,w_0,w_1,z_1,z_2) \in \mathscr{H}, \tag{4.1}$$

where $\varphi,\psi,w$ and $z_1,z_2$ have the regularity

$$\begin{aligned} &\varphi \in C(\mathbb{R}^+;X)\cap C^1(\mathbb{R}^+;Y), \quad \psi \in C(\mathbb{R}^+;X)\cap C^1(\mathbb{R}^+;Y),\\ &w \in C(\mathbb{R}^+;X)\cap C^1(\mathbb{R}^+;Y), \quad z_1,\ z_2 \in C(\mathbb{R}^+;Z). \end{aligned} \tag{4.2}$$

The dynamical system $(\mathscr{H}, S(t))$ is quasi-stable on a set $B \subset \mathscr{H}$ if there exists a compact semi-norm $n_X$ on $X$ and nonnegative scalar functions $a(t)$ and $c(t)$, which are locally bounded in $[0,\infty)$, and $b(t) \in \mathbb{L}^1(\mathbb{R}^+)$ with $\lim_{t\to\infty} b(t) = 0$ such that

$$\|S(t)U_1 - S(t)U_2\|^2_{\mathscr{H}} \leq a(t)\|U_1 - U_2\|^2_{\mathscr{H}},$$

and

$$\begin{aligned} \|S(t)U_1 - S(t)U_2\|^2_{\mathscr{H}} \quad &\leq b(t)\|U_1-U_2\|^2_{\mathscr{H}} + c(t)\sup_{0<s<t}\left[n_X(\varphi^1(s)-\varphi^2(s))\right.\\ &\left.+n_X(\psi^1(s)-\psi^2(s)) + n_X(w^1(s)-w^2(s))\right]^2, \end{aligned} \tag{4.3}$$

for any $U_1, U_2 \in B$.

**Theorem 4.3.** *Let $(\mathscr{H}, S(t))$ be given by (4.1) and satisfy (4.2). If $(\mathscr{H}, S(t))$ possesses a compact global attractor $\mathfrak{A}$ and is quasi-stable on $\mathscr{H}$, then the attractor $\mathfrak{A}$ has finite fractal dimension.*

In order to prove the existence of global attractors, we need to prove the existence of absorbing sets in Section 4.2, and the dissipation and asymptotical smoothness for compactness in Section 4.3 by using Theorems 4.1 and 4.2.

### 4.2. Existence of absorbing sets.

**Theorem 4.4.** *(Absorbing set) Under the assumptions of Theorem 3.4, the semigroup $S(t)$ defined by (3.13) has a bounded absorbing set $B \subset \mathscr{H}$.*

The proof of Theorem 4.3 will be given through several lemmas. First, we define the functional energy of solutions for problem (2.3)-(2.4) by

$$\begin{aligned} E(t) \quad &= \frac{1}{2}\int_0^1 [\rho_1\varphi_t^2+\rho_2\psi_t^2+\rho_1 w_t^2+b\psi_x^2+k|\varphi_x+\psi+lw|^2 \\ &+k_0|w_x-l\varphi|^2]dx+\frac{\xi_1}{2}\int_0^1\int_0^1 z_1^2(x,\rho,t)d\rho dx \\ &+\frac{\xi_2}{2}\int_0^1\int_0^1 z_2^2(x,\rho,t)d\rho dx+\int_0^1 \widehat{f}_1(\psi)dx \\ &+\int_0^1 \widehat{f}_2(w)dx-\int_0^1 (g_1\varphi+g_2\psi+g_3 w)\,dx. \end{aligned}$$

**Lemma 4.5.** *Let $(\varphi,\varphi_t,\psi,\psi_t,w,w_t,z_1,z_2)$ be the solution of problem (2.3)-(2.4). We define the functional $I_1(t)$ by*

$$I_1(t) = -\int_0^1 (\rho_1\varphi\varphi_t+\rho_2\psi\psi_t+\rho_1 w w_t)\,dx-\frac{\gamma_1}{2}\int_0^1 \psi^2 dx-\frac{\gamma_2}{2}\int_0^1 w^2 dx.$$

*Then, for any $\varepsilon>0$,*

$$\begin{aligned} \frac{d}{dt}I_1(t) \quad &\le -\int_0^1 \left(\rho_1\varphi_t^2+\rho_2\psi_t^2+\rho_1 w_t^2\right)dx+b\varsigma_1\int_0^1 \psi_x^2 dx \\ &+k(1+\varsigma_1)\int_0^1 (\varphi_x+\psi+lw)^2dx+k_0(1+\varsigma_1)\int_0^1 (w_x-l\varphi)^2dx \\ &+\frac{\mu_1^2}{4\varepsilon\lambda_1}\int_0^1 z_1^2(x,1,t)dx+\frac{\mu_2^2}{4\varepsilon\lambda_1}\int_0^1 z_2^2(x,1,t)dx \\ &+\frac{1}{4\varepsilon\lambda_1^2}\int_0^1 \left(g_1^2+g_3^2\right)dx+\frac{1}{4\varepsilon\lambda_1}\int_0^1 g_2^2 dx, \end{aligned} \tag{4.4}$$

*where $\lambda_1>0$ denotes the first eigenvalue of $-\Delta$ in $H_0^1(0,1)$.*

*Proof.* It is obvious that

$$\begin{aligned} \frac{dI_1}{dt} \quad &= -\int_0^1 (\rho_1\varphi_{tt}\varphi+\rho_2\psi_{tt}\psi+\rho_1 w_{tt}w)\,dx-\int_0^1 \left(\rho_1\varphi_t^2+\rho_2\psi_t^2+\rho_1 w_t^2\right)dx \\ &-\gamma_1\int_0^1 \psi\psi_t dx-\gamma_2\int_0^1 w w_t dx. \end{aligned}$$

Using (2.3), we have

$$\begin{aligned} \frac{dI_1}{dt} \quad &= -\int_0^1 \left(\rho_1\varphi_t^2+\rho_2\psi_t^2+\rho_1 w_t^2\right)dx+b\int_0^1 \psi_x^2dx+k\int_0^1 (\varphi_x+\psi+lw)^2dx \\ &+k_0\int_0^1 (w_x-l\varphi)^2dx+\mu_1\int_0^1 z_1(x,1,t)\psi dx+\mu_2\int_0^1 z_2(x,1,t)w dx \\ &+\int_0^1 f_1(\psi)\psi dx+\int_0^1 f_2(w)w dx-\int_0^1 (g_1\varphi+g_2\psi+g_3 w)\,dx. \end{aligned} \tag{4.5}$$

Recalling the Young's inequality and the Poincaré's inequality, we get, for any $\varepsilon > 0$,

$$\begin{aligned}\int_0^1 |z_1(x,1,t)\psi|dx &\leq \varepsilon\lambda_1 \int_0^1 \psi^2 dx + \frac{1}{4\varepsilon\lambda_1}\int_0^1 z_1^2(x,1,t)dx \\ &\leq \varepsilon \int_0^1 \psi_x^2 dx + \frac{1}{4\varepsilon\lambda_1}\int_0^1 z_1^2(x,1,t)dx.\end{aligned} \tag{4.6}$$

Similarly, we obtain

$$\int_0^1 |z_2(x,1,t)w|dx \leq \varepsilon \int_0^1 w_x^2 dx + \frac{1}{4\varepsilon\lambda_1}\int_0^1 z_2^2(x,1,t)dx, \tag{4.7}$$

$$\int_0^1 |f_1(\psi)\psi|dx \leq \int_0^1 |\psi|^\theta|\psi||\psi|dx \leq \|\psi\|_{2(\theta+1)}^\theta \|\psi\|_{2(\theta+1)}\|\psi\| \leq c_0 \int_0^1 \psi_x^2 dx, \tag{4.8}$$

$$\int_0^1 |f_2(w)w|dx \quad \leq c_0 \int_0^1 w_x^2 dx, \tag{4.9}$$

and

$$\begin{aligned}\int_0^1 |g_1\varphi + g_2\psi + g_3 w|dx &\leq \lambda_1^2\varepsilon \int_0^1 \varphi^2 dx + \frac{1}{4\varepsilon\lambda_1^2}\int_0^1 g_1^2 dx + \lambda_1\varepsilon\int_0^1 \psi^2 dx + \frac{1}{4\varepsilon\lambda_1}\int_0^1 g_2^2 dx \\ &+\lambda_1^2\varepsilon\int_0^1 w^2 dx + \frac{1}{4\varepsilon\lambda_1^2}\int_0^1 g_3^2 dx \leq \lambda_1\varepsilon\int_0^1 \varphi_x^2 + \frac{1}{4\varepsilon\lambda_1^2}\int_0^1 g_1^2 dx \\ &+\varepsilon\int_0^1 \psi_x^2 + \frac{1}{4\varepsilon\lambda_1}\int_0^1 g_2^2 dx + \lambda_1\varepsilon\int_0^1 w_x^2 + \frac{1}{4\varepsilon\lambda_1^2}\int_0^1 g_3^2 dx.\end{aligned} \tag{4.10}$$

From Lemma 2.1, we have

$$\lambda_1\varepsilon\varphi_x^2 + (b+2\varepsilon+c_0)\,\psi_x^2 + (\varepsilon+c_0+\lambda_1\varepsilon)\,w_x^2 \leq \varsigma_1\left(b\psi_x^2 + k(\varphi_x+\psi+lw)^2 + k_0(w_x-l\varphi)^2\right),$$

with

$$\varsigma_1 = \mathbf{C}\max\left\{\lambda_1\varepsilon, \quad b+2\varepsilon+c_0, \quad \varepsilon+c_0+\lambda_1\varepsilon\right\}.$$

Substituting (4.6)-(4.10) into (4.5), we obtain (4.4). □

**Lemma 4.6.** *Let $(\varphi,\varphi_t,\psi,\psi_t,w,w_t,z_1,z_2)$ be the solution of problem (2.3)-(2.4). Then the functional $I_2(t)$ defined by*

$$I_2(t) = \int_0^1 (\rho_2\psi_t\psi + \rho_1\varphi_t j)\,dx + \frac{\gamma_1}{2}\int_0^1 \psi^2 dx,$$

*satisfies, for any $\eta,\tilde{\eta} > 0$,*

$$\begin{aligned}\frac{d}{dt}I_2(t) &\leq \frac{\rho_1\tilde{\eta}}{\lambda_1}\int_0^1 \varphi_t^2 dx + \left(\rho_2 + \frac{\rho_1}{4\tilde{\eta}}\right)\int_0^1 \psi_t^2 dx \\ &+(-b+\eta(\mu_1+2k+k_0 l+2)+c_0)\int_0^1 \psi_x^2 dx \\ &+\frac{k}{2\eta\lambda_1}\int_0^1 (\varphi_x+\psi+lw)^2 dx + \frac{k_0 l}{4\eta\lambda_1}\int_0^1 (w_x-l\varphi)^2 dx \\ &+\frac{\mu_1}{4\eta\lambda_1}\int_0^1 z_1^2(x,1,t)dx + \frac{1}{4\eta\lambda_1}\int_0^1 (g_1^2+g_2^2)dx,\end{aligned} \tag{4.11}$$

*where $j$ is the solution of*

$$-j_{xx} = \psi_x, \quad j|_{x=0,1} = 0. \tag{4.12}$$

*Proof.* It is clear that

$$\begin{aligned}\frac{d}{dt}I_2(t) &= \int_0^1 \left(\rho_2\psi_t^2+\rho_1\varphi_t j_t\right)dx - b\int_0^1 \psi_x^2 dx - \mu_1\int_0^1 z_1(x,1,t)\psi dx\\ &-\int_0^1 f_1(\psi)\psi dx - k\int_0^1(\varphi_x+\psi+lw)\psi dx + \int_0^1 g_2\psi dx\\ &+k\int_0^1(\varphi_x+\psi+lw)_x j dx + k_0 l\int_0^1(w_x-l\varphi) j dx + \int_0^1 g_1 j dx.\end{aligned}$$

It follows from (4.12) that

$$\begin{aligned}\int_0^1 j_x^2 dx \le \int_0^1 \psi^2 dx \le \int_0^1 \psi_x^2 dx,\\ \int_0^1 j_t^2 dx \le \int_0^1 j_{xt}^2 dx \le \int_0^1 \psi_t^2 dx.\end{aligned}$$

From the Young's inequality and the Poincaré's inequality, we find that

$$\begin{aligned}\mu_1\int_0^1 |\psi z_1(x,1,t)|dx &\le \mu_1\eta\lambda_1\int_0^1 \psi^2 dx + \frac{\mu_1}{4\eta\lambda_1}\int_0^1 z_1^2(x,1,t)dx\\ &\le \mu_1\eta\int_0^1 \psi_x^2 dx + \frac{\mu_1}{4\eta\lambda_1}\int_0^1 z_1^2(x,1,t)dx,\end{aligned} \tag{4.13}$$

$$\begin{aligned}\rho_1\int_0^1 |\varphi_t j_t|dx &\le \frac{\rho_1\tilde\eta}{\lambda_1}\int_0^1 \varphi_t^2 dx + \frac{\rho_1\lambda_1}{4\tilde\eta}\int_0^1 j_t^2 dx\\ &\le \frac{\rho_1\tilde\eta}{\lambda_1}\int_0^1 \varphi_t^2 dx + \frac{\rho_1}{4\tilde\eta}\int_0^1 \psi_t^2 dx,\end{aligned} \tag{4.14}$$

$$k\int_0^1 |(\varphi_x+\psi+lw)\psi|dx \le k\eta\int_0^1 \psi_x^2 dx + \frac{k}{4\eta\lambda_1}\int_0^1(\varphi_x+\psi+lw)^2 dx, \tag{4.15}$$

$$k\int_0^1 |(\varphi_x+\psi+lw)j_x|dx \le \frac{k}{4\eta\lambda_1}\int_0^1(\varphi_x+\psi+lw)^2 dx + k\eta\int_0^1 \psi_x^2 dx, \tag{4.16}$$

$$k_0 l\int_0^1 |(w_x-l\varphi)j|dx \le \frac{k_0 l}{4\eta\lambda_1}\int_0^1(w_x-l\varphi)^2 dx + k_0 l\eta\int_0^1 \psi_x^2 dx, \tag{4.17}$$

$$\int_0^1 |g_1 j|dx \le \eta\int_0^1 \psi_x^2 dx + \frac{1}{4\eta\lambda_1}\int_0^1 g_1^2 dx, \tag{4.18}$$

and

$$\int_0^1 |g_2\psi|dx \le \eta\int_0^1 \psi_x^2 dx + \frac{1}{4\eta\lambda_1}\int_0^1 g_2^2 dx. \tag{4.19}$$

From (4.13)-(4.19,) we easily deduce (4.11). The proof is hence complete. □

Next, we define the functional $J_1(t)$ by

$$J_1(t) = \rho_2\int_0^1 \psi_t(\varphi_x+\psi+lw)dx + \rho_2\int_0^1 \psi_x\varphi_t dx.$$

**Lemma 4.7.** *Let $(\varphi,\varphi_t,\psi,\psi_t,z_1,z_2)$ be the solution of the problem (2.3)-(2.4). Then, for any $\varepsilon>0$, the functional $J_1(t)$ satisfies*

$$\begin{aligned}\frac{d}{dt}J_1(t) &\le b[\varphi_x\psi_x]_{x=0}^{x=1} + k\left(\varsigma_2-\frac{1}{2}\right)\int_0^1(\varphi_x+\psi+lw)^2 dx\\ &+\left(\rho_2(1+l\varepsilon)+\frac{3\gamma_1^2}{2k}\right)\int_0^1 \psi_t^2 dx + \frac{\rho_2 l}{4\varepsilon}\int_0^1 w_t^2 dx\\ &+\varsigma_2 b\int_0^1 \psi_x^2 dx + \left(\frac{bl}{4\varepsilon}+\varsigma_1 k_0\right)\int_0^1(w_x-l\varphi)^2 dx\\ &+\frac{3\mu_1^2}{2k}\int_0^1 z_1^2(x,1,t)dx + \frac{1}{4\rho_1^2\varepsilon}\int_0^1 g_1^2 dx + \frac{3}{2k}\int_0^1 g_2^2 dx.\end{aligned} \tag{4.20}$$

*Proof.* Taking the derivatives of $J_1(t)$, we easily get

$$\begin{aligned}\frac{d}{dt}J_1(t) &= b[\psi_x\varphi_x]_{x=0}^{x=1} + \rho_2\int_0^1 \psi_t^2 dx - k\int_0^1 (\varphi_x+\psi+lw)^2 dx \\ &-\gamma_1\int_0^1 \psi_t(\varphi_x+\psi+lw)dx - \mu_1\int_0^1 z_1(x,1,t)(\varphi_x+\psi+lw)dx \\ &-\int_0^1 f_1(\psi)(\varphi_x+\psi+lw)dx + \int_0^1 g_2(\varphi_x+\psi+lw)dx \\ &+\frac{k_0 l\rho_2}{\rho_1}\int_0^1 (w_x-l\varphi)\psi_x dx + \rho_2 l\int_0^1 \psi_t w_t dx + \frac{\rho_2}{\rho_1}\int_0^1 g_1\psi_x dx.\end{aligned} \tag{4.21}$$

By using the Young's inequality and the Poincaré's inequality, one conclude that, for any $\varepsilon > 0$,

$$\int_0^1 |\varphi_x f_1(\psi)|dx \leq \|\varphi_x\|\|\psi\|_{2(\theta+1)}^{\theta}\|\psi\|_{2(\theta+1)} \leq \frac{\varepsilon}{2b^2}\int_0^1 \varphi_x^2 dx + \frac{b^2}{2\varepsilon\lambda_1}\int_0^1 \psi_x^2 dx, \tag{4.22}$$

and

$$l\int_0^1 |wf_1(\psi)|dx \leq \frac{\varepsilon l}{2b^2\lambda_1}\int_0^1 w_x^2 dx + \frac{b^2 l}{2\varepsilon\lambda_1}\int_0^1 \psi_x^2 dx. \tag{4.23}$$

Choosing $\varepsilon > 0$ small enough and summing (4.22) and (4.23), we get (4.20). The proof is complete. □

Thanks to Said Houari and Laskri [9], we can define the following function to handle the boundary term in (4.21)

$$q(x) = 2-4x, \quad x\in(0,1).$$

**Lemma 4.8.** *Let $(\varphi,\varphi_t,\psi,\psi_t,w,w_t,z_1,z_2)$ be the solution of problem (2.3)-(2.4). Then, for any $\varepsilon>0$,*

$$\begin{aligned}b[\psi_x\varphi_x]_{x=0}^{x=1} &\leq \frac{-\varepsilon\rho_1}{k}\frac{d}{dt}\int_0^1 q\varphi_t\varphi_x dx - \frac{\rho_2 b}{4\varepsilon}\frac{d}{dt}\int_0^1 q\psi_t\psi_x dx \\ &+\left(\frac{\varepsilon\varsigma_3 b}{k} + \frac{b}{4\varepsilon} + \frac{b^2}{2\varepsilon} + \frac{3b^2}{4} + \frac{b^2}{4\varepsilon^3}\right)\int_0^1 \psi_x^2 dx \\ &+\left(\frac{\rho_2 b}{2\varepsilon} + \frac{\gamma_1^2}{4\varepsilon^2}\right)\int_0^1 \psi_t^2 dx + \frac{\mu_1^2}{4\varepsilon^2}\int_0^1 z_1^2(x,1,t)dx \\ &+\left(\varepsilon\frac{k^2}{4} + \varsigma_3\varepsilon\right)\int_0^1 (\varphi_x+\psi+lw)^2 dx + +\varsigma_3\varepsilon\int_0^1 (w_x-l\varphi)^2 dx \\ &+\frac{\rho_1}{2\varepsilon}\int_0^1 \varphi_t^2 dx + \frac{\varepsilon}{k^2}\int_0^1 g_1^2 dx + \frac{1}{4\varepsilon^2}\int_0^1 g_2^2 dx.\end{aligned}$$

*Proof.* Following [9], we obtain that, for any $\varepsilon > 0$,

$$b[\psi_x\varphi_x]_{x=0}^{x=1} \leq \varepsilon\left[\varphi_x^2(1)+\varphi_x^2(0)\right] + \frac{b^2}{4\varepsilon}\left[\psi_x^2(1)+\psi_x^2(0)\right]. \tag{4.24}$$

By (2.3), the Young's inequality, and

$$\frac{d}{dt}\int_0^1 b\rho_2 q\psi_t\psi_x dx = \int_0^1 b\rho_2\psi_{tt}\psi_x dx + \int_0^1 b\rho_2 q\psi_t\psi_{xt} dx,$$

we infer that

$$\begin{aligned}\frac{d}{dt}\int_0^1 b\rho_2 q\psi_t\psi_x dx &\leq -b^2\left[\psi_x^2(1)+\psi_x^2(0)\right] + \left(2\rho_2 b + \frac{\gamma_1^2}{\varepsilon}\right)\int_0^1 \psi_t^2 dx \\ &+\left(2b^2+b+3b^2\varepsilon+\frac{b^2}{\varepsilon}\right)\int_0^1 \psi_x^2 dx \\ &+\varepsilon^2 k^2\int_0^1 (\varphi_x+\psi+lw)^2 dx \\ &+\frac{\mu_1^2}{\varepsilon}\int_0^1 z_1^2(x,1,t)dx + \frac{1}{\varepsilon}\int_0^1 g_2^2 dx,\end{aligned} \tag{4.25}$$

and

$$\begin{aligned}\frac{d}{dt}\int_0^1 \rho_1 q\varphi_t\varphi_x dx \quad &\leq -k\left[\varphi_x^2(1)+\varphi_x^2(0)\right]+\varsigma_3 b\int_0^1 \psi_x^2 dx \\ &+\varsigma_3 k_0\int_0^1 (w_x-l\varphi)^2 dx+2\rho_1\int_0^1 \varphi_t^2 dx \\ &+\varsigma_3 k\int_0^1 (\varphi_x+\psi+lw)^2 dx+\frac{1}{k}\int_0^1 g_1^2 dx.\end{aligned} \tag{4.26}$$

From (4.24), (4.25) and (4.26), we get the desired result. The proof is complete. □

We now define the following functional

$$J_2(t)=\rho_1\int_0^1 \left(\varphi_t(w_x-l\varphi)+w_t(\varphi_x+\psi+lw)\right)dx.$$

**Lemma 4.9.** *Let $(\varphi,\varphi_t,\psi,\psi_t,w,w_t,z_1,z_2)$ be the solution of problem (2.3)- (2.4). Then the following estimate holds*

$$\begin{aligned}\frac{d}{dt}J_2(t) \quad &\leq k\left[w_x\varphi_x\right]_{x=0}^{x=1}-\rho_1 l\int_0^1 \varphi_t^2 dx+\left(\rho_1 l+\frac{\gamma_2}{4\varepsilon}+\rho_1\varepsilon\right)\int_0^1 w_t^2 \\ &+\left(\varepsilon(\gamma_2+\mu_2+1)+k(\varsigma_4-l)\right)\int_0^1 (\varphi_x+\psi+lw)^2 dx \\ &+\frac{\rho_1}{4\varepsilon}\int_0^1 \psi_t^2 dx+\varsigma_4 b\int_0^1 \psi_x^2 dx+(k_0(\varsigma_4+l)+\varepsilon)\int_0^1 (w_x-l\varphi)^2 dx \\ &+\frac{\mu_2}{4\varepsilon}\int_0^1 z_2^2(x,1,t)dx+\frac{1}{4\varepsilon}\int_0^1 (g_1^2+g_3^2)dx.\end{aligned}$$

*Proof.* Differentiating $J_2(t)$, we obtain

$$\begin{aligned}\frac{d}{dt}J_2(t) \quad &=\rho_1\int_0^1 \varphi_{tt}(w_x-l\varphi)dx+\rho_1\int_0^1 \varphi_t(w_x-l\varphi)_t dx \\ &+\rho_1\int_0^1 w_{tt}(\varphi_x+\psi+lw)dx+\rho_1\int_0^1 w_t(\varphi_x+\psi+lw)_t dx.\end{aligned}$$

It follows from (2.3) that

$$\begin{aligned}\frac{d}{dt}J_2(t) \quad &=k[w_x\varphi_x]_{x=0}^{x=1}+k_0 l\int_0^1 (w_x-l\varphi)^2 dx-kl\int_0^1 (\varphi_x+\psi+lw)^2 dx \\ &-\rho_1 l\int_0^1 \varphi_t^2 dx+\rho_1\int_0^1 w_t\psi_t dx+\rho_1 l\int_0^1 w_t^2 dx \\ &+\int_0^1 g_1(w_x-l\varphi)dx-\gamma_2\int_0^1 w_t(\varphi_x+\psi+lw)dx \\ &-\mu_2\int_0^1 z_2(x,1,t)(\varphi_x+\psi+lw)dx \\ &-\int_0^1 f_2(w)(\varphi_x+\psi+lw)+\int_0^1 g_3(\varphi_x+\psi+lw)dx.\end{aligned}$$

Using the Young's inequality and the Poincaré's inequality and choosing $\varepsilon>0$ small enough, one can conclude the desired conclusion immediately. □

**Lemma 4.10.** *Let $(\varphi,\varphi_t,\psi,\psi_t,z_1,z_2)$ be the solution of problem (2.3)- (2.4). Then, for any $\varepsilon>0$,*

$$\begin{aligned} k[w_x\varphi_x]_{x=0}^{x=1} &\leq \frac{-\rho_1 k}{4\varepsilon}\int_0^1 q w_t w_x dx - \frac{\rho_1\varepsilon}{k}\int_0^1 q\varphi_t\varphi_x dx \\ &+\left(\frac{\rho_1 k}{2\varepsilon}+\frac{\gamma_2^2}{4\varepsilon^2}\right)\int_0^1 w_t^2 dx + (\varsigma_5 k_0+\varepsilon)\int_0^1 (w_x-l\varphi)^2 dx \\ &+\left(\frac{k^2 l}{4\varepsilon^2}+\varsigma_5 k\right)\int_0^1 (\varphi_x+\psi+lw)^2 dx + \frac{2\rho_1\varepsilon}{k}\int_0^1 \varphi_t^2 dx \\ &+\varsigma_5 b\int_0^1 \psi_x^2 dx + \frac{\mu_2^2}{4\varepsilon^2}\int_0^1 z_2^2(x,1,t)dx + \frac{\varepsilon}{k^2}\int_0^1 g_1^2 dx + \frac{1}{4\varepsilon^2}\int_0^1 g_3^2 dx. \end{aligned} \tag{4.27}$$

*Proof.* We find that, for any $\varepsilon>0$,

$$k[w_x\varphi_x]_{x=0}^{x=1} \leq \varepsilon\left[\varphi_x^2(1)+\varphi_x^2(0)\right] + \frac{k^2}{4\varepsilon}\left[w_x^2(1)+w_x^2(0)\right]. \tag{4.28}$$

Invoking (2.3) and the Young's inequality, we have

$$\frac{d}{dt}\int_0^1 k\rho_1 q w_t w_x dx = \int_0^1 k\rho_1 w_{tt} w_x dx + \int_0^1 k\rho_1 q w_t w_{xt} dx.$$

It follows that

$$\begin{aligned} \frac{d}{dt}\int_0^1 k\rho_1 q w_t w_x dx &\leq -k^2\left[w_x^2(1)+w_x^2(0)\right] + \left(3k^2+2k^2\varepsilon l+k^2\varepsilon+k\right)\int_0^1 w_x^2 dx \\ &+\left(2\rho_1 k+\frac{\gamma_2^2}{\varepsilon}\right)\int_0^1 w_t^2 dx + \frac{k^2 l}{\varepsilon}\int_0^1 \varphi_x^2 dx \\ &+\frac{k^2 l}{\varepsilon}\int_0^1 (\varphi_x^2+\psi+lw)^2 dx + \frac{\mu_2^2}{\varepsilon}\int_0^1 z_2^2(x,1,t)^2 dx + \frac{1}{\varepsilon}\int_0^1 g_3^2 dx, \end{aligned} \tag{4.29}$$

and

$$\begin{aligned} \frac{d}{dt}\int_0^1 \rho_1 q\varphi_t\varphi_x dx &\leq -k\left[\varphi_x^2(1)+\varphi_x^2(0)\right] + 5k\int_0^1 \varphi_x^2 dx \\ &+k\int_0^1 (w_x-l\varphi)^2 dx + 2\rho_1\int_0^1 \varphi_t^2 dx \\ &+k\int_0^1 \psi_x^2 dx + k\int_0^1 w_x^2 dx + \frac{1}{k}\int_0^1 g_1^2 dx. \end{aligned} \tag{4.30}$$

Putting

$$\varsigma_5 = \mathbf{C}\max\left\{\frac{k^2 l}{4\varepsilon^2}+5\varepsilon,\ \varepsilon,\ \frac{3k^2}{4\varepsilon}+\frac{k^2 l}{2}+\frac{k^2}{4}+\frac{k}{4\varepsilon}+\varepsilon\right\}$$

and using (4.28), (4.29) and (4.30), one concludes (4.27). This completes the proof. □

Now, we define the following functional

$$I_3(t) = \int_0^1\int_0^1 e^{-2\tau_1\rho} z_1^2(x,\rho,t)d\rho dx. \tag{4.31}$$

**Lemma 4.11.** *Let $(\varphi,\varphi_t,\psi,\psi_t,w,w_t,z_1,z_2)$ be the solution of problem (2.3)- (2.4). Then*

$$\frac{d}{dt}I_3(t) \leq -I_3(t) - \frac{c}{2\tau_1}\int_0^1 z_1^2(x,\rho,t)dx + \frac{1}{2\tau_1}\int_0^1 \psi_t^2 dx, \tag{4.32}$$

*where $c$ is a positive constant.*

*Proof.* Differentiating (4.31) with respect to $t$ and using the third equation in (2.3), we have

$$\begin{aligned}\frac{d}{dt}\left(\int_0^1\int_0^1 e^{-2\tau_1\rho}z_1^2(x,\rho,t)d\rho dx\right) &= -\frac{1}{\tau_1}\int_0^1\int_0^1 e^{-2\tau_1\rho}z_1z_{1\rho}(x,\rho,t)d\rho dx\\ = -\int_0^1\int_0^1 e^{-2\tau_1\rho}z_1^2(x,\rho,t)d\rho dx &- \frac{1}{2\tau_1}\int_0^1\int_0^1 e^{-2\tau_1\rho}\frac{\partial}{\partial\rho}(z_1^2(x,\rho,t))d\rho dx.\end{aligned}$$

This implies that there exists a positive constant $c$ such that (4.32) holds. □

Now, we define the functional

$$I_4(t) = \int_0^1\int_0^1 e^{-2\tau_2\rho}z_2^2(x,\rho,t)d\rho dx.$$

**Lemma 4.12.** *Let $(\varphi,\varphi_t,\psi,\psi_t,w,w_t,z_1,z_2)$ be the solution of problem (2.3)-(2.4). Then*

$$\frac{d}{dt}I_4(t) \le -I_4(t) - \frac{c}{2\tau_2}\int_0^1 z_1^2(x,\rho,t)dx + \frac{1}{2\tau_2}\int_0^1 w_t^2dx,$$

*where $c$ is a positive constant.*

Next, we define the functional

$$\begin{aligned}L(t) &= ME(t) + \frac{1}{8}I_1(t) + NI_2(t) + J_1(t) + J_2(t) + \frac{2\varepsilon}{k}\int_0^1 \rho_1 q\varphi_t\varphi_x dx\\ &+\frac{\rho_1 b}{4\varepsilon}\int_0^1 q\psi_t\psi_x dx + \frac{\rho_1 k}{4\varepsilon}\int_0^1 qw_tw_x dx + I_3(t) + I_4(t).\end{aligned}$$

**Lemma 4.13.** *Let $(\varphi,\varphi_t,\psi,\psi_t,w,w_t,z_1,z_2)$ be the solution of problem (2.3)- (2.4). For $M$ large enough, there exist two positive constants $\alpha_1$ and $\alpha_2$ depending on $M$, $N$ and $\varepsilon$ such that, for any $t \ge 0$,*

$$\alpha_1 E(t) - C_1\left(\|g_1\|^2 + \|g_2\|^2 + \|g_3\|^2\right) \le L(t) \le \alpha_2 E(t) + C_1\left(\|g_1\|^2 + \|g_2\|^2 + \|g_3\|^2\right). \tag{4.33}$$

*Proof.* We consider the functional

$$\begin{aligned}H(t) &= \frac{1}{8}I_1(t) + NI_2(t) + J_1(t) + J_2(t) + \frac{2\varepsilon}{k}\int_0^1 \rho_1 q\varphi_t\varphi_x dx\\ &+\frac{\rho_1 b}{4\varepsilon}\int_0^1 q\psi_t\psi_x dx + \frac{\rho_1 k}{4\varepsilon}\int_0^1 qw_tw_x dx + I_3(t) + I_4(t).\end{aligned}$$

So,

$$\begin{aligned}|H(t)| = \frac{1}{8}\bigg| &-\int_0^1(\rho_1\varphi_t\varphi + \rho_2\psi_t\psi + \rho_1 w_t w)dx - \frac{\gamma_1}{2}\int_0^1\psi^2dx\\ &- \frac{\gamma_2}{2}\int_0^1 w^2dx\bigg| + N\left|\int_0^1(\rho_2\psi_t\psi + \rho_1\varphi_t j)dx + \frac{\gamma_1}{2}\int_0^1\psi^2dx\right|\\ &+\left|\rho_2\int_0^1\psi_t(\varphi_x+\psi+lw)dx + \rho_2\int_0^1\psi_x\varphi_t dx\right|\\ &+\left|\rho_1\int_0^1\varphi_t(w_x-l\varphi)dx + \rho_1\int_0^1 w_t(\varphi_x+\psi+lw)dx\right|\\ &+\left|\frac{2\rho_1\varepsilon}{k}\int_0^1 q\varphi_t\varphi_x dx + \frac{\rho_2 b}{4\varepsilon}\int_0^1 q\psi_t\psi_x dx + \frac{k\rho_1}{4\varepsilon}\int_0^1 qw_tw_x dx\right|\\ &+\left|\int_0^1\int_0^1 e^{-2\tau_1\rho}z_1^2(x,\rho,t)dxd\rho + \int_0^1\int_0^1 e^{-2\tau_2\rho}z_2^2(x,\rho,t)dxd\rho\right|.\end{aligned}$$

From the Young's and Poincaré's inequalities, we get

$$\begin{aligned} |H(t)| &\leq \nu_1 \int_0^1 \varphi_t^2 dx + \nu_2 \int_0^1 \psi_t^2 dx + \nu_3 \int_0^1 w_t^2 dx + \nu_4 \int_0^1 \psi_x^2 dx \\ &+\nu_5 \int_0^1 (\varphi_x + \psi + lw)^2 dx + \nu_6 \int_0^1 (w_x - l\varphi)^2 dx \\ &+ \int_0^1 \int_0^1 z_1^2(x,\rho,t) d\rho dt + \int_0^1 \int_0^1 z_2^2(x,\rho,t) d\rho dt \\ &+ \int_0^1 \hat{f}_1(\psi) dx + \int_0^1 \hat{f}_2(w) dx, \end{aligned} \tag{4.34}$$

since

$$n_0(\varphi_x^2 + \psi_x^2 + w_x^2) \leq b\psi_x^2 + k(\varphi_x + \psi + lw)^2 + k_0(w_x - l\varphi)^2,$$

with $n_0 = \mathbf{C} \max\left\{ \frac{\rho_1}{16} + \frac{2\varepsilon\rho_1}{k}, \frac{\rho_2}{2}, \frac{\rho_1}{16} + \frac{\gamma_2}{16} + \frac{k\rho_1}{2\varepsilon} \right\}$. The positive constants $\nu_i (i = 1,2,3,4,5,6)$ are given by

$$\begin{aligned} &\nu_1 = \tfrac{\rho_1}{16} + \tfrac{N\rho_1}{2} + \tfrac{\rho_2}{2} + \tfrac{\rho_1}{2} + \tfrac{2\varepsilon\rho_1}{k}, \quad \nu_2 = \tfrac{\rho_2}{16} + \tfrac{N\rho_2}{2} + \tfrac{\rho_2}{2} + \tfrac{\rho_2 b}{2\varepsilon}, \\ &\nu_3 = \tfrac{\rho_1}{16} + \tfrac{\rho_1}{2} + \tfrac{k\rho_1}{2\varepsilon}, \quad \nu_4 = \tfrac{\rho_2}{16} + \tfrac{\gamma_1}{16} + n_0 b + \tfrac{N\rho_2}{2} + \tfrac{N\rho_1}{2} + \tfrac{N\gamma_1}{2} + \tfrac{\rho_2 b}{2}, \\ &\nu_5 = n_0 k + \tfrac{\rho_2}{2} + \tfrac{\rho_1}{2}, \quad \nu_6 = n_0 k_0 + \tfrac{\rho_1}{2}. \end{aligned}$$

Using the Young's inequality, we find

$$\begin{aligned} E(t) &\geq \frac{1}{4} \min\{1, \xi\} \left( \int_0^1 \varphi_t^2 dx + \int_0^1 \psi_t^2 dx + \int_0^1 w_t^2 dx + \int_0^1 \psi_x^2 dx \right. \\ &+ \int_0^1 (\varphi_x + \psi + lw)^2 dx + \int_0^1 (w_x - l\varphi)^2 dx \\ &+ \int_0^1 \int_0^1 z_1^2(x,\rho,t) d\rho dt + \int_0^1 \int_0^1 z_2^2(x,\rho,t) d\rho dt \\ &+ \left. \int_0^1 \hat{f}_1(\psi) dx + \int_0^1 \hat{f}_2(w) dx \right) - C_1 \left( \|g_1\|^2 + \|g_2\|^2 + \|g_3\|^2 \right). \end{aligned} \tag{4.35}$$

Combining (4.34) and (4.35), we see that there exists a positive constant $\tilde{C} > 0$ such that

$$|H(t)| \leq \tilde{C} E(t) + C_1 \left( \|g_1\|^2 + \|g_2\|^2 + \|g_3\|^2 \right).$$

Then we choose $M$ large enough such that $\alpha_1 = M - \tilde{C} > 0$ and $\alpha_2 = M + \tilde{C} > 0$. This complete the proof. □

*Proof of Theorem 4.4.* It follows from the previous Lemmas that

$$
\begin{aligned}
&\frac{d}{dt}L(t) \le \Bigg(-MC-\frac{\rho_2}{8}+N\left(\rho_2+\frac{\rho_1}{4\tilde{\eta}}\right)+\frac{\rho_2 b}{2\varepsilon}+\frac{\gamma_1^2}{4\varepsilon^2}+\rho_2(1+l\varepsilon)\\
&+\frac{3\gamma_1^2}{4\varepsilon^2}+\frac{\rho_1}{4\varepsilon}+\frac{1}{2\tau_1}\Bigg)\int_0^1 \psi_t^2 dx\\
&+\left(\frac{-\rho_1}{8}+\frac{N\rho_1\tilde{\eta}}{\lambda_1}+\frac{\rho_1}{2\varepsilon}+\frac{2\rho_1\varepsilon}{k}-\rho_1 l\right)\int_0^1 \varphi_t^2 dx\\
&+\left(-MC-\frac{\rho_1}{8}+\frac{\rho_2 l}{4\varepsilon}+\frac{\rho_1 k}{2\varepsilon}+\frac{\gamma_2^2}{4\varepsilon^2}+\rho_1 l++\frac{\gamma_2}{4\varepsilon}+\rho_1\varepsilon+\frac{1}{2\tau_2}\right)\int_0^1 w_t^2 dx\\
&+\Bigg(\frac{b\varsigma_1}{8}+N(-b+\eta(\mu_1+k(2+l)+2))+c_0+\frac{\varepsilon\varsigma_3 b}{k}+\frac{b}{4\varepsilon}+\frac{b^2}{2\varepsilon}\\
&+\frac{3b^2}{4}+\frac{b^2}{4\varepsilon^3}+b(\varsigma_2+\varsigma_4+\varsigma_5)\Bigg)\int_0^1 \psi_x^2 dx\\
&+\Bigg(\frac{-3k}{8}+N\frac{k}{4\eta\lambda_1}+\varepsilon\left(\frac{k^2}{4}+\gamma_2+\mu_1+2+\varsigma_3\right)-kl\\
&+k(\frac{\varsigma_1}{8}+\varsigma_2+\varsigma_4+\varsigma_5)\Bigg)\int_0^1 (\varphi_x+\psi+lw)^2 dx\\
&+\Bigg(\frac{k_0}{8}+N\frac{kl}{4\eta\lambda_1}+\varepsilon(\varsigma_3+2)+k_0(\frac{\varsigma_1}{8}+\varsigma_2+\varsigma_4+\varsigma_5)\\
&+\frac{bl}{4\varepsilon}+k_0 l\Bigg)\int_0^1 (w_x-l\varphi)^2 dx\\
&\left(-MC+\frac{\mu_1^2}{32\varepsilon\lambda_1}+N\frac{\mu_1}{4\eta\lambda_1}+\frac{\mu_1^2}{4\varepsilon^2}+\frac{3\mu_1^2}{2k}\right)\int_0^1 z_1^2(x,1,t)dx\\
&+\left(-MC+\frac{\mu_2^2}{32\varepsilon\lambda_1}+\frac{\mu_2^2}{4\varepsilon}\right)\int_0^1 z_2^2(x,1,t)dx\\
&+\left(\frac{1}{32\varepsilon\lambda_1^2}+N\frac{1}{4\eta\lambda_1}+\frac{\varepsilon}{k^2}+\frac{1}{4\rho_1^2\varepsilon}+\varepsilon+\frac{1}{4\varepsilon}\right)\int_0^1 g_1^2 dx\\
&+\left(\frac{1}{32\varepsilon\lambda_1}+N\frac{1}{4\eta\lambda_1}+\frac{1}{4\varepsilon^2}+\frac{3}{2k}\right)\int_0^1 g_2^2 dx+\left(\frac{1}{32\varepsilon\lambda_1^2}+\frac{1}{4\varepsilon^2}+\frac{1}{4\varepsilon}\right)\int_0^1 g_3^2 dx.
\end{aligned}
$$

We choose $\eta$ and $\varepsilon > 0$ small such that

$$\eta \le \frac{b}{2(\mu_1+k(2+l)+2)}\,,$$

and

$$\varepsilon \le \frac{k\left(\frac{3}{8}-\frac{N}{4\eta\lambda_1}+l-(\frac{\varsigma_1}{8}+\varsigma_2+\varsigma_4+\varsigma_5)\right)}{\frac{k^2}{2}+2\gamma_2+2\mu_1+4+2\varsigma_3}\,.$$

We pick $N$ large enough and $\tilde{\eta}$ small enough such that

$$\frac{Nb}{4} \ge \frac{b\varsigma_1}{8}+c_0+\frac{\varepsilon\varsigma_3 b}{k}+\frac{b}{4\varepsilon}+\frac{b^2}{2\varepsilon}+\frac{3b^2}{4}+\frac{b^2}{4\varepsilon^3}+b(\varsigma_2+\varsigma_4+\varsigma_5),$$

and

$$\tilde{\eta} \le \frac{1}{32N}\,.$$

Finally, we select $M$ large enough such that there exists a constant $\delta > 0$ with

$$\begin{aligned}\frac{d}{dt}L(t) &\leq -\delta \int_0^1 \left(\varphi_t^2 + \psi_t^2 + \psi_x^2 + (\varphi_x + \psi + lw)^2 + (w_x - l\varphi)^2 \right. \\ &\quad \left. + z_1^2(x,1,t) + z_2^2(x,1,t)\right) dx - \delta \int_0^1 \int_0^1 \left(z_{1\rho}(x,\rho,t)\right. \\ &\quad \left. + z_{2\rho}(x,\rho,t)\right) d\rho dx + C_2\|g_1\|^2 + C_3\|g_2\|^2 + C_4\|g_3\|^2 + C_5,\end{aligned}$$

which together with (2.6) yields that there exists a positive constant $\beta$ such that

$$\frac{d}{dt}L(t) \leq -\beta E(t) C_2\|g_1\|^2 + C_3\|g_2\|^2 + C_4\|g_3\|^2 + C_5.$$

Using (4.33), one has

$$\frac{d}{dt}L(t) \leq -\frac{\beta}{\alpha_2}L(t) + C_2'\|g_1\|^2 + C_3'\|g_2\|^2 + C_4'\|g_3\|^2 + C_5',$$

and

$$L(t) \leq L(0)e^{-\frac{\beta}{\alpha_2}t} + C_2'\|g_1\|^2 + C_3'\|g_2\|^2 + C_4'\|g_3\|^2 + C_5'.$$

Using (4.33), we obtain

$$\begin{aligned}E(t) &\leq \frac{1}{\alpha_1}\left(\alpha_1 E(0) C_2\|g_1\|^2 + C_3\|g_2\|^2 + C_4\|g_3\|^2\right) e^{\frac{\beta}{\alpha_2}t} \\ &\quad + C_2''\|g_1\|^2 + C_3''\|g_2\|^2 + C_4''\|g_3\|^2 + C_5''.\end{aligned}$$

In view of (2.6), we find

$$\|(\varphi,\varphi_t,\psi,\psi_t,w,w_t,z_1,z_1)\|_{\mathscr{H}}^2 \leq C_0 e^{-\frac{\beta}{\alpha_2}t} + C_1'\|g_1\|^2 + C_1''\|g_2\|^2 + C_1'''\|g_3\|^2 + C_1'''.$$

Then there exists an absorbing ball $B(0,R)$ which radius $R > \sqrt{C_1'\|g_1\|^2 + C_1''\|g_2\|^2 + C_1'''\|g_3\|^2 + C_1'''}$, for the dynamical system $(\mathscr{H}, S(t))$. □

## 4.3. **The quasi-stability.**

**Theorem 4.14.** [2] *Under the hypotheses of Theorem 3.4, the semigroup $S(t)$ defined by (3.13) is asymptotically smooth in $\mathscr{H}$, which is deduced by the quasi-stability inequality.*

*Proof.* For any $(\varphi_0^i, \varphi_1^i, \psi_0^i, \psi_1^i, w_0^i, w_1^i, f_0^i, \tilde{f}_0^i) \in B$, let $(\varphi^i, \varphi_t^i, \psi^i, \psi_t^i, w^i, w_t^i, z_1^i, z_2^i)$ be the corresponding solutions with respect to initial data $(\varphi_0^i, \varphi_1^i, \psi_0^i, \psi_1^i, w_0^i, w_1^i, f_0^i, \tilde{f}_0^i), i = 1,2$. Let

$$W(t) = (\Phi, \Psi, \varpi, \chi_1, \chi_2)^T = U_1(t) - U_2(t) = (\varphi^1 - \varphi^2, \psi^1 - \psi^2, w^1 - w^2, z_1^1 - z_1^2 . z_2^1 - z_2^2),$$

Then, $W(t)$ verifies

$$\begin{cases}\rho_1\Phi_{tt} - k(\Phi_x + \Psi + l\varpi)_x + k_0 l(\varpi_x - l\Phi) = 0, \\ \rho_2\Psi_{tt} - b\Psi_{xx} + k(\Phi_x + \Psi + l\varpi) + \gamma_1\Psi_t + \mu_1\chi_1(x,1,t)(f_1(\Psi^1(t)) - f_1(\Psi^2(t))) = 0, \\ \tau_1\chi_{1t}(x,\rho,t) + \chi_{1\rho}(x,\rho,t) = 0, \\ \rho_1\varpi_{tt} - k_0(\varpi_x - l\Phi)_x + kl(\Phi_x + \Psi + l\varpi) + \gamma_2\varpi_t + \mu_2\chi_2(x,1,t) + (f_2(\varpi^1(t)) - f_2(\varpi^2(t))) = 0, \\ \tau_2\chi_{2t}(x,\rho,t) + \chi_{2\rho}(x,\rho,t) = 0.\end{cases}$$

We denote the associated energy functional by

$$\begin{aligned} F(t) &= \frac{1}{2}\int_0^1 \left(\rho_1\Phi_t^2+\rho_1\Psi_t^2+\rho_1\varpi_t^2+b\Psi_x^2+k|\Phi_x+\Psi+l\varpi|^2\right.\\ &\left.+k_0|\varpi_x-l\Phi|^2\right)dx+\frac{\xi_1}{2}\int_0^1\int_0^1 \chi_1^2(x,\rho,t)d\rho dx\\ &+\frac{\xi_2}{2}\int_0^1\int_0^1 \chi_2^2(x,\rho,t)d\rho dx. \end{aligned}$$

Choosing constant $\eta>0$, we get

$$\begin{aligned} \frac{d}{dt}F(t) &\le -\eta\int_0^1\left(\Psi_t^2+\varpi_t^2+\chi_1^2(x,1,t)+\chi_2^2(x,1,t)\right)dx\\ &-\int_0^1 (f_1(\Psi^1(t)-f_1(\Psi^2(t))))\Psi_t dx-\int_0^1 (f_2(\varpi^1(t))-f_2(\varpi^2(t)))\varpi_t dx\\ &\le -\frac{\eta}{2}\int_0^1\left(\Psi_t^2+\varpi_t^2+\chi_1^2(x,\rho,t)+\chi_2^2(x,\rho,t)\right)dx\\ &+C_B\left(\|\Psi(t)\|_{2(\theta+1)}^2+\|\varpi(t)\|_{2(\theta+1)}^2\right). \end{aligned}$$

We introduce the following multiplies

$$\begin{aligned} A_1(t) &= -\int_0^1 (\rho_1\Phi\Phi_t+\rho_2\Psi\Psi_t+\rho_1\varpi\varpi_t)\,dx-\frac{\gamma_1}{2}\int_0^1\Psi^2dx-\frac{\gamma_2}{2}\int_0^1\varpi^2dx,\\ A_2(t) &= \int_0^1 (\rho_2\Psi_t\Psi+\rho_1\Phi_t j)\,dx+\frac{\gamma_1}{2}\int_0^1\Psi^2dx,\\ A_3(t) &= \rho_2\int_0^1\Psi_t(\Phi_x+\Psi)dx+\rho_2\int_0^1\Psi_x\Phi_t dx,\\ A_4(t) &= \rho_1\int_0^1 (\Phi_t(\varpi_x-l\Phi)+\varpi_t(\Phi_x+\Psi+l\varpi))\,dx,\\ A_5(t) &= \int_0^1 e^{-2\tau_1\rho}\chi_1^2(x,\rho,t)d\rho dx,\\ A_6(t) &= \int_0^1 e^{-2\tau_2\rho}\chi_2^2(x,\rho,t)d\rho dx, \end{aligned}$$

where $j$ is the solution of $-j_{xx}=\Psi_x, j|_{x=0,1}=0$. We define the Lyapunov functional $\mathscr{F}(t)$ by

$$\begin{aligned} \mathscr{F}(t) &= MF(t)+\frac{1}{8}A_1(t)+NA_2(t)+A_3(t)+A_4(t)\\ &+\frac{2\varepsilon}{k}\int_0^1\rho_1 q\Phi_t\Phi_x dx+\frac{\rho_2 b}{4\varepsilon}\int_0^1 q\Psi_t\Psi_x dx+\frac{\rho_1 k}{4\varepsilon}\int_0^1 q\varpi_t\varpi_x dx+A_5(t)+A_6(t), \end{aligned}$$

with $q=2-4x$, $x\in(0,1)$. Then it is easy to verify that there exist two positive constants $\alpha_1,\alpha_2$ such that

$$\alpha_1 F(t)\le\mathscr{F}(t)\le\alpha_2 F(t). \tag{4.36}$$

Note that

$$\begin{aligned} \int_0^1 |f_1(\Psi^1)-f_1(\Psi^2)\Psi|dx &\le \int_0^1\left(|\Psi^1|^\theta+|\Psi|^\theta\right)||\Psi||\Psi|dx\\ &\left(\|\Psi^1\|_{2(\theta+1)}^2+\|\Psi^2\|_{2(\theta+1)}^2\right)\|\Psi\|_{2(\theta+1)}\|\Psi\|\\ &\le\varepsilon\|\Psi_x\|^2+C_B\|\Psi\|_{2(\theta+1)}^2,\quad\forall\varepsilon>0, \end{aligned}$$

$$\int_0^1 |f_2(\varpi^1)-f_2(\varpi^2)\varpi|dx \le \varepsilon\|\varpi_x\|^2+C_B\|\varpi\|_{2(\theta+1)}^2,\quad\forall\varepsilon>0,$$

$$\begin{aligned}\int_0^1 |(f_1(\Psi)-f_1(\Psi^2))\Psi_t|dx &\le \int_0^1 (|\Psi^1|\theta+|\Psi^2|^{\theta})|\Psi||\Psi_t|dx\\ &\left(\|\Psi^1\|^2_{2(\theta+1)}+\|\Psi^2\|^2_{2(\theta+1)}\right)\|\Psi\|_{2(\theta+1)}\|\Psi_t\|\\ &\le \varepsilon\|\Psi_t\|^2+C_B\|\Psi\|^2_{2(\theta+1)},\quad \forall\varepsilon>0,\end{aligned}$$

and

$$\int_0^1 |f_2(\varpi^1)-f_2(\varpi^2)\varpi|dx \quad \le \varepsilon\|\varpi_t\|^2+C_B\|\varpi\|^2_{2(\theta+1)},\quad \forall\varepsilon>0.$$

Choosing appropriate constants, we can derive that there exists a positive constant $\alpha$ such that

$$\frac{d}{dt}\mathscr{F}(t)\le -\alpha F(t)+C_B\left(\|\Psi(t)\|^2_{2(\theta+1)}+\|\varpi(t)\|^2_{2(\theta+1)}\right),$$

which along with (4.36) gives us

$$\mathscr{F}(t)\le -\frac{\alpha}{\alpha_2}\mathscr{F}(t)+C_B\left(\|\Psi(t)\|^2_{2(\theta+1)}+\|\varpi(t)\|_{2(\theta+1)}.^2\right). \tag{4.37}$$

Combining (4.36) and (4.37), we can deduce

$$F(t)\le F(0)e^{-\frac{\alpha}{\alpha_2}t}+C_B'\int_0^t\left(\|\Psi(t)\|^2_{2(\theta+1)}+\|\varpi(t)\|^2_{2(\theta+1)}\right)ds.$$

From Theorems 4.2 and 4.3, we can obtain the semigroup $S(t)$ defined by (3.13) is asymptotically smooth in $\mathscr{H}$. The proof is complete. □

*Proof of Theorem 3.4.* From Theorems 4.4 -4.14, we can conclude the desired conclusion immediately. □

4.4. **The finite-dimensional attractor.** In this subsection, we study the finite-dimensional attractor to complete the proof of Theorem 3.5

*Proof of Theorem 3.5.* Let $X=H^1$, $Y=\mathbb{L}^2$, and $Z=\mathbb{L}^2$. Then the dynamical system $(\mathscr{H},S(t))$ obtained from the solution operator of problem (3.1) satisfies (4.1) and (4.2). Let $B\subset\mathscr{H}$ be a bounded set positively invariant with respect to $S(t)$. Let $(\varphi^i,\varphi^i_t,\psi^i,\psi^i_t,w^i,w^i_t,z^i_1,z^i_2)\in B$ be the corresponding solution with respect to initial data $(\varphi^i_0,\varphi^i_1,\psi^i_0,\psi^i_1,w^i_0,w^i_1,f^i_0,\tilde{f}^i_0),(i=1,2)$. It is easy to show that there exists a constant $c>0$ such that

$$\|S(t)U_1-S(t)U_2\|^2_{\mathscr{H}}\le e^{ct}\|U_1-U_2\|^2_{\mathscr{H}}.$$

We denote

$$S(t)(\varphi^i_0,\varphi^i_1,\psi^i_0,\psi^i_1,\psi^i_0,\psi^i_1,w^i_0,w^i_1,f^i_0,\tilde{f}^i_0)=(\varphi^i,\varphi^i_t,\psi^i,\psi^i_t,w^i,w^i_t,z^i_1,z^i_2),(i=1,2).$$

We consider the semi-norm

$$n_X(\psi)=\|\psi\|_{2(\theta+1)},\qquad n_X(w)=\|w\|_{2(\theta+1)}.$$

Since the embedding $H^1\hookrightarrow\mathbb{L}^{2(\theta+1)}$ is compact, it follows that $n_X(.)$ is a compact semi-norm on $H^1$. We conclude from (4.37) that

$$\begin{aligned}\|S(t)U_1-S(t)U_2\|^2_{\mathscr{H}} &\le F(0)e^{-\frac{\alpha}{\alpha_2}t}+C_B'\int_0^t e^{-\frac{\alpha}{\alpha_2}(t-s)}\left(\|\psi\|^2_{2(\theta+1)}+\|w\|^2_{2(\theta+1)}\right)ds\\ &\le C_0e^{-\frac{\alpha}{\alpha_2}t}\|U_1-U_2\|^2_{\mathscr{H}}+C_B'\int_0^t e^{-\frac{\alpha}{\alpha_2}(t-s)}ds\sup\left[n_X(\psi^1(s)-\psi^2(s))\right]^2\\ &+C_B'\int_0^t e^{-\frac{\alpha}{\alpha_2}(t-s)}ds\sup\left[n_X(w^1(s)-w^2(s)\right]^2\\ &=b(t)\|U_1-U_2\|^2_{\mathscr{H}}+c(t)\sup\left[n_X(\psi^1(s)-\psi^2(s))+n_X(w^1(s)-w^2(s))\right]^2,\end{aligned}$$

with

$$b(t) = C_0 e^{-\frac{\alpha}{\alpha_2}t}, \quad c(t) = C'_B \int_0^t e^{-\frac{\alpha}{\alpha_2}(t-s)} ds, \quad t \geq 0.$$

Finally, we note that

$$b(t) \in \mathbb{L}^1(\mathbb{R}^+) \quad \textit{with} \quad \lim_{t\to\infty} b(t) = 0.$$

Since $B \subset \mathscr{H}$ is bounded, we obtain that $c(t)$ is locally bounded on $[0,\infty)$. Thus, inequality (4.3) holds, which shows that dynamics system $(\mathscr{H}, S(t))$ is quasi-stable on $\mathfrak{A}$. By theorem 3.4, we know dynamics system $(\mathscr{H}, S(t))$ has a compact global attractor $\mathfrak{A}$, which is a bounded positively invariant set of $\mathscr{H}$. Then Theorem 4.3 yields that $\mathfrak{A}$ has finite fractal dimension. The proof is complete. □

4.5. **The exponential attractor.** In this subsection, we establish the existence of exponential attractors to problem (3.1).

**Definition 4.15.** A compact set $\mathfrak{A}_{exp} \subset X$ is said to be a fractal exponential attractor of dynamical system $(X, S(t)$ if $\mathfrak{A}_{exp}$ is a positively invariant set of finite fractal dimension in $x$ and for every bounded set $B \subset X$ there exist positive constants $t_B, C_B$ and $\alpha_B$ such that, for all $t \geq t_B$,

$$dist_{\mathscr{H}}(S(t)B, \mathfrak{A}_{exp}) \leq C_B exp(-\alpha_B(t - t_B)).$$

If there exists an exponential attractor only having finite dimension in some extended space $\tilde{\mathscr{H}} \supseteq \mathscr{H}$, then this exponentially attracting set is called a generalized fractal exponential attractor.

**Theorem 4.16.** [2] *Under the hypotheses of Theorem 3.1, the corresponding dynamical system* $(\mathscr{H}, S(t))$ *has a generalized exponential attractor* $\mathfrak{A}_{exp} \subset \mathscr{H}$ *with fractal dimension in*

$$\tilde{\mathscr{H}} = (X' \times Y)^2 \times Z^2,$$

*where $X'$ is the topological dual of $X$.*

**Theorem 4.17.** *Let* $(\mathscr{H}, S(t))$ *be a dynamical system satisfying (4.1) and (4.2) and quasi-stable on some bounded absorbing set* **B**. *Assume further that there exists an extended space* $\tilde{\mathscr{H}} \supseteq \mathscr{H}$ *such that, for any* $T > 0$,

$$\|S(t_1)y - S(t_2)y\|_{\tilde{\mathscr{H}}} \leq C_{\mathbf{B}T} |t_1 - t_2|^{\alpha}, t_1, t_2 \in [0, T], y \in \mathbf{B}, \tag{4.38}$$

*where* $C_{\mathbf{B}T} > 0$ *and* $\alpha \in (0, 1]$ *are constants. Then* $(\mathscr{H}, S(t))$ *has a generalized exponential attractor* $\mathfrak{A}_{exp} \subset \mathscr{H}$ *with finite fractal dimension in* $\tilde{\mathscr{H}}$.

*Proof of Theorem 4.16.* We take

$$\mathbf{B} = \left\{ (\varphi, \varphi_t, \psi, \psi_t, w, w_t, z_1, z_2) \quad \text{such that} \quad \|(\varphi, \varphi_t, \psi, \psi_t, w, w_t, z_1, z_2)\|^2_{\mathscr{H}} \leq R^2 \right\}.$$

Then, **B** is a positively invariant bounded absorbing set for $R$ large. Therefore the system is quasi-stable on **B**. Taking a weak solution $U(t) = (\varphi(t), \varphi_t(t), \psi(t), \psi_t(t), w(t), w_t(t), z_1(t), z_2(t)) \in \mathscr{H}$, we easily deduce from (2.3)-(2.4) that

$$(\varphi_t, \varphi_{tt}, \psi_t, \psi_{tt}, w_t, w_{tt}, z_{1t}, z_{2t}) \in \mathbb{L}^2_{loc}(\mathbb{R}^+, \tilde{\mathscr{H}}), \quad \text{with} \quad \tilde{\mathscr{H}} = (X' \times Y)^2 \times Z^2.$$

Hence, for the solutions $U(t)$ with initial data $y = U(0) \in \mathbf{B}$, we easily conclude that

$$\int_0^T \|U_t(s)\|^2_{\tilde{\mathscr{H}}} ds \leq C^2_{\mathbf{B}T},$$

where $C_{\mathbf{B}T}$ is positive constant. Thus

$$\|S(t_1)y - S(t_2)y\|_{\tilde{\mathscr{H}}} \leq \int_{t_1}^{t_2} \|U_t(s)\|_{\tilde{\mathscr{H}}} ds \leq C_{\mathbf{B}T}|t_1 - t_2|^{\frac{1}{2}}, \quad 0 \leq t_1 \leq t_2 \leq T.$$

We easily observe from the inequality above that for any $y \in \mathbf{B}$, the map $t \mapsto S(t)y$ is Hölder continuous in $\tilde{\mathscr{H}}$ with exponent $\alpha = \frac{1}{2}$. Hence (4.38) holds and Theorem 4.17 guarantees the existence of a generalized exponential attractor whose fractal dimension is finite in $\mathscr{H}$. This completes the proof. □

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