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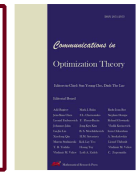

EXISTENCE OF SOLUTIONS AND THE ULAM STABILITY FOR A CLASS OF SINGULAR NONLINEAR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we consider a singular system of nonlinear fractional integro-differential equations. Using the Schauder fixed point theorem and the contraction mapping principle, we establish sufficient conditions for new existence and uniqueness results. Moreover, we define and investigate the Ulam-Hyers stability and the generalized Ulam-Hyers stability of solutions for the system. Some examples are presented to illustrate the applications of our main results.

Keywords. Caputo derivative; Fixed point; Existence and uniqueness; Generalized Ulam-Hyers stability.

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1. INTRODUCTION

Recent progress in the area of fractional derivatives and integrals involves promising potential for future developments and application of the theory. For instance see [1, 2, 3]. On the other hand, the study of the Ulam-Hyers stability problems has grown to be one of the most important topics in the mathematical analysis area. In [4], Wang, Lv and Zhou studied different types of Ulam stability for a fractional differential equation. In [5], Li investigate the existence of solutions for singular nonlinear fractional differential Equations. In [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16], Taïeb and Dahmani obtained new results on the existence and uniqueness of solutions in addition to some types of Ulam stability for the fractional differential equations. In [17, 18, 19, 20], Taïeb discussed some other results concerning the Ulam stability.

In this paper, we consider a coupled system of singular fractional integro-differential equations and investigate the existence and uniqueness of solutions in addition to the existence of at least one solution. Moreover, we define and prove some types of the Ulam stability for the following problem:

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$$\begin{cases} D^{\alpha_k}u_k(t) = f_k\left(\begin{array}{l} t, u_1(t), ..., u_n(t), D^{\delta_1}u_1(t), ..., D^{\delta_n}u_n(t), \\ \int_0^t h_1(\tau, u_1(\tau))\,d\tau, ..., \int_0^t h_n(\tau, u_n(\tau))\,d\tau \end{array}\right), \\ 0 < t \leq 1, n-1 < \alpha_k < n,\ n-2 < \delta_k < n-1,\ k = 1,2,...,n, \\ u_k^{(j)}(0) = \omega_j^k,\ j = 0,1,...,n-2,\ k = 1,2,...,n, \\ u_k^{(n-1)}(0) = J^{\beta_k}u_k(\upsilon_k),\ \beta_k > 0, 0 < \upsilon_k < 1,\ k = 1,2,...,n, \end{cases} \tag{1.1}$$

where $n \in \mathbb{N} - \{0,1\}$, $h_k : [0,1] \times \mathbb{R} \to [-1,1]$ are continuous, $f_k : (0,1] \times \mathbb{R}^{3n} \to \mathbb{R}$ are continuous functions, singular at $t = 0$, and $\lim_{t \to 0^+} f_k(t) = \infty$. The operators D^{α_k}, D^{δ_k} stand for the Caputo fractional derivatives, defined by

$$D^{\gamma}u(t) = \frac{1}{\Gamma(m-\gamma)} \int_0^t (t-s)^{m-\gamma-1} u^{(m)}(s)\,ds = J^{m-\gamma}u^{(m)}(t), \tag{1.2}$$

with $m-1 < \gamma < m$, $m \in \mathbb{N} - \{0\}$.

The Riemann-Liouville fractional integral J^{α} of order $\alpha \geq 0$ for a continuous function f on $[0, +\infty)$ is defined by

$$J^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds,\ \alpha > 0,\ t \geq 0, \tag{1.3}$$

with

$$\Gamma(\alpha) := \int_0^{\infty} e^{-x}x^{\alpha-1}dx. \tag{1.4}$$

2. Preliminaries

We present some well known properties of the fractional calculus theory which can be found in [2].

(i) : For $\alpha, \beta > 0$; $n-1 < \alpha < n$, we have $D^{\alpha}t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}t^{\beta-\alpha-1}$, $\beta > n$, and $D^{\alpha}t^j = 0$, $j = 0,1,...,n-1$.

(ii) : $D^pJ^qf(t) = J^{q-p}f(t)$, where $q > p > 0$ and $f \in L^1([a,b])$.

(iii) : Let $n \in \mathbb{N} - \{0\}$, $n-1 < \alpha < n$, and $D^{\alpha}u(t) = 0$. Then, $u(t) = \sum_{j=0}^{n-1} c_jt^j$, and $J^{\alpha}D^{\alpha}u(t) = u(t) + \sum_{j=0}^{n-1} c_jt^j$, $(c_j)_{j=0,1,...,n-1} \in \mathbb{R}$.

The following Lemma is fundamental to prove our existence results

Lemma 2.1. [2] *(Shauder fixed point theorem) Let (E,d) be a complete metric space, let X be a closed convex subset of E, and let $A : E \to E$ be a mapping such that the set $Y := \{Ax : x \in X\}$ is relatively compact in E. Then, A has at least one fixed point.*

Now, we give the following auxiliary result.

Lemma 2.2. *Let $n \in \mathbb{N} - \{0,1\}$, $n-1 < \alpha_k < n$, $k = 1,2,...,n$, and $U_k \in C([0,1], \mathbb{R})$. Then, the problem*

$$D^{\alpha_k}u_k(t) = U_k(t),\ k = 1,2,...,n,\ 0 < t < 1,$$

associated with the conditions:

$$\begin{cases} u_k^{(j)}(0) = \omega_j^k, \ j = 0,1,...,n-2, \\ u_k^{(n-1)}(0) = J^{\beta_k} u_k(\upsilon_k), \ \beta_k > 0, \ 0 < \ \upsilon_k < 1, \end{cases}$$

has a unique solution $(u_1, u_2 ..., u_n)(t)$;

$$\begin{aligned} u_k(t) &= \int_0^t \frac{(t-s)^{\alpha_k - 1}}{\Gamma(\alpha_k)} U_k(s)\, ds \\ &+ \sum_{j=0}^{n-2} \frac{\omega_j^k}{j!} t^j + \frac{\Gamma(n+\beta_k)\, t^{n-1}}{(n-1)!\left(\Gamma(n+\beta_k) - \upsilon_k^{n-1+\beta_k}\right)} \\ &\times \left(\int_0^{\upsilon_k} \frac{(\upsilon_k - s)^{\alpha_k + \beta_k - 1}}{\Gamma(\alpha_k + \beta_k)} U_k(s)\, ds + \sum_{j=0}^{n-2} \frac{\omega_j^k \upsilon_k^{j+\beta_k}}{\Gamma(j+1+\beta_k)} \right), \end{aligned} \tag{2.1}$$

where $k = 1,2,...,n$.

Proof. The property (iii) allows us to write the above problem to an equivalent integral equations:

$$u_k(t) = \int_0^t \frac{(t-s)^{\alpha_k - 1}}{\Gamma(\alpha_k)} U_k(s)\, ds - \sum_{j=0}^{n-1} c_j^k t^j, \ k = 1,2,...,n, \tag{2.2}$$

with

$$\begin{pmatrix} c_0^1 & c_1^1 & \dots & c_{n-1}^1 \\ c_0^2 & c_1^2 & \dots & c_{n-1}^2 \\ \vdots & \vdots & \vdots & \vdots \\ c_0^n & c_1^n & \dots & c_{n-1}^n \end{pmatrix} \in M_n(\mathbb{R}).$$

We observe that

$$u_k^{(j)}(0) = -j! c_j^k, \ j = 0,1,...,n-2. \tag{2.3}$$

By the condition $u_k^{(j)}(0) = \omega_j^k$, $j = 0,1,...,n-2$, we obtain

$$c_j^k = -\frac{\omega_j^k}{j!}, \ j = 0,1,...,n-2. \tag{2.4}$$

Indeed,

$$\begin{cases} u_k^{(n-1)}(0) = -(n-1)! c_{n-1}^k, \\ J^{\beta_k} u_k(\upsilon_k) = \int_0^{\upsilon_k} \frac{(\upsilon_k - s)^{\alpha_k + \beta_k - 1}}{\Gamma(\alpha_k + \beta_k)} U_k(s)\, ds + \sum_{j=0}^{n-2} \frac{\omega_j^k \upsilon_k^{j+\beta_k}}{\Gamma(j+1+\beta_k)} - \frac{c_{n-1}^k \Gamma(n) \upsilon_k^{n-1+\beta_k}}{\Gamma(n+\beta_k)}, \end{cases} \tag{2.5}$$

which implies that

$$c_{n-1}^k = \frac{\Gamma(n+\beta_k)}{\Gamma(n)\left(\upsilon_k^{n-1+\beta_k} - \Gamma(n+\beta_k)\right)} \begin{pmatrix} \int_0^{\upsilon_k} \frac{(\upsilon_k - s)^{\alpha_k + \beta_k - 1}}{\Gamma(\alpha_k + \beta_k)} U_k(s)\, ds \\ + \sum_{j=0}^{n-2} \frac{\omega_j^k \upsilon_k^{j+\beta_k}}{\Gamma(j+1+\beta_k)} \end{pmatrix}. \tag{2.6}$$

Substituting (2.4) and (2.6) in (2.2), we get (2.1). This ends the proof. □

Now, we introduce the Banach space

$$B := \left\{ (u_1, ..., u_n) : u_k \in C([0,1], \mathbb{R}),\ D^{\delta_k} u_k \in C([0,1], \mathbb{R}), k = 1, 2, ..., n \right\},$$

where $n \in \mathbb{N} - \{0, 1\}$, endowed with the norm:

$$\|(u_1, ..., u_n)\|_B = \max_{1 \le k \le n} \left(\|u_k\|_\infty, \left\| D^{\delta_k} u_k \right\|_\infty \right);$$

$\|u_k\|_\infty = \max_{t \in [0,1]} |u_k(t)|$, $\left\| D^{\delta_k} u_k \right\|_\infty = \max_{t \in [0,1]} \left| D^{\delta_k} u_k \right|$.

3. Existence and Uniqueness

In this section, we will establish sufficient conditions for the existence and uniqueness of solution in addition to the existence of at least one solution to system (1.1). Furtheremore, we give some examples to demonstrate the application of our Theorems.

Define the nonlinear operator $T : B \to B$ by

$$T(u_1, ..., u_n)(t) := (T_1(u_1, ..., u_n)(t), ..., T_n(u_1, ..., u_n)(t)),$$

$\forall\, t \in [0, 1]$,

$$\begin{aligned}
& T_k(u_1, ..., u_n)(t) \\
:\ =\ & \int_0^t \frac{(t-s)^{\alpha_k - 1}}{\Gamma(\alpha_k)} f_k \left(\begin{array}{c} s, u_1(s), ..., u_n(s), D^{\delta_1} u_1(s), ..., D^{\delta_n} u_n(s), \\ \int_0^s h_1(\tau, u_1(\tau)) d\tau, ..., \int_0^s h_n(\tau, u_n(\tau)) d\tau \end{array} \right) ds \\
& + \sum_{j=0}^{n-2} \frac{\omega_j^k}{j!} t^j + \frac{\Gamma(n+\beta_k) t^{n-1}}{(n-1)! \left(\Gamma(n+\beta_k) - \upsilon_k^{n-1+\beta_k} \right)} \\
& \times \left(\int_0^{\upsilon_k} \frac{(\upsilon_k - s)^{\alpha_k + \beta_k - 1}}{\Gamma(\alpha_k + \beta_k)} f_k \left(\begin{array}{l} s, u_1(s), ..., u_n(s), \\ D^{\delta_1} u_1(s), ..., D^{\delta_n} u_n(s), \\ \int_0^s h_1(\tau, u_1(\tau)) d\tau, ..., \\ \int_0^s h_n(\tau, u_n(\tau)) d\tau \end{array} \right) ds + \sum_{j=0}^{n-2} \frac{\omega_j^k \upsilon_k^{j+\beta_k}}{\Gamma(j+1+\beta_k)} \right).
\end{aligned} \tag{3.1}$$

Lemma 3.1. *Let $n-1 < \alpha_k < n$, $k = 1, 2, ..., n$, $n \in \mathbb{N} - \{0, 1\}$. Assume that $G_k : (0, 1] \to \mathbb{R}$ are continuous, $\lim_{t \to 0^+} G_k(t) = \infty$, and there exist constants $0 < \mu_k < 1$, such that $t^{\mu_k} G_k(t)$ are continuous for each $t \in [0, 1]$. Then,*

$$\begin{aligned}
u_k(t) \ =\ & \int_0^t \frac{(t-s)^{\alpha_k - 1}}{\Gamma(\alpha_k)} G_k(s) ds + \sum_{j=0}^{n-2} \frac{\omega_j^k}{j!} t^j \\
& + \frac{\Gamma(n+\beta_k) t^{n-1}}{(n-1)! \left(\Gamma(n+\beta_k) - \upsilon_k^{n-1+\beta_k} \right)} \\
& \times \left(\int_0^{\upsilon_k} \frac{(\upsilon_k - s)^{\alpha_k + \beta_k - 1}}{\Gamma(\alpha_k + \beta_k)} G_k(s) ds + \sum_{j=0}^{n-2} \frac{\omega_j^k \upsilon_k^{j+\beta_k}}{\Gamma(j+1+\beta_k)} \right),
\end{aligned}$$

are continuous on $[0,1]$.

Proof. From the continuity of $t^{\mu_k}G_k(t)$, and

$$
\begin{aligned}
u_k(t) &= \int_0^t \frac{(t-s)^{\alpha_k-1}s^{-\mu_k}}{\Gamma(\alpha_k)}s^{\mu_k}G_k(s)\,ds+\sum_{j=0}^{n-2}\frac{\omega_j^k}{j!}t^j \\
&\quad+\frac{\Gamma(n+\beta_k)t^{n-1}}{(n-1)!\left(\Gamma(n+\beta_k)-\upsilon_k^{n-1+\beta_k}\right)} \\
&\quad\times\left(\int_0^{\upsilon_k}\frac{(\upsilon_k-s)^{\alpha_k+\beta_k-1}s^{-\mu_k}}{\Gamma(\alpha_k+\beta_k)}s^{\mu_k}G_k(s)\,ds+\sum_{j=0}^{n-2}\frac{\omega_j^k\upsilon_k^{j+\beta_k}}{\Gamma(j+1+\beta_k)}\right),
\end{aligned}
$$

it is known that $u_k(0)=\omega_0^k$. The proof will be divided into three cases.

Case 1: For $t_0=0$ and $\forall t\in(0,1]$, since $t^{\mu_k}G_k(t)$ are continuous on $[0,1]$, there exist $A_k>0$: $|t^{\mu_k}G_k(t)|\le A_k$, $\forall t\in[0,1]$. Then,

$$
\begin{aligned}
&|u_k(t)-u_k(0)| \\
=&\left|\begin{array}{l}\int_0^t \frac{(t-s)^{\alpha_k-1}s^{-\mu_k}}{\Gamma(\alpha_k)}s^{\mu_k}G_k(s)\,ds+\sum_{j=1}^{n-2}\frac{\omega_j^k}{j!}t^j+\frac{\Gamma(n+\beta_k)t^{n-1}}{(n-1)!\left(\Gamma(n+\beta_k)-\upsilon_k^{n-1+\beta_k}\right)} \\ \times\left(\int_0^{\upsilon_k}\frac{(\upsilon_k-s)^{\alpha_k+\beta_k-1}s^{-\mu_k}}{\Gamma(\alpha_k+\beta_k)}s^{\mu_k}G_k(s)\,ds+\sum_{j=0}^{n-2}\frac{\omega_j^k\upsilon_k^{j+\beta_k}}{\Gamma(j+1+\beta_k)}\right)\end{array}\right| \\
\le&\frac{A_k}{\Gamma(\alpha_k)}\int_0^t(t-s)^{\alpha_k-1}s^{-\mu_k}ds+\sum_{j=1}^{n-2}\frac{\left|\omega_j^k\right|}{j!}t^j+\frac{\Gamma(n+\beta_k)t^{n-1}}{(n-1)!\left|\Gamma(n+\beta_k)-\upsilon_k^{n-1+\beta_k}\right|} \\
&\times\left(\frac{A_k}{\Gamma(\alpha_k+\beta_k)}\int_0^{\upsilon_k}(\upsilon_k-s)^{\alpha_k+\beta_k-1}s^{-\mu_k}ds+\sum_{j=0}^{n-2}\frac{\left|\omega_j^k\right|\upsilon_k^{j+\beta_k}}{\Gamma(j+1+\beta_k)}\right) \\
\le&\frac{A_kt^{\alpha_k-\mu_k}}{\Gamma(\alpha_k)}\int_0^1(1-v)^{\alpha_k-1}v^{-\mu_k}dv+\sum_{j=1}^{n-2}\frac{\left|\omega_j^k\right|}{j!}t^j+\frac{\Gamma(n+\beta_k)t^{n-1}}{(n-1)!\left|\Gamma(n+\beta_k)-\upsilon_k^{n-1+\beta_k}\right|} \\
&\times\left(\frac{A_k\upsilon_k^{\alpha_k+\beta_k-\mu_k}}{\Gamma(\alpha_k+\beta_k)}\int_0^1(1-w)^{\alpha_k+\beta_k-1}w^{-\mu_k}dw+\sum_{j=0}^{n-2}\frac{\left|\omega_j^k\right|\upsilon_k^{j+\beta_k}}{\Gamma(j+1+\beta_k)}\right).
\end{aligned}
$$

So,

$$
\begin{aligned}
&|u_k(t)-u_k(0)| \\
\le&\frac{A_kBe(\alpha_k,1-\mu_k)t^{\alpha_k-\mu_k}}{\Gamma(\alpha_k)}+\sum_{j=1}^{n-2}\frac{\left|\omega_j^k\right|}{j!}t^j \\
&+\frac{\Gamma(n+\beta_k)t^{n-1}}{(n-1)!\left|\Gamma(n+\beta_k)-\upsilon_k^{n-1+\beta_k}\right|} \\
&\times\left(\frac{A_k\upsilon_k^{\alpha_k+\beta_k-\mu_k}Be(\alpha_k+\beta_k,1-\mu_k)}{\Gamma(\alpha_k+\beta_k)}+\sum_{j=0}^{n-2}\frac{\left|\omega_j^k\right|\upsilon_k^{j+\beta_k}}{\Gamma(j+1+\beta_k)}\right),
\end{aligned}
$$

where $Be(.,.)$ denotes Beta function. Thus,

$$\begin{aligned}
&\left|u_k(t)-u_k(0)\right| \\
\leq &\left(\begin{array}{c}
\frac{A_k\Gamma(1-\mu_k)t^{\alpha_k-\mu_k}}{\Gamma(\alpha_k+1-\mu_k)}+\sum_{j=1}^{n-2}\frac{\left|\omega_j^k\right|t^j}{j!}+\frac{\Gamma(n+\beta_k)t^{n-1}}{(n-1)!\left|\Gamma(n+\beta_k)-\upsilon_k^{n-1+\beta_k}\right|} \\
\times\left(\frac{A_k\upsilon_k^{\alpha_k+\beta_k-\mu_k}\Gamma(1-\mu_k)}{\Gamma(\alpha_k+\beta_k+1-\mu_k)}+\sum_{j=0}^{n-2}\frac{\left|\omega_j^k\right|\upsilon_k^{j+\beta_k}}{\Gamma(j+1+\beta_k)}\right)
\end{array}\right) \\
\rightarrow &\ 0,\ \text{as } t\rightarrow 0. \qquad (3.2)
\end{aligned}$$

Case 2: For $t_0\in(0,1)$ and $\forall t\in(t_0,1]$,

$$\begin{aligned}
&\left|u_k(t)-u_k(t_0)\right| \\
\leq &\left|\int_0^t\frac{(t-s)^{\alpha_k-1}s^{-\mu_k}}{\Gamma(\alpha_k)}s^{\mu_k}G_k(s)\,ds-\int_0^{t_0}\frac{(t_0-s)^{\alpha_k-1}s^{-\mu_k}}{\Gamma(\alpha_k)}s^{\mu_k}G_k(s)\,ds\right| \\
&+\sum_{j=1}^{n-2}\frac{\omega_j^k}{j!}\left(t^j-t_0^j\right)+\frac{\Gamma(n+\beta_k)\left(t^{n-1}-t_0^{n-1}\right)}{(n-1)!\left|\Gamma(n+\beta_k)-\upsilon_k^{n-1+\beta_k}\right|} \\
&\times\left(\int_0^{\upsilon_k}\frac{(\upsilon_k-s)^{\alpha_k+\beta_k-1}s^{-\mu_k}}{\Gamma(\alpha_k+\beta_k)}s^{\mu_k}G_k(s)\,ds+\sum_{j=0}^{n-2}\frac{\omega_j^k\upsilon_k^{j+\beta_k}}{\Gamma(j+1+\beta_k)}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\left|u_k(t)-u_k(t_0)\right| \leq &\left(\begin{array}{l}
\frac{A_k\Gamma(1-\mu_k)\left(t^{\alpha_k-\mu_k}-t_0^{\alpha_k-\mu_k}\right)}{\Gamma(\alpha_k+1-\mu_k)}+\sum_{j=1}^{n-2}\frac{\left|\omega_j^k\right|\left(t^j-t_0^j\right)}{j!} \\
+\frac{\Gamma(n+\beta_k)\left(t^{n-1}-t_0^{n-1}\right)}{(n-1)!\left|\Gamma(n+\beta_k)-\upsilon_k^{n-1+\beta_k}\right|} \\
\times\left(\frac{A_k\upsilon_k^{\alpha_k+\beta_k-\mu_k}\Gamma(1-\mu_k)}{\Gamma(\alpha_k+\beta_k+1-\mu)}+\sum_{j=0}^{n-2}\frac{\left|\omega_j^k\right|\upsilon_k^{j+\beta_k}}{\Gamma(j+1+\beta_k)}\right)
\end{array}\right) \\
\rightarrow &\ 0,\ \text{as } t\rightarrow t_0. \qquad (3.3)
\end{aligned}$$

Case 3: For t_0 $(0,1]$ and $\forall t\in[0,t_0)$, the proof is identically to that of case 2, we omit it here. This completes the proof. □

Lemma 3.2. *Let $n-1<\alpha_k<n$, $n-2<\delta_k<n-1$, $k=1,2,...,n$, $n\in\mathbb{N}-\{0,1\}$, $h_k:[0,1]\times\mathbb{R}\rightarrow[-1,1]$ and $f_k:(0,1]\times\mathbb{R}^{3n}\rightarrow\mathbb{R}$ are continuous, $\lim_{t\rightarrow 0^+}f_k(t,...)=\infty$. Assume that there exist constants $0<\mu_k<1$,* *such that $t^{\mu_k} f_k(t,\ldots)$ are continuous on $[0,1]\times\mathbb{R}^{3n}$. Then,*

$$\begin{aligned}
&D^{\delta_k} T_k(u_1,\ldots,u_n)(t)\\
=\;& \int_0^t \frac{(t-s)^{\alpha_k-\delta_k-1}}{\Gamma(\alpha_k-\delta_k)} f_k\left(\begin{array}{l} s,u_1(s),\ldots,u_n(s),D^{\delta_1}u_1(s),\ldots,D^{\delta_n}u_n(s),\\ \int_0^s h_1(\tau,u_1(\tau))\,d\tau,\ldots,\int_0^s h_n(\tau,u_n(\tau))\,d\tau \end{array}\right) ds\\
&+\frac{\Gamma(n+\beta_k)\,t^{n-1-\delta_k}}{\Gamma(n-\delta_k)\left(\Gamma(n+\beta_k)-\upsilon_k^{n-1+\beta_k}\right)}\\
&\times\left(\int_0^{\upsilon_k} \frac{(\upsilon_k-s)^{\alpha_k+\beta_k-1}}{\Gamma(\alpha_k+\beta_k)} f_k\left(\begin{array}{l} s,u_1(s),\ldots,u_n(s),\\ D^{\delta_1}u_1(s),\ldots,D^{\delta_n}u_n(s),\\ \int_0^s h_1(\tau,u_1(\tau))\,d\tau,\ldots,\\ \int_0^s h_n(\tau,u_n(\tau))\,d\tau \end{array}\right) ds+\sum_{j=0}^{n-2}\frac{\omega_j^k\upsilon_k^{j+\beta_k}}{\Gamma(j+1+\beta_k)}\right),
\end{aligned}$$

are continuous on $[0,1]\times\mathbb{R}^{3n}$.

Proof. Let $(u_1,\ldots,u_n)\in B$. Then $u_k(t)\in C([0,1])$, and $D^{\delta_k}u_k(t)\in C([0,1])$, $k=1,2,\ldots,n$. Also, we have $h_k\in C([0,1]\times\mathbb{R})$. Hence, there exist $l_k,m_k,n_k>0$,

$$|u_k(t)|\le l_k,\;\left|D^{\delta_k}u_k(t)\right|\le m_k, |h_k(t,u_k(t))|\le n_k, \forall t\in[0,1].$$

Since $t^{\mu_k} f_k(t,\ldots)$ are continuous on $[0,1]\times\mathbb{R}^{3n}$, there exist $M_k>0$

$$M_k=\left\|t^{\mu_k} f_k\left(\begin{array}{l} t,u_1,\ldots,u_n,D^{\delta_1}u_1,\ldots,D^{\delta_n}u_n,\\ \int_0^t h_1(\tau,u_1(\tau))\,d\tau,\ldots,\int_0^t h_n(\tau,u_n(\tau))\,d\tau \end{array}\right)\right\|_\infty, \tag{3.4}$$

for $-l_k\le u_k\le l_k$, $-m_k\le D^{\delta_k}u_k\le m_k$, and $-n_k\le h_k(t,u_k)\le n_k$. By (3.4), we get

$$\begin{aligned}
&\left|D^{\delta_k} T_k(u_1,u_2,\ldots,u_n)(t)\right|\\
\le\;& \frac{M_k}{\Gamma(\alpha_k-\delta_k)}\int_0^t (t-s)^{\alpha_k-\delta_k-1} s^{-\mu_k}ds+\frac{\Gamma(n+\beta_k)\,t^{n-1-\delta_k}}{\Gamma(n-\delta_k)\left|\Gamma(n+\beta_k)-\upsilon_k^{n-1+\beta_k}\right|}\\
&\times\left(\frac{M_k}{\Gamma(\alpha_k+\beta_k)}\int_0^{\upsilon_k}(\upsilon_k-s)^{\alpha_k+\beta_k-1}s^{-\mu_k}ds+\sum_{j=0}^{n-2}\frac{\omega_j^k\upsilon_k^{j+\beta_k}}{\Gamma(j+1+\beta_k)}\right)\\
\le\;& \frac{M_k\Gamma(1-\mu_k)\,t^{\alpha_k-\delta_k-\mu_k}}{\Gamma(\alpha_k-\delta_k+1-\mu_k)}+\frac{\Gamma(n+\beta_k)\,t^{n-1-\delta_k}}{\Gamma(n-\delta_k)\left|\Gamma(n+\beta_k)-\upsilon_k^{n-1+\beta_k}\right|}\\
&\times\left(\frac{M_k\Gamma(1-\mu_k)\,\upsilon_k^{\alpha_k+\beta_k-\mu_k}}{\Gamma(\alpha_k+\beta_k+1-\mu_k)}+\sum_{j=0}^{n-2}\frac{\omega_j^k\upsilon_k^{j+\beta_k}}{\Gamma(j+1+\beta_k)}\right).
\end{aligned}\tag{3.5}$$

From inequality (3.5), we see that: $t^{\alpha_k-\delta_k-\mu_k}$ and $t^{n-1-\delta_k}$ are continuous on $[0,1]$. So, we can show that $D^{\delta_k}T_k(u_1,u_2,\ldots,u_n)$ are continuous on $[0,1]$, by the same method as in Lemma 3.1 □

Lemma 3.3. *Let $n-1<\alpha_k<n$, $k=1,2,...,n$, $n\in\mathbb{N}-\{0,1\}$, $h_k:[0,1]\times\mathbb{R}\to[-1,1]$ and $f_k:(0,1]\times\mathbb{R}^{3n}\to\mathbb{R}$ are continuous; $\lim\limits_{t\to 0^+} f_k(t,...)=\infty$. Assume that there exist $0<\mu_k<1$, such that $t^{\mu_k}f_k(t,...)$ are continuous on $[0,1]\times\mathbb{R}^{3n}$. Then, the operator $T:B\to B$ is completely continuous.*

Proof. For $(u_1,...,u_n)\in B$, we have

$$T(u_1,...,u_n)(t)=(T_1(u_1,...,u_n)(t),...,T_n(u_1,...,u_n)(t)), \tag{3.6}$$

where $T_k(u_1,...,u_n)(t)$ is given by (3.1). Lemma 3.1 and Lemma 3.2 imply that $T:B\to B$. Let us devide the proof into three steps.

(1) : We prove that $T:B\to B$ is a continuous operator.

Let $\left(u_1^0,...,u_n^0\right)\in B:\left\|\left(u_1^0,...,u_n^0\right)\right\|_B=\lambda_0$, and let $(u_1,...,u_n)\in B:\left\|(u_1,...,u_n)-\left(u_1^0,...,u_n^0\right)\right\|_B<1$. Then, $\|(u_1,...,u_n)\|_B<1+\lambda_0=\lambda$. The continuity of $t^{\mu_k}f_k(t,...)$, implies that $t^{\mu_k}f_k(t,...)$ are uniformly continuous on $[0,1]\times[-\lambda,\lambda]^{3n}$. Then, $\forall\, t\in[0,1]$, $\forall\,\varepsilon>0$, there exist $\gamma>0$ $(\gamma<1)$:

$$\left|\begin{array}{l} t^{\mu_k}f_k\left(\begin{array}{c} t,u_1,...,u_n,D^{\delta_1}u_1,...,D^{\delta_n}u_n,\\ \int_0^t h_1(\tau,u_1(\tau))\,d\tau,...,\int_0^t h_n(\tau,u_n(\tau))\,d\tau \end{array}\right)\\ -t^{\mu_k}f_k\left(\begin{array}{c} t,u_1^0,...,u_n^0,D^{\delta_1}u_1^0,...,D^{\delta_n}u_n^0,\\ \int_0^t h_1\left(\tau,u_1^0(\tau)\right)d\tau,...,\int_0^t h_n\left(\tau,u_n^0(\tau)\right)d\tau \end{array}\right) \end{array}\right|<\varepsilon, \tag{3.7}$$

for $(u_1,...,u_n)\in B$, and $\left\|(u_1,...,u_n)-\left(u_1^0,...,u_n^0\right)\right\|_B<\gamma$. Therefore,

$$\begin{aligned} &\left\|T_k(u_1,...,u_n)-T_k\left(u_1^0,...,u_n^0\right)\right\|_\infty\\ \le\;& \max_{t\in[0,1]}\int_0^t \frac{(t-s)^{\alpha_k-1}s^{-\mu_k}}{\Gamma(\alpha_k)}\left|\begin{array}{l} s^{\mu_k}f_k\left(\begin{array}{c} t,u_1,...,u_n,D^{\delta_1}u_1,...,D^{\delta_n}u_n,\\ \int_0^t h_1(\tau,u_1(\tau))\,d\tau,...,\int_0^t h_n(\tau,u_n(\tau))\,d\tau \end{array}\right)\\ -s^{\mu_k}f_k\left(\begin{array}{c} t,u_1^0,...,u_n^0,D^{\delta_1}u_1^0,...,D^{\delta_n}u_n^0,\\ \int_0^t h_1\left(\tau,u_1^0(\tau)\right)d\tau,...,\int_0^t h_n\left(\tau,u_n^0(\tau)\right)d\tau \end{array}\right) \end{array}\right|ds\\ &+\frac{\Gamma(n+\beta_k)}{(n-1)!\left|\Gamma(n+\beta_k)-\upsilon_k^{n-1+\beta_k}\right|}\max_{t\in[0,1]}t^{n-1}\\ &\times\int_0^{\upsilon_k}\frac{(\upsilon_k-s)^{\alpha_k+\beta_k-1}s^{-\mu_k}}{\Gamma(\alpha_k+\beta_k)}\left|\begin{array}{l} s^{\mu_k}f_k\left(\begin{array}{c} t,u_1,...,u_n,D^{\delta_1}u_1,...,D^{\delta_n}u_n,\\ \int_0^t h_1(\tau,u_1(\tau))\,d\tau,...,\int_0^t h_n(\tau,u_n(\tau))\,d\tau \end{array}\right)\\ -s^{\mu_k}f_k\left(\begin{array}{c} t,u_1^0,...,u_n^0,D^{\delta_1}u_1^0,...,D^{\delta_n}u_n^0,\\ \int_0^t h_1\left(\tau,u_1^0(\tau)\right)d\tau,...,\int_0^t h_n\left(\tau,u_n^0(\tau)\right)d\tau \end{array}\right) \end{array}\right|ds. \end{aligned}$$

Thanks to (3.7), we get

$$\begin{aligned}&\left\|T_k(u_1,\dots,u_n)-T_k\left(u_1^0,\dots,u_n^0\right)\right\|_\infty\\ \leq\ &\frac{\varepsilon\Gamma(1-\mu_k)}{\Gamma(\alpha_k+1-\mu_k)}\max_{t\in[0,1]}t^{\alpha_k-\mu_k}\\ &+\frac{\varepsilon\Gamma(n+\beta_k)}{(n-1)!\left|\Gamma(n+\beta_k)-\upsilon_k^{n-1+\beta_k}\right|}\left(\frac{\Gamma(1-\mu_k)\,\upsilon_k^{\alpha_k+\beta_k-\mu_k}}{\Gamma(\alpha_k+\beta_k+1-\mu_k)}\right).\end{aligned}\tag{3.8}$$

We set

$$\begin{aligned}F_k^1\ &:=\frac{\Gamma(1-\mu_k)}{\Gamma(\alpha_k+1-\mu_k)},\ F_k^2:=\frac{\Gamma(n+\beta_k)}{(n-1)!\left|\Gamma(n+\beta_k)-\upsilon_k^{n-1+\beta_k}\right|},\\ \Lambda_k\ &:=\frac{\Gamma(1-\mu_k)\,\upsilon_k^{\alpha_k+\beta_k-\mu_k}}{\Gamma(\alpha_k+\beta_k+1-\mu_k)}.\end{aligned}\tag{3.9}$$

Consequently,

$$\left\|T_k(u_1,\dots,u_n)-T_k\left(u_1^0,\dots,u_n^0\right)\right\|_\infty\leq\varepsilon\left(F_k^1+F_k^2\Lambda_k\right).\tag{3.10}$$

On the other hand, we have

$$\begin{aligned}&\left\|D^{\delta_k}\left(T_k(u_1,\dots,u_n)-T_k\left(u_1^0,\dots,u_n^0\right)\right)\right\|_\infty\\ \leq\ &\max_{t\in[0,1]}\int_0^t\frac{(t-s)^{\alpha_k-\delta_k-1}s^{-\mu_k}}{\Gamma(\alpha_k-\delta_k)}\left|\begin{array}{l}s^{\mu_k}f_k\left(\begin{array}{l}t,u_1,\dots,u_n,D^{\delta_1}u_1,\dots,D^{\delta_n}u_n,\\ \int_0^t h_1(\tau,u_1(\tau))\,d\tau,\dots,\int_0^t h_n(\tau,u_n(\tau))\,d\tau\end{array}\right)\\ -s^{\mu_k}f_k\left(\begin{array}{l}t,u_1^0,\dots,u_n^0,D^{\delta_1}u_1^0,\dots,D^{\delta_n}u_n^0,\\ \int_0^t h_1\left(\tau,u_1^0(\tau)\right)d\tau,\dots,\int_0^t h_n\left(\tau,u_n^0(\tau)\right)d\tau\end{array}\right)\end{array}\right|ds\\ &+\frac{\Gamma(n+\beta_k)}{\Gamma(n-\delta_k)\left|\Gamma(n+\beta_k)-\upsilon_k^{n-1+\beta_k}\right|}\max_{t\in[0,1]}t^{n-1-\delta_k}\\ &\times\int_0^{\upsilon_k}\frac{(\upsilon_k-s)^{\alpha_k+\beta_k-1}}{\Gamma(\alpha_k+\beta_k)}\left|\begin{array}{l}s^{\mu_k}f_k\left(\begin{array}{l}t,u_1,\dots,u_n,D^{\delta_1}u_1,\dots,D^{\delta_n}u_n,\\ \int_0^t h_1(\tau,u_1(\tau))\,d\tau,\dots,\int_0^t h_n(\tau,u_n(\tau))\,d\tau\end{array}\right)\\ -s^{\mu_k}f_k\left(\begin{array}{l}t,u_1^0,\dots,u_n^0,D^{\delta_1}u_1^0,\dots,D^{\delta_n}u_n^0,\\ \int_0^t h_1\left(\tau,u_1^0(\tau)\right)d\tau,\dots,\int_0^t h_n\left(\tau,u_n^0(\tau)\right)d\tau\end{array}\right)\end{array}\right|ds.\end{aligned}$$

Using (3.7), we get

$$\begin{aligned}&\left\|D^{\delta_k}\left(T_k(u_1,\dots,u_n)-T_k\left(u_1^0,\dots,u_n^0\right)\right)\right\|_\infty\\ \leq\ &\frac{\varepsilon\Gamma(1-\mu_k)}{\Gamma(\alpha_k-\delta_k+1-\mu_k)}\max_{t\in[0,1]}t^{\alpha_k-\delta_k-\mu_k}\\ &+\frac{\varepsilon\Gamma(n+\beta_k)}{\Gamma(n-\delta_k)\left|\Gamma(n+\beta_k)-\upsilon_k^{n-1+\beta_k}\right|}\left(\frac{\Gamma(1-\mu_k)\,\upsilon_k^{\alpha_k+\beta_k-\mu_k}}{\Gamma(\alpha_k+\beta_k+1-\mu_k)}\right).\end{aligned}\tag{3.11}$$

Set

$$F_k^3 := \frac{\Gamma(1-\mu_k)}{\Gamma(\alpha_k-\delta_k+1-\mu_k)},\ F_k^4 := \frac{\Gamma(n+\beta_k)}{\Gamma(n-\delta_k)\left|\Gamma(n+\beta_k)-\upsilon_k^{n-1+\beta_k}\right|}. \tag{3.12}$$

It follows that

$$\left\|D^{\delta_k}\left(T_k(u_1,...,u_n)-T_k\left(u_1^0,...,u_n^0\right)\right)\right\|_\infty \leq \varepsilon\left(F_k^3+F_k^4\Lambda_k\right). \tag{3.13}$$

Thanks to (3.10) and (3.13), we get

$$\left\|T(u_1,...,u_n)-T\left(u_1^0,...,u_n^0\right)\right\|_B \leq \varepsilon \max_{1\leq k\leq n}\left(F_k^1+F_k^2\Lambda_k, F_k^3+F_k^4\Lambda_k\right). \tag{3.14}$$

Hence,

$$\left\|T(u_1,...,u_n)-T\left(u_1^0,...,u_n^0\right)\right\|_B \to 0 \text{ as } \left\|(u_1,...,u_n)-\left(u_1^0,...,u_n^0\right)\right\|_B \to 0.$$

So, $T: B\to B$ is continuous.

(2) : Let

$$\Omega := \left\{(u_1,...,u_n)\in B : \|(u_1,...,u_n)\|_B \leq \xi\right\}; \xi>0.$$

We prove that $T(\Omega)$ is bounded. Since $t^{\mu_k} f_k(t,...)$ are continuous on $[0,1]\times[-\xi,\xi]^{3n}$, there exist $L_k>0 : \forall t\in[0,1], \forall(u_1,u_2,...,u_n)\in\Omega$,

$$\left|t^{\mu_k} f_k\left(\begin{array}{l} t, u_1,...,u_n, D^{\delta_1}u_1,...,D^{\delta_n}u_n, \\ \int_0^t h_1(\tau,u_1(\tau))\,d\tau,...,\int_0^t h_n(\tau,u_n(\tau))\,d\tau \end{array}\right)\right| \leq L_k. \tag{3.15}$$

Using (3.15), we get

$$\begin{aligned} &\|T_k(u_1,...,u_n)\|_\infty \\ \leq\ & \int_0^t \frac{(t-s)^{\alpha_k-1}s^{-\mu_k}}{\Gamma(\alpha_k)}\left|s^{\mu_k} f_k\left(\begin{array}{l} s, u_1(s),...,u_n, \\ D^{\delta_1}u_1(s),...,D^{\delta_n}u_n(s), \\ \int_0^s h_1(\tau,u_1(\tau))\,d\tau,..., \\ \int_0^s h_n(\tau,u_n(\tau))\,d\tau \end{array}\right)\right| ds \\ & + \sum_{j=0}^{n-2}\frac{\left|\omega_j^k\right|}{j!}\max_{t\in[0,1]} t^j + \frac{\Gamma(n+\beta_k)}{(n-1)!\left|\Gamma(n+\beta_k)-\upsilon_k^{n-1+\beta_k}\right|}\max_{t\in[0,1]} t^{n-1} \\ & \times\left(\int_0^{\upsilon_k}\frac{(\upsilon_k-s)^{\alpha_k+\beta_k-1}s^{-\mu_k}}{\Gamma(\alpha_k+\beta_k)}\left|s^{\mu_k} f_k\left(\begin{array}{l} s, u_1(s),...,u_n, \\ D^{\delta_1}u_1(s),...,D^{\delta_n}u_n(s), \\ \int_0^s h_1(\tau,u_1(\tau))\,d\tau,..., \\ \int_0^s h_n(\tau,u_n(\tau))\,d\tau \end{array}\right)\right| ds + \sum_{j=0}^{n-2}\frac{\omega_j^k\upsilon_k^{j+\beta_k}}{\Gamma(j+1+\beta_k)}\right). \end{aligned}$$

Then,

$$\begin{aligned}
&\left\| T_k\left(u_1,\ldots,u_n\right)\right\|_\infty \\
\leq\ & \frac{L_k\Gamma\left(1-\mu_k\right)}{\Gamma\left(\alpha_k+1-\mu_k\right)}\max_{t\in[0,1]} t^{\alpha_k-\mu_k}+\sum_{j=0}^{n-2}\frac{\left|\omega_j^k\right|}{j!}+\frac{\Gamma\left(n+\beta_k\right)}{(n-1)!\left|\Gamma\left(n+\beta_k\right)-\upsilon_k^{n-1+\beta_k}\right|} \\
&\times\left(\frac{L_k\Gamma\left(1-\mu_k\right)\upsilon_k^{\alpha_k+\beta_k-\mu_k}}{\Gamma\left(\alpha_k+\beta_k+1-\mu_k\right)}+\sum_{j=0}^{n-2}\frac{\omega_j^k\upsilon_k^{j+\beta_k}}{\Gamma\left(j+1+\beta_k\right)}\right).
\end{aligned} \tag{3.16}$$

Letting

$$\Lambda_k^*=\sum_{j=0}^{n-2}\frac{\omega_j^k\upsilon_k^{j+\beta_k}}{\Gamma\left(j+1+\beta_k\right)}, \tag{3.17}$$

we obtain

$$\left\| T_k\left(u_1,\ldots,u_n\right)\right\|_\infty\leq L_kF_k^1+F_k^2\left(L_k\Lambda_k+\Lambda_k^*\right)+\sum_{j=0}^{n-2}\frac{\left|\omega_j^k\right|}{j!}. \tag{3.18}$$

Similarly, we conclude from (3.15) that

$$\begin{aligned}
&\left\| D^{\delta_k}T_k\left(u_1,\ldots,u_n\right)\right\|_\infty \\
\leq\ & \frac{L_k\Gamma\left(1-\mu_k\right)}{\Gamma\left(\alpha_k-\delta_k+1-\mu_k\right)}\max_{t\in[0,1]} t^{\alpha_k-\delta_k-\mu_k}+\frac{\Gamma\left(n+\beta_k\right)}{\Gamma\left(n-\delta_k\right)\left|\Gamma\left(n+\beta_k\right)-\upsilon_k^{n-1+\beta_k}\right|} \\
&\times\left(\frac{L_k\Gamma\left(1-\mu_k\right)\upsilon_k^{\alpha_k+\beta_k-\mu_k}}{\Gamma\left(\alpha_k+\beta_k+1-\mu_k\right)}+\sum_{j=0}^{n-2}\frac{\omega_j^k\upsilon_k^{j+\beta_k}}{\Gamma\left(j+1+\beta_k\right)}\right).
\end{aligned} \tag{3.19}$$

Therefore,

$$\left\| D^{\delta_k}T_k\left(u_1,\ldots,u_n\right)\right\|_\infty\leq L_kF_k^3+F_k^4\left(L_k\Lambda_k+\Lambda_k^*\right). \tag{3.20}$$

By (3.18) and (3.20), we state

$$\left\| T\left(u_1,\ldots,u_n\right)\right\|_B\leq\max_{1\leq k\leq n}\left(\begin{array}{c} L_kF_k^1+F_k^2\left(L_k\Lambda_k+\Lambda_k^*\right)+\sum_{j=0}^{n-2}\frac{\left|\omega_j^k\right|}{j!}, \\ L_kF_k^3+F_k^4\left(L_k\Lambda_k+\Lambda_k^*\right)\end{array}\right). \tag{3.21}$$

So, $T\left(\Omega\right)$ is bounded.

(3) : We show that $T(\Omega)$ is equicontinuous. Let $(u_1,...,u_n)\in\Omega$, and $t_1,t_2\in[0,1]$; $t_1<t_2$. Therefore,

$$\begin{aligned}
&\left\| T_k(u_1,...,u_n)(t_2)-T_k(u_1,...,u_n)(t_1)\right\|_\infty \\
\leq\ & \max_{t\in[0,1]}\left|\begin{array}{l}
\int_0^{t_2}\frac{(t_2-s)^{\alpha_k-1}s^{-\mu_k}}{\Gamma(\alpha_k)}s^{\mu_k}f_k\left(\begin{array}{l} s,u_1(s),...,u_n,D^{\delta_1}u_1(s),...,D^{\delta_n}u_n(s),\\ \int\limits_0^s h_1(\tau,u_1(\tau))\,d\tau,...,\int\limits_0^s h_n(\tau,u_n(\tau))\,d\tau\end{array}\right)ds \\
-\int_0^{t_1}\frac{(t_1-s)^{\alpha_k-1}s^{-\mu_k}}{\Gamma(\alpha_k)}s^{\mu_k}f_k\left(\begin{array}{l} s,u_1(s),...,u_n,D^{\delta_1}u_1(s),...,D^{\delta_n}u_n(s),\\ \int\limits_0^s h_1(\tau,u_1(\tau))\,d\tau,...,\int\limits_0^s h_n(\tau,u_n(\tau))\,d\tau\end{array}\right)ds
\end{array}\right| \\
&+\sum_{j=0}^{n-2}\frac{\left|\omega_j^k\right|\left(t_2^j-t_1^j\right)}{j!}+\frac{\Gamma(n+\beta_k)\left(t_2^{n-1}-t_1^{n-1}\right)}{(n-1)!\left|\Gamma(n+\beta_k)-\upsilon_k^{n-1+\beta_k}\right|} \\
&\times\left(\begin{array}{l}
\int\limits_0^{\upsilon_k}\frac{(\upsilon_k-s)^{\alpha_k+\beta_k-1}s^{-\mu_k}}{\Gamma(\alpha_k+\beta_k)}s^{\mu_k}\left|f_k\left(\begin{array}{l} s,u_1(s),...,u_n,D^{\delta_1}u_1(s),...,D^{\delta_n}u_n(s),\\ \int\limits_0^s h_1(\tau,u_1(\tau))\,d\tau,...,\int\limits_0^s h_n(\tau,u_n(\tau))\,d\tau\end{array}\right)\right|ds \\
+\sum_{j=0}^{n-2}\frac{\omega_j^k\upsilon_k^{j+\beta_k}}{\Gamma(j+1+\beta_k)}
\end{array}\right).
\end{aligned}\tag{3.22}$$

Thus,

$$\begin{aligned}
&\left\| T_k(u_1,...,u_n)(t_2)-T_k(u_1,...,u_n)(t_1)\right\|_\infty \\
\leq\ & \frac{L_k\Gamma(1-\mu_k)}{\Gamma(\alpha_k+1-\mu_k)}\left(t_2^{\alpha_k-\mu_k}-t_1^{\alpha_k-\mu_k}\right)+\sum_{j=0}^{n-2}\frac{\left|\omega_j^k\right|\left(t_2^j-t_1^j\right)}{j!} \\
&+\frac{\Gamma(n+\beta_k)\left(t_2^{n-1}-t_1^{n-1}\right)}{(n-1)!\left|\Gamma(n+\beta_k)-\upsilon_k^{n-1+\beta_k}\right|} \\
&\times\left(\frac{L_k\Gamma(1-\mu_k)\,\upsilon_k^{\alpha_k+\beta_k-\mu_k}}{\Gamma(\alpha_k+\beta_k+1-\mu_k)}+\sum_{j=0}^{n-2}\frac{\omega_j^k\upsilon_k^{j+\beta_k}}{\Gamma(j+1+\beta_k)}\right).
\end{aligned}\tag{3.23}$$

Similarly, we obtain

$$\begin{aligned}
&\left\| D^{\delta_k}\left(T_k(u_1,...,u_n)(t_2)-T_k(u_1,...,u_n)(t_1)\right)\right\|_\infty \\
\leq\ & \frac{L_k\Gamma(1-\mu_k)\left(t_2^{\alpha_k-\delta_k-\mu_k}-t_1^{\alpha_k-\delta_k-\mu_k}\right)}{\Gamma(\alpha_k-\delta_k+1-\mu_k)} \\
&+\frac{\Gamma(n+\beta_k)\left(t_2^{n-1-\delta_k}-t_1^{n-1-\delta_k}\right)}{\Gamma(n-\delta_k)\left|\Gamma(n+\beta_k)-\upsilon_k^{n-1+\beta_k}\right|} \\
&\times\left(\frac{L_k\Gamma(1-\mu_k)\,\upsilon_k^{\alpha_k+\beta_k-\mu_k}}{\Gamma(\alpha_k+\beta_k+1-\mu_k)}+\sum_{j=0}^{n-2}\frac{\omega_j^k\upsilon_k^{j+\beta_k}}{\Gamma(j+1+\beta_k)}\right).
\end{aligned}\tag{3.24}$$

The right-hand sides of (3.23) and (3.24) are independent of $(u_1,...,u_n)$ and tend to zero as $t_1 \to t_2$, so, we deduce that $T(\Omega)$ is equicontinuous. Hence, by Arzela-Ascoli theorem, we state that T is completely continuous. □

Theorem 3.4. *Assume that*

(H_1) *There exist nonnegative constants* $\left(a_j^k\right)_{j=1,2,...,3n}^{k=1,,...,n}$, $n \in \mathbb{N} - \{0,1\}$, *satisfying*

$$t^{\mu_k} \left| f_k(t,x_1,...,x_{3n}) - f_k(t,y_1,...,y_{3n}) \right| \le \sum_{j=1}^{3n} a_j^k \left| x_j - y_j \right|,$$

$\forall\, t \in [0,1]$, $\forall\, (x_1,...,x_{3n}),\ (y_1,...,y_{3n}) \in \mathbb{R}^{3n}$.

(H_2) *There exist positive constants* $(b_j)_{j=1,...,n}$:

$$\left| h_j(t,x_j) - h_j(t,y_j) \right| \le b_j \left| x_j - y_j \right|,$$

$\forall\, t \in [0,1]$, $\forall\, x_j, y_j \in \mathbb{R}$, $j = 1,2,...,n$.

(H_3) $\Delta := \max_{1 \le k \le n} \Sigma_k \left(F_k^1 + F_k^2 \Lambda_k, F_k^3 + F_k^4 \Lambda_k\right) < 1;\ \Sigma_k = \sum_{j=1}^{n} \left(a_j^k + a_{n+j}^k + a_{2n+j}^k b_j\right).$

Then, system (1.1) has a unique solution on $[0,1]$.

Proof. Let $(u_1,...,u_n), (v_1,...,v_n) \in B$ and $t \in [0,1]$. Then,

$$\begin{aligned}
&\left\| T_k(u_1,...,u_n) - T_k(v_1,...,v_n) \right\|_\infty \\
\le\ & \int_0^t \frac{(t-s)^{\alpha_k-1} s^{-\mu_k}}{\Gamma(\alpha_k)} s^{\mu_k} \left| \begin{array}{l} f_k\left(\begin{array}{l} s, u_1(s),...,u_n, D^{\delta_1} u_1(s),...,D^{\delta_n} u_n(s), \\ \int_0^s h_1(\tau,u_1(\tau))\,d\tau,...,\int_0^s h_n(\tau,u_n(\tau))\,d\tau \end{array} \right) \\ -f_k\left(\begin{array}{l} s, v_1(s),...,v_n, D^{\delta_1} v_1(s),...,D^{\delta_n} v_n(s), \\ \int_0^s h_1(\tau,v_1(\tau))\,d\tau,...,\int_0^s h_n(\tau,v_n(\tau))\,d\tau \end{array} \right) \end{array} \right| ds \\
& + \frac{\Gamma(n+\beta_k)}{(n-1)! \left| \Gamma(n+\beta_k) - \upsilon_k^{n-1+\beta_k} \right|} \max_{t \in [0,1]} t^{n-1} \\
& \times \int_0^{\upsilon_k} \frac{(\upsilon_k - s)^{\alpha_k+\beta_k-1} s^{-\mu_k}}{\Gamma(\alpha_k+\beta_k)} s^{\mu_k} \left| \begin{array}{l} f_k\left(\begin{array}{l} s, u_1(s),...,u_n, D^{\delta_1} u_1(s),...,D^{\delta_n} u_n(s), \\ \int_0^s h_1(\tau,u_1(\tau))\,d\tau,...,\int_0^s h_n(\tau,u_n(\tau))\,d\tau \end{array} \right) \\ -f_k\left(\begin{array}{l} s, v_1(s),...,v_n, D^{\delta_1} v_1(s),...,D^{\delta_n} v_n(s), \\ \int_0^s h_1(\tau,v_1(\tau))\,d\tau,...,\int_0^s h_n(\tau,v_n(\tau))\,d\tau \end{array} \right) \end{array} \right| ds.
\end{aligned}$$

By (H_1) and (H_2), we obtain

$$
\begin{aligned}
&\left\|T_k(u_1,u_2,...,u_n)-T_k(v_1,v_2,...,v_n)\right\|_\infty \\
\leq\ & \left(\begin{array}{l} a_1^k\left\|u_1-v_1\right\|_\infty+...+a_n^k\left\|u_n-v_n\right\|_\infty \\ +a_{n+1}^k\left\|D^{\delta_1}(u_1-v_1)\right\|_\infty+...+a_{2n}^k\left\|D^{\delta_n}(u_n-v_n)\right\|_\infty \\ +a_{2n+1}^k b_1\left\|u_1-v_1\right\|_\infty \max\limits_{s\in[0,1]}\int_0^s d\tau \\ +...+a_{3n}^k b_n\left\|u_n-v_n\right\|_\infty \max\limits_{s\in[0,1]}\int_0^s d\tau \end{array}\right) \\
&\times\left(\begin{array}{l} \int_0^t \frac{(t-s)^{\alpha_k-1}s^{-\mu_k}}{\Gamma(\alpha_k)}ds \\ +\frac{\Gamma(n+\beta_k)}{(n-1)!\left|\Gamma(n+\beta_k)-\upsilon_k^{n-1+\beta_k}\right|}\int_0^{\upsilon_k}\frac{(\upsilon_k-s)^{\alpha_k+\beta_k-1}s^{-\mu_k}}{\Gamma(\alpha_k+\beta_k)}ds \end{array}\right) \\
\leq\ & \left(\begin{array}{l} \left(a_1^k+a_{2n+1}^k b_1^1\right)\left\|u_1-v_1\right\|_\infty+...+\left(a_n^k+a_{3n}^k b_n^n\right)\left\|u_n-v_n\right\|_\infty \\ +a_{n+1}^k\left\|D^{\delta_1}(u_1-v_1)\right\|_\infty+...+a_{2n}^k\left\|D^{\delta_n}(u_n-v_n)\right\|_\infty \end{array}\right) \\
&\times\left(\begin{array}{l} \frac{\Gamma(1-\mu_k)}{\Gamma(\alpha_k+1-\mu_k)}\max\limits_{t\in[0,1]}t^{\alpha_k-\mu_k} \\ +\frac{\Gamma(n+\beta_k)}{(n-1)!\left|\Gamma(n+\beta_k)-\upsilon_k^{n-1+\beta_k}\right|}\left(\frac{\Gamma(1-\mu_k)\upsilon_k^{\alpha_k+\beta_k-\mu_k}}{\Gamma(\alpha_k+\beta_k+1-\mu_k)}\right) \end{array}\right) \\
\leq\ & \sum_{j=1}^n\left(a_j^k+a_{n+j}^k+a_{2n+j}^k b_j\right)\left(F_k^1+F_k^2\Lambda_k\right)\max_{1\leq k\leq n}\left(\begin{array}{l}\left\|u_k-v_k\right\|_\infty, \\ \left\|D^{\delta_k}(u_k-v_k)\right\|_\infty\end{array}\right).
\end{aligned}
\tag{3.25}
$$

Hence,

$$
\begin{aligned}
&\left\|T_k(u_1,...,u_n)-T_k(v_1,...,v_n)\right\|_\infty \\
\leq\ & \Sigma_k\left(F_k^1+F_k^2\Lambda_k\right)\left\|(u_1-v_1,...,u_n-v_n)\right\|_B.
\end{aligned}
\tag{3.26}
$$

On the other hand, we have

$$
\begin{aligned}
&\left\|D^{\delta_k}\left(T_k(u_1,...,u_n)-T_k(v_1,...,v_n)\right)\right\|_\infty \\
\leq\ & \int_0^t\frac{(t-s)^{\alpha_k-\delta_k-1}s^{-\mu_k}}{\Gamma(\alpha_k-\delta_k)}s^{\mu_k}\left|\begin{array}{l} f_k\left(\begin{array}{l} s,u_1(s),...,u_n,D^{\delta_1}u_1(s),...,D^{\delta_n}u_n(s), \\ \int_0^s h_1(\tau,u_1(\tau))d\tau,...,\int_0^s h_n(\tau,u_n(\tau))d\tau \end{array}\right) \\ -f_k\left(\begin{array}{l} s,v_1(s),...,v_n,D^{\delta_1}v_1(s),...,D^{\delta_n}v_n(s), \\ \int_0^s h_1(\tau,v_1(\tau))d\tau,...,\int_0^s h_n(\tau,v_n(\tau))d\tau \end{array}\right)\end{array}\right|ds \\
&+\frac{\Gamma(n+\beta_k)}{\Gamma(n-\delta_k)\left|\Gamma(n+\beta_k)-\upsilon_k^{n-1+\beta_k}\right|}\max_{t\in[0,1]}t^{n-\delta_k-1} \\
&\times\int_0^{\upsilon_k}\frac{(\upsilon_k-s)^{\alpha_k+\beta_k-1}s^{-\mu_k}}{\Gamma(\alpha_k+\beta_k)}s^{\mu_k}\left|\begin{array}{l} f_k\left(\begin{array}{l} s,u_1(s),...,u_n,D^{\delta_1}u_1(s),...,D^{\delta_n}u_n(s), \\ \int_0^s h_1(\tau,u_1(\tau))d\tau,...,\int_0^s h_n(\tau,u_n(\tau))d\tau \end{array}\right) \\ -f_k\left(\begin{array}{l} s,v_1(s),...,v_n,D^{\delta_1}v_1(s),...,D^{\delta_n}v_n(s), \\ \int_0^s h_1(\tau,v_1(\tau))d\tau,...,\int_0^s h_n(\tau,v_n(\tau))d\tau \end{array}\right)\end{array}\right|ds.
\end{aligned}
$$

Similarly, using (H_1) and (H_2), we obtain

$$\left\| D^{\delta_k}\left(T_k\left(u_1,u_2,\dots,u_n\right)-T_k\left(v_1,v_2,\dots,v_n\right)\right)\right\|_\infty \leq$$
$$\sum_{j=1}^{n}\left(a_j^k+a_{n+j}^k+a_{2n+j}^k b_j^j\right)\left(F_k^3+F_k^4\Lambda_k\right)\left\|\left(u_1-v_1,\dots,u_n-v_n\right)\right\|_B . \tag{3.27}$$

Thanks to (3.26) and (3.27), we get

$$\begin{aligned} & \left\|T\left(u_1,u_2,\dots,u_n\right)-T\left(v_1,v_2,\dots,v_n\right)\right\|_B \\ \leq\ & \max_{1\leq k\leq n}\Sigma_k\left(F_k^1+F_k^2\Lambda_k,F_k^3+F_k^4\Lambda_k\right)\left\|\left(u_1-v_1,\dots,u_n-v_n\right)\right\|_B . \end{aligned} \tag{3.28}$$

Using the hypothesis (H_3), we conclude that T is contractive. By Banach fixed point theorem, we state that T has a fixed point which is the unique solution of system (1.1). □

Example 3.5. Consider the following singular fractional system:

$$\begin{cases} D^{\frac{7}{3}}u_1(t)=\dfrac{t^{-\frac{1}{4}}\left|u_1(t)+u_2(t)+u_3(t)+D^{\frac{3}{2}}u_1(t)+D^{\frac{4}{3}}u_2(t)+D^{\frac{7}{4}}u_3(t)\right|}{64\pi^2\left(1+\left|u_1(t)+u_2(t)+u_3(t)+D^{\frac{3}{2}}u_1(t)+D^{\frac{4}{3}}u_2(t)+D^{\frac{7}{4}}u_3(t)\right|\right)} \\ +\frac{t^{-\frac{1}{4}}}{16\pi}\left(\int_0^t\frac{\cos u_1(\tau)}{2\pi}d\tau+\int_0^t\frac{\sin u_2(\tau)}{2\pi}d\tau-\int_0^t\frac{\sin u_3(\tau)}{2\pi}d\tau\right), \\ D^{\frac{5}{2}}u_2(t)=\dfrac{\cos u_1(t)+\cos u_2(t)+\cos u_3(t)+\sin D^{\frac{3}{2}}u_1(t)+\sin D^{\frac{4}{3}}u_2(t)+\sin D^{\frac{7}{4}}u_3(t)}{120\pi t^{\frac{1}{3}}} \\ +\frac{t^{-\frac{1}{3}}}{6}\left(\int_0^t\frac{\cos u_1(\tau)}{2\pi}d\tau+\int_0^t\frac{\sin u_2(\tau)}{2\pi}d\tau+\int_0^t\frac{\sin u_3(\tau)}{2\pi}d\tau\right), \\ D^{\frac{9}{4}}u_3(t)=\dfrac{t^{-\frac{1}{6}}\left|u_1(t)+u_2(t)+u_3(t)\right|}{8\pi^2\left(1+\left|u_1(t)+u_2(t)+u_3(t)\right|\right)} \\ +\frac{t^{-\frac{1}{6}}}{16\pi^2}\left(\cos D^{\frac{3}{2}}u_1(t)+\cos D^{\frac{4}{3}}u_2(t)+\cos D^{\frac{7}{4}}u_3(t)\right) \\ +\frac{t^{-\frac{1}{6}}}{8\pi}\left(\int_0^t\frac{\cos u_1(\tau)}{2\pi}d\tau-\int_0^t\frac{\sin u_2(\tau)}{2\pi}d\tau-\int_0^t\frac{\sin u_3(\tau)}{2\pi}d\tau\right), \\ 0<t\leq 1,\ u_1(0)=\sqrt{3},\ u_1'(0)=1,\ u_1''(0)=J^{\frac{11}{3}}u_1\left(\frac{1}{2}\right), \\ u_2(0)=2\sqrt{2},\ u_2'(0)=-1,\ u_2''(0)=J^{\frac{8}{3}}u_2\left(\frac{1}{3}\right), \\ u_3(0)=-\sqrt{2},\ u_3'(0)=\sqrt{3},\ u_3''(0)=J^{\frac{1}{4}}u_3\left(\frac{3}{4}\right). \end{cases} \tag{3.29}$$

We have: $n=3$, $\alpha_1=\frac{7}{3}$, $\alpha_2=\frac{5}{2}$, $\alpha_3=\frac{9}{4}$, $\delta_1=\frac{3}{2}$, $\delta_2=\frac{4}{3}$, $\delta_3=\frac{7}{4}$, $\beta_1=\frac{11}{3}$, $\beta_2=\frac{8}{3}$, $\beta_3=\frac{1}{4}$, $\upsilon_1=\frac{1}{2}$, $\upsilon_2=\frac{1}{3}$, $\upsilon_3=\frac{3}{4}$.

$$h_1(t,u_1(t))=\frac{\cos u_1(t)}{2\pi},\ h_2(t,u_2(t))=\frac{\sin u_2(t)}{2\pi},\ h_3(t,u_3(t))=\frac{\sin u_3(t)}{2\pi}.$$

For all $t\in[0,1]$ and $x_j, y_j\in\mathbb{R}$, $j=1,2,\dots,9$, we get

$$t^{\frac{1}{2}}\left|f_1(t,x_1,\dots,x_9)-f_1(t,y_1,\dots,y_9)\right|\leq\frac{t^{\frac{1}{4}}}{64\pi^2}\sum_{i=1}^{6}|x_i-y_i|+\frac{t^{\frac{1}{4}}}{16\pi}\sum_{i=7}^{9}\frac{1}{2\pi}|x_i-y_i|,$$

$$t^{\frac{2}{3}}\left|f_2(t,x_1,\dots,x_9)-f_2(t,y_1,\dots,y_9)\right|\leq\frac{t^{\frac{1}{3}}}{120\pi}\sum_{i=1}^{6}|x_i-y_i|+\frac{t^{\frac{1}{3}}}{6}\sum_{i=7}^{9}\frac{1}{2\pi}|x_i-y_i|,$$

and

$$t^{\frac{1}{3}}\left|f_3(t,x_1,\dots,x_9)-f_3(t,y_1,\dots,y_9)\right|\leq\frac{t^{\frac{1}{6}}}{16\pi^2}\sum_{i=1}^{6}|x_i-y_i|+\frac{t^{\frac{1}{6}}}{8\pi}\sum_{i=7}^{9}\frac{1}{2\pi}|x_i-y_i|,$$

where $\mu_1 = \frac{1}{2}$, $\mu_2 = \frac{2}{3}$ and $\mu_3 = \frac{1}{3}$. Then, we can take

$$b_1 = b_2 = b_3 = \frac{1}{2\pi},$$

$$\begin{aligned}
\left(a_j^1\right)_{j=1,2,\ldots,6} &= \frac{1}{64\pi^2},\ \left(a_j^1\right)_{j=7,8,9} = \frac{1}{16\pi},\ \Sigma_1 = \frac{3}{16\pi^2},\\
\left(a_j^2\right)_{j=1,2,\ldots,6} &= \frac{1}{120\pi},\ \left(a_j^2\right)_{j=7,8,9} = \frac{1}{6},\ \Sigma_2 = \frac{3}{10\pi},\\
\left(a_j^3\right)_{j=1,2,\ldots,6} &= \frac{1}{16\pi^2},\ \left(a_j^3\right)_{j=7,8,9} = \frac{1}{8\pi},\ \Sigma_3 = \frac{9}{16\pi^2}.
\end{aligned}$$

Indeed,

$$F_1^1 = 1.0278,\ F_1^2 = 0.5000,\ \Lambda_1 = 1.3617e-004,\ F_1^3 = 1.9849,\ F_1^4 = 1.1285.$$

$$F_2^1 = 1.5534,\ F_2^2 = 0.5000,\ \Lambda_2 = 3.6480e-004,\ F_2^3 = 3.0229,\ F_2^4 = 1.1078,$$

$$F_3^1 = 0.7302,\ F_3^2 = 0.6292,\ \Lambda_3 = 0.3096,\ F_3^3 = 1.4596,\ F_3^4 = 1.3883.$$

It is clear that $\Delta < 1$. So, the system (3.29) has a unique solution on $[0,1]$.

Theorem 3.6. *Assume that $n-1 < \alpha_k < n$, $n-2 < \delta_k < n-1$, $k = 1,2,\ldots,n$, $n \in \mathbb{N} - \{0,1\}$, $h_k : [0,1] \times \mathbb{R} \to [-1,1]$ and $f_k : (0,1] \times \mathbb{R}^{3n} \to \mathbb{R}$ are continuous, $\lim_{t\to 0^+} f_k(t,\ldots) = \infty$. Let $0 < \mu_k < 1$, and $t^{\mu_k} f_k(t,\ldots)$ are continuous on $[0,1] \times \mathbb{R}^{3n}$. Then, the fractional coupled system (1.1) has at least one solution on $[0,1]$.*

Proof. Let

$$C_k = \max_{t\in[0,1]} \left| t^{\mu_k} f_k \begin{pmatrix} t, u_1, \ldots, u_n, D^{\delta_1} u_1, \ldots, D^{\delta_n} u_n, \\ \int_0^t h_1(\tau, u_1(\tau))\, d\tau, \ldots, \int_0^t h_n(\tau, u_n(\tau))\, d\tau \end{pmatrix} \right|, \tag{3.30}$$

and $\Phi := \{(u_1, u_2, \ldots, u_n) \in B : \|(u_1, u_2, \ldots, u_n)\|_B \le r\}$, where

$$r = \max_{1\le k\le n} \begin{pmatrix} C_k F_k^1 + F_k^2 \left(C_k \Lambda_k + \Lambda_k^*\right) + \sum_{j=0}^{n-2} \frac{\left|\omega_j^k\right|}{j!}, \\ C_k F_k^3 + F_k^4 \left(C_k \Lambda_k + \Lambda_k^*\right) \end{pmatrix}. \tag{3.31}$$

We shall show that $T : \Phi \to \Phi$. Let $(u_1, u_2, \ldots, u_n) \in \Phi$, $t \in [0,1]$. By (3.30), we obtain

$$\|T_k(u_1, \ldots, u_n)\|_\infty \le C_k F_k^1 + F_k^2 (C_k \Lambda_k + \Lambda_k^*) + \sum_{j=0}^{n-2} \frac{\left|\omega_j^k\right|}{j!}, \tag{3.32}$$

and

$$\left\| D^{\delta_k} T_k(u_1, \ldots, u_n) \right\|_\infty \le C_k F_k^3 + F_k^4 (C_k \Lambda_k + \Lambda_k^*). \tag{3.33}$$

By (3.32) and (3.33), we obtain

$$\begin{aligned}
&\|T(u_1, \ldots, u_n)\|_B \\
\le\ &\max_{1\le k\le n} \begin{pmatrix} C_k F_k^1 + F_k^2 \left(C_k \Lambda_k + \Lambda_k^*\right) + \sum_{j=0}^{n-2} \frac{\left|\omega_j^k\right|}{j!}, \\ C_k F_k^3 + F_k^4 \left(C_k \Lambda_k + \Lambda_k^*\right) \end{pmatrix},
\end{aligned} \tag{3.34}$$

which implies that $\|T(u_1, \ldots, u_n)\|_B \le r$. So for $(u_1, \ldots, u_n) \in \Phi$, we get $T(u_1, \ldots, u_n) \in \Phi$. Then, Lemma 3.1 and Lemma 3.2 imply that $T_k(u_1, \ldots, u_n), D^{\delta_k} T_k(u_1, \ldots, u_n) \in C([0,1])$. Hence, $T : \Phi \to \Phi$. Moreover, by Lemma 3.3, we see that T is completely continuous. Consequently, using Lemma 2.1, we state that problem (1.1) has at least one solution on $[0,1]$. □

Example 3.7. Consider the following system:

$$\begin{cases} D^{\frac{10}{3}}u_1(t)=\frac{t^{-\frac{1}{6}}\cos(u_1(t)u_2(t)-u_3(t)u_4(t))}{e^t+\left|\sin\left(D^{\frac{5}{2}}u_1(t)D^{\frac{8}{3}}u_2(t)D^{\frac{9}{4}}u_3(t)D^{\frac{10}{4}}u_4(t)\right)\right|} \\ +t^{-\frac{1}{6}}\left(\int_0^t\sin u_1(\tau)d\tau+\int_0^t\frac{|u_2(\tau)|}{|1+u_2(\tau)|}d\tau-\int_0^t\cos u_3(\tau)d\tau\right), \\ D^{\frac{7}{2}}u_2(t)=\frac{t^{-\frac{2}{9}}}{2\pi}\sin\frac{(u_1(t)+u_2(t)+u_3t+u_4(t))}{\pi+\left|D^{\frac{5}{2}}u_1(t)+D^{\frac{8}{3}}u_2(t)+D^{\frac{9}{4}}u_3(t)+D^{\frac{10}{4}}u_4(t)\right|} \\ +t^{-\frac{2}{9}}\left(\int_0^t\sin u_1(\tau)d\tau+\int_0^t\frac{|u_2(\tau)|}{|1+u_2(\tau)|}d\tau-\int_0^t\cos u_3(\tau)d\tau\right), \\ D^{\frac{13}{4}}u_3(t)=\frac{\left(u_1(t)-D^{\frac{5}{2}}u_1(t)\right)\left(u_3(t)+D^{\frac{9}{4}}u_3(t)\right)}{\sqrt{t}e^t\left|u_2(t)+D^{\frac{8}{3}}u_2(t)\right|+\left|u_4(t)+D^{\frac{10}{4}}u_4(t)\right|} \\ +t^{-\frac{1}{2}}\left(\int_0^t\sin u_1(\tau)d\tau+\int_0^t\frac{|u_2(\tau)|}{|1+u_2(\tau)|}d\tau+\int_0^t\cos u_3(\tau)d\tau\right), \\ D^{\frac{16}{5}}u_4(t)=\frac{t^{-\frac{1}{4}}\left(u_1(t)+D^{\frac{5}{2}}u_1(t)\right)\left(u_2(t)+D^{\frac{8}{3}}u_2(t)\right)}{\pi e+\left|u_3(t)+D^{\frac{9}{4}}u_3(t)\right|\left|u_4(t)+D^{\frac{10}{4}}u_4(t)\right|} \\ +t^{-\frac{1}{4}}\left(\int_0^t\sin u_1(\tau)d\tau+\int_0^t\frac{|u_2(\tau)|}{|1+u_2(\tau)|}d\tau+\int_0^t\cos u_3(\tau)d\tau\right), \\ 0<t\leq 1, \\ u_1(0)=\pi,\ u_1'(0)=-1,\ u_1''(0)=\sqrt{2},\ u_1^{(3)}(0)=J^{\frac{15}{4}}u_1\left(\frac{4}{5}\right), \\ u_2(0)=3\sqrt{2},\ u_2'(0)=1,\ u_2''(0)=-1,\ u_2^{(3)}(0)=J^{\frac{8}{3}}u_2\left(\frac{1}{2}\right), \\ u_3(0)=-1,\ u_3'(0)=2\sqrt{3},\ u_3''(0)=\frac{1}{\pi},\ u_3^{(3)}(0)=J^{\frac{12}{5}}u_3\left(\frac{1}{4}\right), \\ u_4(0)=2\sqrt{3},\ u_4'(0)=-\sqrt{2},\ u_4''(0)=1,\ u_4^{(3)}(0)=J^{\frac{11}{4}}u_4\left(\frac{3}{8}\right). \end{cases} \tag{3.35}$$

Let $\mu_1=\frac{1}{3}$, $\mu_2=\frac{4}{9}$, $\mu_3=\frac{3}{4}$,and $\mu_4=\frac{1}{2}$. Then all the assumptions of Theorem 3.6 are satisfied. So, (3.35) has at least one solution on $[0,1]$.

4. Generalized Ulam-Hyers Stability

In this section, we study the Ulam-Hyers stability and the generalized Ulam-Hyers stability for system (1.1).

Definition 4.1. System (1.1) is Ulam-Hyers stable if there exists a constant $\eta_{f_k}>0$ such that for all $(\varepsilon_1,...,\varepsilon_n)>0$, and for all solution $(v_1,...,v_n)\in B$ of

$$\begin{cases} \left|D^{\alpha_k}v_k(t)-f_k\begin{pmatrix} s,v_1(s),...,v_n(s),D^{\delta_1}v_1(s),...,D^{\delta_n}v_n(s),\int_0^s h_1(\tau,v_1(\tau))d\tau \\ ,...,\int_0^s h_n(\tau,v_n(\tau))d\tau \end{pmatrix}\right|\leq\varepsilon_k, \\ 0<t\leq 1,\ n-1<\alpha_k<n,\ n-2<\delta_k<n-1,\ v_k^{(j)}(0)=\omega_j^k,\ j=0,1,...,n-2, \\ v_k^{(n-1)}(0)=J^{\beta_k}v_k(\upsilon_k),\ \beta_k>0,0<\upsilon_k<1,\ k=1,2,...,n, \end{cases} \tag{4.1}$$

there exists $(u_1,...,u_n)\in B$ of (1.1) with

$$\|(v_1-u_1,...,v_n-u_n)\|_B\leq\eta_{f_k}\varepsilon,\ \varepsilon>0. \tag{4.2}$$

Definition 4.2. System (1.1) is generalized Ulam-Hyers stable if there exists $\phi_{f_k} \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\phi_{f_k}(0) = 0$, such that for all $\varepsilon > 0$, and for each solution $(v_1, ..., v_n) \in B$ of (4.1), there exists $(u_1, ..., u_n) \in B$ of (1.1) with

$$\|(v_1 - u_1, ..., v_n - u_n)\|_B \leq \phi_{f_k}(\varepsilon), \ \varepsilon > 0. \tag{4.3}$$

Theorem 4.3. *Let $n-1 < \alpha_k < n$, $n \in \mathbb{N} - \{0,1\}$. Assume that the hypotheses $(H_i)_{i=1,2,3}$ are satisfied. Then, system (1.1) is generalized Ulam-Hyers stable in B.*

Proof. Let $(v_1, ..., v_n) \in B$ be solution of Ineq. (4.1). By integrating inequality (4.1), we obtain

$$\begin{aligned}
&\left| \begin{array}{l} v_k(t) - \int_0^t \frac{(t-s)^{\alpha_k-1} s^{-\mu_k}}{\Gamma(\alpha_k)} s^{\mu_k} f_k \left(\begin{array}{l} s, v_1(s), ..., v_n(s), D^{\delta_1} v_1(s), ..., D^{\delta_n} v_n(s), \\ \int\limits_0^s h_1(\tau, v_1(\tau)) d\tau, ..., \int\limits_0^s h_n(\tau, v_n(\tau)) d\tau \end{array} \right) ds \\ - \sum\limits_{j=0}^{n-2} \frac{\omega_j^k}{j!} t^j - \frac{\Gamma(n+\beta_k) t^{n-1}}{(n-1)!\left(\Gamma(n+\beta_k) - \upsilon_k^{n-1+\beta_k}\right)} \\ \times \left(\begin{array}{l} \int\limits_0^{\upsilon_k} \frac{(\upsilon_k - s)^{\alpha_k+\beta_k-1} s^{-\mu_k}}{\Gamma(\alpha_k+\beta_k)} s^{\mu_k} f_k \left(\begin{array}{l} s, v_1(s), ..., v_n(s), D^{\delta_1} v_1(s), ..., D^{\delta_n} v_n(s), \\ \int\limits_0^s h_1(\tau, v_1(\tau)) d\tau, ..., \int\limits_0^s h_n(\tau, v_n(\tau)) d\tau \end{array} \right) ds \\ + \sum\limits_{j=0}^{n-2} \frac{\omega_j^k \upsilon_k^{j+\beta_k}}{\Gamma(j+1+\beta_k)} \end{array} \right) \end{array} \right| \\
\leq \;& J^{\alpha_k} \varepsilon_k \\
\leq \;& \frac{t^{\alpha_k}}{\Gamma(\alpha_k+1)} \varepsilon_k, \ k = 1, 2, .., n.
\end{aligned} \tag{4.4}$$

Using (H_1) and (H_2), there exists a solution $(u_1, ..., u_n) \in B$ of Eq. (1.1):

$$\begin{aligned}
u_k(t) &= \int_0^t \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)} f_k \left(\begin{array}{l} s, u_1(s), ..., u_n(s), D^{\delta_1} u_1(s), ..., D^{\delta_n} u_n(s), \\ \int\limits_0^s h_1(\tau, u_1(\tau)) d\tau, ..., \int\limits_0^s h_n(\tau, u_n(\tau)) d\tau \end{array} \right) ds \\
&\quad + \sum_{j=0}^{n-2} \frac{\omega_j^k}{j!} t^j + \frac{\Gamma(n+\beta_k) t^{n-1}}{(n-1)!\left(\Gamma(n+\beta_k) - \upsilon_k^{n-1+\beta_k}\right)} \\
&\quad \times \left(\begin{array}{l} \int\limits_0^{\upsilon_k} \frac{(\upsilon_k - s)^{\alpha_k+\beta_k-1}}{\Gamma(\alpha_k+\beta_k)} f_k \left(\begin{array}{l} s, u_1(s), ..., u_n(s), D^{\delta_1} u_1(s), ..., D^{\delta_n} u_n(s), \\ \int\limits_0^s h_1(\tau, u_1(\tau)) d\tau, ..., \int\limits_0^s h_n(\tau, u_n(\tau)) d\tau \end{array} \right) ds \\ + \sum\limits_{j=0}^{n-2} \frac{\omega_j^k \upsilon_k^{j+\beta_k}}{\Gamma(j+1+\beta_k)} \end{array} \right), \\
k &= 1, ..., n.
\end{aligned}$$

Then, we get

$$
\begin{aligned}
&|v_k(t)-u_k(t)| \\
&= \left| \begin{array}{l}
v_k(t) - \sum_{j=0}^{n-2} \frac{\omega_j^k}{j!} t^j - \int_0^t \frac{(t-s)^{\alpha_k-1} s^{-\mu_k}}{\Gamma(\alpha_k)} s^{\mu_k} f_k \left(\begin{array}{l} s, v_1(s), \dots, v_n(s), D^{\delta_1} v_1(s), \dots, D^{\delta_n} v_n(s), \\ \int_0^s h_1(\tau, v_1(\tau))\, d\tau, \dots, \int_0^s h_n(\tau, v_n(\tau))\, d\tau \end{array} \right) ds \\
- \frac{\Gamma(n+\beta_k) t^{n-1}}{(n-1)!\left(\Gamma(n+\beta_k) - \upsilon_k^{n-1+\beta_k}\right)} \\
\times \left(\begin{array}{l} \int_0^{\upsilon_k} \frac{(\upsilon_k-s)^{\alpha_k+\beta_k-1} s^{-\mu_k}}{\Gamma(\alpha_k+\beta_k)} s^{\mu_k} f_k \left(\begin{array}{l} s, v_1(s), \dots, v_n(s), D^{\delta_1} v_1(s), \dots, D^{\delta_n} v_n(s), \\ \int_0^s h_1(\tau, v_1(\tau))\, d\tau, \dots, \int_0^s h_n(\tau, v_n(\tau))\, d\tau \end{array} \right) ds \\ + \sum_{j=0}^{n-2} \frac{\omega_j^k \upsilon_k^{j+\beta_k}}{\Gamma(j+1+\beta_k)} \end{array} \right) \\
+ \int_0^t \frac{(t-s)^{\alpha_k-1} s^{-\mu_k}}{\Gamma(\alpha_k)} s^{\mu_k} \left(\begin{array}{l} f_k \left(\begin{array}{l} s, v_1(s), \dots, v_n(s), D^{\delta_1} v_1(s), \dots, D^{\delta_n} v_n(s), \\ \int_0^s h_1(\tau, v_1(\tau))\, d\tau, \dots, \int_0^s h_n(\tau, v_n(\tau))\, d\tau \end{array} \right) \\ - f_k \left(\begin{array}{l} s, u_1(s), \dots, u_n(s), D^{\delta_1} u_1(s), \dots, D^{\delta_n} u_n(s), \\ \int_0^s h_1(\tau, u_1(\tau))\, d\tau, \dots, \int_0^s h_n(\tau, u_n(\tau))\, d\tau \end{array} \right) \end{array} \right) ds \\
+ \frac{\Gamma(n+\beta_k) t^{n-1}}{(n-1)!\left(\Gamma(n+\beta_k) - \upsilon_k^{n-1+\beta_k}\right)} \\
\times \int_0^{\upsilon_k} \frac{(\upsilon_k-s)^{\alpha_k+\beta_k-1}}{\Gamma(\alpha_k+\beta_k)} \left(\begin{array}{l} f_k \left(\begin{array}{l} s, v_1(s), \dots, v_n(s), D^{\delta_1} v_1(s), \dots, D^{\delta_n} v_n(s), \\ \int_0^s h_1(\tau, v_1(\tau))\, d\tau, \dots, \int_0^s h_n(\tau, v_n(\tau))\, d\tau \end{array} \right) \\ - f_k \left(\begin{array}{l} s, u_1(s), \dots, u_n(s), D^{\delta_1} u_1(s), \dots, D^{\delta_n} u_n(s), \\ \int_0^s h_1(\tau, u_1(\tau))\, d\tau, \dots, \int_0^s h_n(\tau, u_n(\tau))\, d\tau \end{array} \right) \end{array} \right) ds
\end{array} \right|.
\end{aligned}
\tag{4.5}
$$

By using (4.4), we obtain

$$
\begin{aligned}
&\max_{t\in[0,1]}\left|v_k(t)-u_k(t)\right| \\
\leq\ & \frac{\varepsilon_k}{\Gamma(\alpha_k+1)}+\int_0^t \frac{(t-s)^{\alpha_k-1}s^{-\mu_k}}{\Gamma(\alpha_k)}s^{\mu_k}\left|\begin{array}{l} f_k\left(\begin{array}{l} s,v_1(s),\dots,v_n(s),D^{\delta_1}v_1(s),\dots,D^{\delta_n}v_n(s), \\ \int\limits_0^s h_1(\tau,v_1(\tau))\,d\tau,\dots,\int\limits_0^s h_n(\tau,v_n(\tau))\,d\tau \end{array}\right) \\ -f_k\left(\begin{array}{l} s,u_1(s),\dots,u_n(s),D^{\delta_1}u_1(s),\dots,D^{\delta_n}u_n(s), \\ \int\limits_0^s h_1(\tau,u_1(\tau))\,d\tau,\dots,\int\limits_0^s h_n(\tau,u_n(\tau))\,d\tau \end{array}\right)\end{array}\right|ds \\
&+\frac{\Gamma(n+\beta_k)t^{n-1}}{(n-1)!\left|\Gamma(n+\beta_k)-\upsilon_k^{n-1+\beta_k}\right|} \\
&\times\int\limits_0^{\upsilon_k}\frac{(\upsilon_k-s)^{\alpha_k+\beta_k-1}s^{-\mu_k}}{\Gamma(\alpha_k+\beta_k)}s^{\mu_k}\left|\begin{array}{l} f_k\left(\begin{array}{l} s,v_1(s),\dots,v_n(s),D^{\delta_1}v_1(s),\dots,D^{\delta_n}v_n(s), \\ \int\limits_0^s h_1(\tau,v_1(\tau))\,d\tau,\dots,\int\limits_0^s h_n(\tau,v_n(\tau))\,d\tau \end{array}\right) \\ -f_k\left(\begin{array}{l} s,u_1(s),\dots,u_n(s),D^{\delta_1}u_1(s),\dots,D^{\delta_n}u_n(s), \\ \int\limits_0^s h_1(\tau,u_1(\tau))\,d\tau,\dots,\int\limits_0^s h_n(\tau,u_n(\tau))\,d\tau \end{array}\right)\end{array}\right|ds.
\end{aligned} \tag{4.6}
$$

Consequently,

$$
\|v_k-u_k\|_\infty \leq \frac{\varepsilon_k}{\Gamma(\alpha_k+1)}+\Sigma_k\left(F_k^1+F_k^2\Lambda_k\right)\|(v_1-u_1,\dots,v_n-u_n)\|_B. \tag{4.7}
$$

By differentiating inequality (4.4), we have

$$
\begin{aligned}
&\left|\begin{array}{l} D^{\delta_k}v_k(t)-\int_0^t \frac{(t-s)^{\alpha_k-\delta_k-1}s^{-\mu_k}}{\Gamma(\alpha_k-\delta_k)}s^{\mu_k}f_k\left(\begin{array}{l} s,v_1(s),\dots,v_n(s),D^{\delta_1}v_1(s),\dots,D^{\delta_n}v_n(s), \\ \int\limits_0^s h_1(\tau,v_1(\tau))\,d\tau,\dots,\int\limits_0^s h_n(\tau,v_n(\tau))\,d\tau \end{array}\right)ds \\ -\frac{\Gamma(n+\beta_k)t^{n-1-\delta_k}}{\Gamma(n-\delta_k)\left(\Gamma(n+\beta_k)-\upsilon_k^{n-1+\beta_k}\right)} \\ \times\left(\int\limits_0^{\upsilon_k}\frac{(\upsilon_k-s)^{\alpha_k+\beta_k-1}s^{-\mu_k}}{\Gamma(\alpha_k+\beta_k)}s^{\mu_k}f_k\left(\begin{array}{l} s,v_1(s),\dots,v_n(s),D^{\delta_1}v_1(s),\dots,D^{\delta_n}v_n(s), \\ \int\limits_0^s h_1(\tau,v_1(\tau))\,d\tau,\dots,\int\limits_0^s h_n(\tau,v_n(\tau))\,d\tau \end{array}\right)ds+\sum_{j=0}^{n-2}\frac{\omega_j^k\upsilon_k^{j+\beta_k}}{\Gamma(j+1+\beta_k)}\right)\end{array}\right| \\
\leq\ & J^{\alpha_k-\delta_k}\varepsilon_k \\
\leq\ & \frac{t^{\alpha_k-\delta_k}}{\Gamma(\alpha_k-\delta_k+1)}\varepsilon_k,\ k=1,2,..,n.
\end{aligned} \tag{4.8}
$$

Similarly, we can obtain

$$
\left\|D^{\delta_k}(v_k-u_k)\right\|_\infty \leq \frac{\varepsilon_k}{\Gamma(\alpha_k-\delta_k+1)}+\Sigma_k\left(F_k^3+F_k^4\Lambda_k\right)\|(v_1-u_1,\dots,v_n-u_n)\|_B, \tag{4.9}
$$

.

Thanks to (4.7) and (4.9), we get

$$\begin{aligned}\|(v_1-u_1,\ldots,v_n-u_n)\|_B &\le \max_{1\le k\le n}\left(\frac{\varepsilon_k}{\Gamma(\alpha_k+1)},\frac{\varepsilon_k}{\Gamma(\alpha_k-\delta_k+1)}\right)+\Delta\|(v_1-u_1,\ldots,v_n-u_n)\|_B\\ &\le \varepsilon\Lambda+\Delta\|(v_1-u_1,\ldots,v_n-u_n)\|_B, \qquad (4.10)\end{aligned}$$

where

$$\varepsilon=\max_{1\le k\le n}\varepsilon_k,\ \Lambda=\max_{1\le k\le n}\left(\frac{1}{\Gamma(\alpha_k+1)},\frac{1}{\Gamma(\alpha_k-\delta_k+1)}\right). \qquad (4.11)$$

Thus,

$$\|(v_1-u_1,\ldots,v_n-u_n)\|_B\le\frac{\varepsilon\Lambda}{(1-\Delta)}:=\eta_{f_k}\varepsilon,\ \eta_{f_k}=\frac{\Lambda}{(1-\Delta)}. \qquad (4.12)$$

Using (H_3), we have $\eta_{f_k}>0$. Therefore, system (1.1) is Ulam-Hyers stable. By putting $\phi_{f_k}(\varepsilon)=\eta_{f_k}\varepsilon$, we get system (1.1) is generalized Ulam-Hyers stable. □

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