Commun. Optim. Theory 2019 (2019), Article ID 3 https://doi.org/10.23952/cot.2019.3

MATHRES

Communications in Optimization Theory
Available online at http://cot.mathres.org

# COMMON FIXED POINTS OF WEAKLY INCREASING $F$-CONTRACTIONS ON ORDERED PARTIAL METRIC SPACES

MUHAMMAD NAZAM[1], ISMAT BEG[2,*], MUHAMMAD ARSHAD[3]

[1]Department of Mathematics, Allama Iqbal Open University, Islamabad, Pakistan
[2]Department of Mathematics, Lahore School of Economics, Lahore, Pakistan
[3]Department of Mathematics, International Islamic University, Islamabad, Pakistan

**Abstract.** In this paper, we introduce weakly increasing $F$-contractions on an ordered partial metric space and establish a common fixed point theorem for weakly increasing $F$-contractions. This result generalizes some recent results on $F$-contractions. We give an example showing the significance of our main theorem. We also show that our main result can be applied to the existence of solutions of an implicit integral equation.

**Keywords.** Common fixed point; Partial metric space; Weakly increasing $F$-contraction.

**2010 Mathematics Subject Classification.** 47H09, 47H10, 54H25.

## 1. INTRODUCTION

The well known Banach Contraction Principle has many fruitful generalizations in various directions. One of these generalizations is for $F$-contraction presented by Wardowski [1] that every $F$-contraction defined on a complete metric space has a unique fixed point. The concept of an $F$-contraction has been proved to be a milestone in fixed point theory. Several research papers on $F$-contractions have been published (see, for instance, [2, 3, 4, 5, 6]). Recently, Cosentino and Verto [7] established a fixed point result for Hardy-Rogers type $F$-contraction and Minak, Helvaci and Altun [8] presented a fixed point result for Ćirić type generalized $F$-contraction. Partial metric spaces, which were introduced by Mathews [9] and Romaguera [10], obtained a characterization of completeness in partial metric spaces. Recently several researchers [11, 12, 13, 14, 15] studied and obtained the Banach fixed point and its generalizations in partial metric spaces. Here we continue this study for weakly increasing $F$-contraction defined on an ordered partial metric space. In particular, we present a common fixed point result on a complete ordered partial metric space. It is remarked that the notion of an $F$-contraction in partial metric spaces is more

*Corresponding author.

E-mail addresses: muhammad.nazam@aiou.edu.pk (M. Nazam), ibeg@lahoreschool.edu.pk (I. Beg), marshadzia@iiu.edu.pk (M. Arshad).

Received November 14, 2018; Accepted February 6, 2019.

general than that in metric spaces. An example is constructed to illustrate our result and to show that our result generalizes result established by Durmaz, Minak and Altun [4]. As an application we apply the obtained result to show the existence of solutions of implicit type integral equations.

## 2. PRELIMINARIES

We denote the set of natural numbers, $(-\infty,+\infty)$, $(0,+\infty)$ and $[0,+\infty)$ by $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ respectively.

Matthews generalized the notion of metric $d$ as follows:

**Definition 2.1.** [9] Let $M$ be any non-empty set and $p : M\times M \to [0,\infty)$ satisfy

$(p_1)$ $x=y \Leftrightarrow p(x,x)=p(x,y)=p(y,y)$,
$(p_2)$ $p(x,x)\leq p(x,y)$,
$(p_3)$ $p(x,y)=p(y,x)$,
$(p_4)$ $p(x,z)\leq p(x,y)+p(y,z)-p(y,y)$

for all $x,y,z\in M$. The mapping $p$ is called a partial metric on $M$ and pair $(M,p)$ a partial metric space.

**Example 2.2.** Define $p:\mathbb{R}_0^+\times\mathbb{R}_0^+\to\mathbb{R}_0^+$ by $p(x,y)=\max\{x,y\}$. It is easy to check that $p$ satisfies $(p_1)-(p_4)$ and hence $p$ is a partial metric on $\mathbb{R}_0^+$. Note that $p$ does not define a metric on $\mathbb{R}_0^+$. Indeed $p(x,x)=x>0$ for all $x>0$.

Example 2.2 is a classical example of a partial metric. The following is a new and nontrivial example of a partial metric.

**Example 2.3.** Let the set of rational numbers be $\mathbb{Q}=\{x,y,\cdots\}$. We define $p:\mathbb{R}\times\mathbb{R}\to\mathbb{R}^+$ by

$$p(r,s)=\begin{cases} 1, & \text{if } r=s\in\mathbb{R}-\mathbb{Q};\\ \frac{3}{2}, & \text{if } r\neq s\in\mathbb{R}-\mathbb{Q};\\ \frac{1}{3}, & \text{if } r=s\in\mathbb{Q};\\ 1+\frac{1}{m}+\frac{1}{n}, & \text{if } r=r_m, s=r_n \text{ and } m\neq n;\\ 1+\frac{1}{n}, & \text{if } \{r,s\}\cap\mathbb{Q}=\{r_n\} \text{ and } \{r,s\}-\mathbb{Q}\neq\phi. \end{cases}$$

It is easy to see that $p$ is a partial metric but not a metric on $\mathbb{R}$.

A metric on a set $M$ is a partial metric $p$ such that $p(x,x)=0$ for all $x\in M$ and $p(x,y)=0$ implies $x=y$ (using $(p_1)$ and $(p_2)$) but the converse may not be true. The self distance $p(x,x)$ referred to as the size or weight of $x$, is a feature used to describe the amount of information contained in $x$.

Matthews [9] explored the following aspects of a partial metric $p$ on $M$:

(1) The function $d: M\times M\to\mathbb{R}_0^+$ defined by

$$d(x,y)=2p(x,y)-p(x,x)-p(y,y) \tag{2.1}$$

for all $x,y\in M$ defines a metric on $M$ (called induced metric).

(2) Open ball has the structure $B_p(r,\varepsilon)=\{x\in M: p(r,x)<p(r,r)+\varepsilon\}$ for all $r\in M$ and $\varepsilon>0$.

(3) A partial metric $p$ on $M$ generates a $T_0$ topology $\mathscr{T}[p]$ on $M$. The base of topology $\mathscr{T}[p]$ consists of family of open balls $\{B_p(r,\varepsilon) : r \in M,\ \varepsilon > 0\}$.
(4) A sequence $\{r_n\}_{n\in\mathbb{N}}$ in $(M,p)$ converges to a point $r \in M$ if and only if

$$p(r,r) = \lim_{n\to\infty} p(r,r_n).$$

(5) A sequence $\{r_n\}_{n\in\mathbb{N}}$ in $(M,p)$ is called a Cauchy sequence if $\lim_{n,m\to\infty} p(r_n,r_m)$ exists and is finite.
(6) A partial metric space $(M,p)$ is said to be complete if every Cauchy sequence $\{r_n\}_{n\in\mathbb{N}}$ in $M$ converges, with respect to $\mathscr{T}[p]$, to a point $r \in M$ such that

$$p(r,r) = \lim_{n,m\to\infty} p(r_n,r_m).$$

Matthews [9] also established an analogue of the Banach's fixed point theorem in partial metric spaces. This remarkable theorem led numerous authors to obtain various applicable fixed point results in partial metric spaces (see, for example, [2, 13, 16, 17, 18, 19, 20] and references therein).

The following is due to [9].

**Lemma 2.4.** [9]

(1) A sequence $\{r_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in a partial metric space $(M,p)$ if and only if it is a Cauchy sequence in a metric space $(M,d)$.
(2) A partial metric space $(M,p)$ is complete if and only if $(M,d)$ is complete.
(3) A sequence $\{r_n\}_{n\in\mathbb{N}}$ in $M$ converges to a point $r \in M$, with respect to $\tau(d)$ if and only if $\lim_{n\to\infty} p(r,r_n) = p(r,r) = \lim_{n,m\to\infty} p(r_n,r_m)$.

**Definition 2.5.** [21] Let $(M,\preceq)$ be an ordered set and $p$ be a partial metric on $M$. Then the triplet $(M,\preceq,p)$ is known as an ordered partial metric space. If $(M,p)$ is complete, then $(M,\preceq,p)$ is called an ordered complete partial metric space.

**Definition 2.6.** [1] Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a function satisfying the following conditions

($F_1$) F is strictly increasing;
($F_2$) For each sequence $\{r_n\}$ of positive numbers, the following condition holds:

$$\lim_{n\to\infty} r_n = 0 \text{ if and only if } \lim_{n\to\infty} F(r_n) = -\infty;$$

($F_3$) There exists $\theta \in (0,1)$ such that $\lim_{\alpha\to 0^+} (\alpha)^\theta F(\alpha) = 0$.

We denote the set of all functions satisfying the conditions $(F_1)-(F_3)$ by $\Delta_F$.

**Example 2.7.** Let $F : \mathbb{R}^+ \to \mathbb{R}$ be defined by:

(a) $F(r) = \ln(r)$;
(b) $F(r) = r + \ln(r)$;
(c) $F(r) = \ln(r^2 + r)$;
(d) $F(r) = -\dfrac{1}{\sqrt{r}}$.

It is easy to check that (a), (b), (c) and (d) are members of $\Delta_F$.

**Definition 2.8.** [1] Let $(M,d)$ be a metric space. We say the mapping $T:M\to M$ an $F_d$-contraction, if there exist $F\in\Delta_F$ and $\tau>0$ such that

$$(d(T(x),T(y))>0,$$
$$\Rightarrow\ \tau+F(d(T(x),T(y))\leq F(d(x,y))),\ \text{for all } x,y\in M.$$

**Definition 2.9.** Let $(M,p)$ be a partial metric space. We say the mapping $T:M\to M$ an $F_p$-contraction, if there exist $F\in\Delta_F$ and $\tau>0$ such that

$$(p(T(x),T(y))>0,$$
$$\Rightarrow\ \tau+F(p(T(x),T(y))\leq F(p(x,y))),\ \text{for all } x,y\in M.$$

Following example explains that an $F_p$-contraction is more general than an $F_d$-contraction.

**Example 2.10.** Let $M=[0,1]$ and define partial metric by $p(x,y)=\max\{x,y\}$ for all $x,y\in M$. The metric $d$ induced by partial metric $p$ is given by $d(x,y)=|x-y|$ for all $x,y\in M$. Define $F:\mathbb{R}^+\to\mathbb{R}$ by $F(r)=\ln(r)$ and $T$ by

$$T(r)=\begin{cases}\dfrac{r}{5}, & \text{if } r\in[0,1);\\[2ex] 0, & \text{if } r=1.\end{cases}$$

Note that, for all $x,y\in M$ with $x\leq y$ or $y\leq x$, $T$ is an $F_p$-contraction.

$$\tau+F\left(p(T(x),T(y))\right)\leq F\left(p(x,y)\right),$$
$$\Rightarrow\tau+F\left(\frac{x}{5}\right)\leq F(x)\ \text{or}\ \tau+F\left(\frac{y}{5}\right)\leq F(y).$$

But $T$ is not an $F_d$-contraction. Indeed, for $x=1$ and $y=\frac{5}{6}$, $d(T(x),T(y))>0$, we have

$$\begin{aligned}\tau+F\left(d(T(x),T(y))\right) &\leq F\left(d(x,y)\right),\\ \tau+F\left(d\left(T(1),T\left(\frac{5}{6}\right)\right)\right) &\leq F\left(d\left(1,\frac{5}{6}\right)\right),\\ \tau+F\left(d\left(0,\frac{1}{6}\right)\right) &\leq F\left(\frac{1}{6}\right),\\ \frac{1}{6} &< \frac{1}{6},\end{aligned}$$

which is a contradiction for all possible values of $\tau$.

Wardowski [1] proved the following theorem using $F$-contraction.

**Theorem 2.11.** [1] *Let $T:M\to M$ be an $F_d$-contraction on a complete metric space $(M,d)$. The mapping $T$ has a unique fixed point $\upsilon\in M$ and for every $r_0\in M$ the sequence $\{T^n(r_0)\}_{n\in\mathbb{N}}$ is convergent to $\upsilon$.*

Recently, Durmaz, Minak and Altun [4] obtained an ordered version of Theorem 2.11

**Theorem 2.12.** [4] *Let $T:M\to M$ be an ordered $F_d$-contraction on an ordered complete metric space $(M,\preceq,d)$. If $T$ is a nondecreasing mapping and there exists $r_0\in M$ such that $r_0\preceq T(r_0)$ and $T$ is also continuous or $M$ is regular, then $T$ has a fixed point.*

**Definition 2.13.** Let $(M,\preceq)$ be a partially ordered set. Two mappings $S,T: M\to M$ are said to be weakly increasing mappings, if $S(r)\preceq TS(r)$ and $T(r)\preceq ST(r)$ hold for all $r\in M$.

**Example 2.14.** Let $M=\mathbb{R}^+$ be endowed with usual order and usual topology. Define $S,T: M\to M$ by

$$S(r)=\begin{cases} r^{\frac{1}{2}}, & \text{if } r\in[0,1],\\ r^2, & \text{if } r\in(1,\infty),\end{cases} \quad \text{and } T(r)=\begin{cases} r, & \text{if } r\in[0,1],\\ 2r, & \text{if } r\in(1,\infty).\end{cases}$$

Then, $S,T$ are weakly increasing mappings.

## 3. Fixed points

Let

$$\gamma=\{(\alpha,\beta)\in M\times M: S(\alpha)\preceq TS(\alpha) \text{ and } T(\beta)\preceq ST(\beta)\}.$$

**Definition 3.1.** The space $(M,\preceq,p)$ is said to be a $\gamma$-regular space, if the following condition hold:

$$\begin{cases} \text{If } \{r_n\}\subset M \text{ is a nondecreasing (nonincreasing) sequence with } r_n\to r,\\ \text{then } (r_n,r)\in\gamma \ ((r,r_n)\in\gamma) \text{ for all } n\end{cases}$$

**Definition 3.2.** Let $(M,\preceq,p)$ be an ordered partial metric space and $S,T: M\to M$ be two self-mappings. We say that $S,T$ are weakly increasing $F$-contractions, if there exists $F\in\Delta_F$ and $\tau>0$ such that

$$\tau+F(p(S(\alpha),T(\beta))\le F(\mathscr{M}(\alpha,\beta)) \text{ for all } (\alpha,\beta)\in\gamma, \tag{3.1}$$

where

$$\mathscr{M}(\alpha,\beta)=\max\left\{ p(\alpha,\beta), p(\alpha,S(\alpha)), p(\beta,T(\beta)), \quad \frac{p(\beta,S(\alpha))+p(\alpha,T(\beta))}{2}\right\}.$$

The following lemma will be used later.

**Lemma 3.3.** *Let $(M,\preceq,p)$ be an ordered partial metric space and $S,T$ be weakly increasing $F$-contractions. If there exists $r_0\in M$ such that $r_0\preceq S(r_0)$, then $p(r_{2i},r_{2i+1})=0$ implies $p(r_{2i+1},r_{2i+2})=0$ for all $i=0,1,2,3,\ldots$.*

*Proof.* Let $r_0\in M$ be an initial point and take $x=S(r_0)$ and $y=T(x)$. By induction we can construct an iterative sequence $r_n$ of points in $M$ such that $r_{2i+1}=S(r_{2i})$ and $r_{2i+2}=T(r_{2i+1})$, where $i=0,1,2,\ldots$. As there exists $r_0\in M$ such that $r_0\preceq S(r_0)$ and $S,T$ are weakly increasing self-mappings, we obtain

$$x=S(r_0)\preceq TS(r_0)=T(x)=y=T(x)\preceq ST(x)=S(y)=z.$$

Iteratively, we obtain

$$r_0\preceq x\preceq y\preceq\cdots\preceq r_{n-1}\preceq r_n\preceq r_{n+1}\preceq\cdots.$$

We argue by contradiction that $p(r_{2i+1},r_{2i+2})>0$. Note that

$$\begin{aligned}\mathscr{M}(r_{2i},r_{2i+1}) &= \max\left\{\begin{array}{l} p(r_{2i},r_{2i+1}),p(r_{2i},S(r_{2i})),p(r_{2i+1},T(r_{2i+1})),\\ \\ \dfrac{p(r_{2i+1},S(r_{2i}))+p(r_{2i},T(r_{2i+1}))}{2}\end{array}\right\}\\ &= \max\left\{\begin{array}{l} p(r_{2i},r_{2i+1}),p(r_{2i},r_{2i+1}),p(r_{2i+1},r_{2i+2}),\\ \\ \dfrac{p(r_{2i+1},r_{2i+1})+p(r_{2i},r_{2i+2})}{2}\end{array}\right\}\\ &= \max\{0,p(r_{2i+1},r_{2i+2})\}=p(r_{2i+1},r_{2i+2}).\end{aligned}$$

Since, $r_{2i}\preceq r_{2i+1}$, from inequality (3.1), we have

$$\begin{aligned}\tau+F\left(p(r_{2i+1},r_{2i+2})\right) &= \tau+F\left(p(S(r_{2i}),T(r_{2i+1}))\right)\\ &\leq F\left(\mathscr{M}(r_{2i},r_{2i+1})\right)\\ &\leq F\left(p(r_{2i+1},r_{2i+2})\right)\end{aligned}$$

for all $i\in\mathbb{W}$, which gives a contradiction to $(F_1)$. Hence, $p(r_{2i+1},r_{2i+2}))=0$. □

Next we present our main result.

**Theorem 3.4.** *Let $(M,\preceq,p)$ be an ordered complete partial metric space and $S,T:M\to M$ be weakly increasing F-contractions. If there exists $r_0\in M$ such that $r_0\preceq S(r_0)$ and either*

(a) *one of $S,T$ is continuous or*
(b) *$M$ is $\gamma$-regular,*

*then $S,T$ have a common fixed point.*

*Proof.* (a) We begin with the following observation:

$$\mathscr{M}(r_{2i},r_{2i+1})=0 \text{ if and only if } r_{2i}=r_{2i+1} \text{ is a common fixed point of } S,T.$$

Let $\mathscr{M}(r_{2i},r_{2i+1})>0$ for all $i\in\mathbb{W}$. Arguing as in Lemma 3.3, we have

$$r_0\preceq x\preceq y\preceq\cdots\preceq r_{n-1}\preceq r_n\preceq r_{n+1}\preceq\cdots.$$

If $p(S(r_{2i}),T(r_{2i+1}))=0$, using Lemma 3.3, we can conclude that $r_{2i}$ is a common fixed point of $S,T$. Let $p(S(r_{2i}),T(r_{2i+1}))>0$. Since $(r_{2i},r_{2i+1})\in\gamma$, by contractive condition (3.1), we get

$$\begin{aligned}\tau+F\left(p(r_{2i+1},r_{2i+2})\right)&=\tau+F\left(p(S(r_{2i}),T(r_{2i+1}))\right)\\ &\leq F\left(\mathscr{M}(r_{2i},r_{2i+1})\right),\end{aligned}\tag{3.2}$$

for all $i\in\mathbb{W}$, where

$$\begin{aligned}\mathscr{M}(r_{2i},r_{2i+1})&=\max\left\{\begin{array}{l} p(r_{2i},r_{2i+1}),p(r_{2i},S(r_{2i})),p(r_{2i+1},T(r_{2i+1})),\\ \\ \dfrac{p(r_{2i+1},S(r_{2i}))+p(r_{2i},T(r_{2i+1}))}{2}\end{array}\right\}\\ &=\max\{p(r_{2i},r_{2i+1}),p(r_{2i+1},r_{2i+2})\}.\end{aligned}$$

If $\mathscr{M}(r_{2i},r_{2i+1})=p(r_{2i+1},r_{2i+2})$, by $(F_1)$ and (3.2), we get a contradiction. Thus, for $\mathscr{M}(r_{2i},r_{2i+1})=p(r_{2i},r_{2i+1})$, we have

$$F\left(p(r_{2i+1},r_{2i+2})\right)\leq F\left(p(r_{2i},r_{2i+1})\right)-\tau, \tag{3.3}$$

for all $i\in\mathbb{W}$. Let $p(S(r_{2i+2}),T(r_{2i+1}))>0$. Otherwise, by Lemma 3.3, $r_{2i+1}$ is a common fixed point of $S,T$. As $(r_{2i+1},r_{2i+2})\in\gamma$ and

$$\begin{aligned}\mathscr{M}(r_{2i+2},r_{2i+1})&=\max\left\{\begin{array}{c}p(r_{2i+2},r_{2i+1}),p(r_{2i+2},S(r_{2i+2})),p(r_{2i+1},T(r_{2i+1})),\\[2ex] \dfrac{p(r_{2i+1},S(r_{2i+2}))+p(r_{2i+2},T(r_{2i+1}))}{2}\end{array}\right\}\\ &=\max\left\{\begin{array}{c}p(r_{2i+2},r_{2i+1}),p(r_{2i+2},r_{2i+3}),p(r_{2i+1},r_{2i+2}),\\[2ex] \dfrac{p(r_{2i+1},r_{2i+3})+p(r_{2i+2},r_{2i+2})}{2}\end{array}\right\}\\ &=\max\left\{p(r_{2i+2},r_{2i+1}),p(r_{2i+2},r_{2i+3})\right\}.\end{aligned}$$

If

$$\mathscr{M}(r_{2i+2},r_{2i+1})=p(r_{2i+2},r_{2i+3}),$$

then the contractive condition (3.1) leads to a contradiction to $(F_1)$. Thus, contractive condition (3.1) implies

$$F\left(p(r_{2i+2},r_{2i+3})\right)\leq F\left(p(r_{2i+1},r_{2i+2})\right)-\tau,\ \text{for all } i\in\mathbb{W}. \tag{3.4}$$

By (3.3) and (3.4), we have

$$F\left(p(r_{n+1},r_{n+2})\right)\leq F\left(p(r_{n},r_{n+1})\right)-\tau,\ \text{for all } n\in\mathbb{W}. \tag{3.5}$$

By (3.5), we obtain

$$F\left(p(r_{n},r_{n+1})\right)\leq F\left(p(r_{n-2},r_{n-1})\right)-2\tau.$$

Repeating these steps, we get

$$F\left(p(r_{n},r_{n+1})\right)\leq F\left(p(r_{0},x)\right)-n\tau. \tag{3.6}$$

By (3.6), we obtain $\lim_{n\to\infty}F\left(p(r_n,r_{n+1})\right)=-\infty$ and $(F_2)$ which imply

$$\lim_{n\to\infty}p(r_n,r_{n+1})=0. \tag{3.7}$$

By the property $(F_3)$, there exists $\kappa\in(0,1)$ such that

$$\lim_{n\to\infty}\left(\left(p(r_n,r_{n+1})\right)^{\kappa}F\left(p(r_n,r_{n+1})\right)\right)=0. \tag{3.8}$$

Multiplying (3.6) by $\left(p(r_n,r_{n+1})\right)^{\kappa}$, we obtain

$$\left(p(r_n,r_{n+1})\right)^{\kappa}\left(F\left(p(r_n,r_{n+1})\right)-F\left(p(r_0,x_1)\right)\right)\leq-\left(p(r_n,r_{n+1})\right)^{\kappa}n\tau\leq 0. \tag{3.9}$$

By (3.7) and (3.8) and letting $n\to\infty$ in (3.9), we have

$$\lim_{n\to\infty}\left(n\left(p(r_n,r_{n+1})\right)^{\kappa}\right)=0.$$

There exists $n_1\in\mathbb{N}$, such that $n\left(p(r_n,r_{n+1})\right)^{\kappa}\leq 1$ for all $n\geq n_1$ or,

$$p(r_n,r_{n+1})\leq\frac{1}{n^{\frac{1}{\kappa}}}\ \text{for all } n\geq n_1. \tag{3.10}$$

By (3.10), we get, for $m > n \geq n_1$,

$$\begin{aligned} p(r_n, r_m) &\leq p(r_n, r_{n+1}) + p(r_{n+1}, r_{n+2}) + p(r_{n+2}, r_{n+3}) + \ldots + p(r_{m-1}, r_m) \\ &- \sum_{j=n+1}^{m-1} p(r_j, r_j) \\ &\leq p(r_n, r_{n+1}) + p(r_{n+1}, r_{n+2}) + p(r_{n+2}, r_{n+3}) + \ldots + p(r_{m-1}, r_m) \\ &= \sum_{i=n}^{m-1} p(r_i, r_{i+1}) \leq \sum_{i=n}^{\infty} p(r_i, r_{i+1}) \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

The convergence of the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{\kappa}}}$ entails $\lim_{n,m\to\infty} p(r_n, r_m) = 0$. Hence $\{r_n\}$ is a Cauchy sequence in $(M, p)$. By Lemma 2.4 (1), $\{r_n\}$ is a Cauchy sequence in $(M, d)$. Since $(M, p)$ is a complete partial metric space, one has that $(M, d)$ is a complete metric space. As a result, there exists $\upsilon \in M$ such that $\lim_{n\to\infty} d(r_n, \upsilon) = 0$. By Lemma 2.4 (3),

$$\lim_{n\to\infty} p(\upsilon, r_n) = p(\upsilon, \upsilon) = \lim_{n,m\to\infty} p(r_n, r_m). \tag{3.11}$$

Since $\lim_{n,m\to\infty} p(r_n, r_m) = 0$, by (3.11) we have

$$p(\upsilon, \upsilon) = 0 = \lim_{n\to\infty} p(\upsilon, r_n). \tag{3.12}$$

The equation (3.12) implies $r_{2n+1} \to \upsilon$ and $r_{2n+2} \to \upsilon$ as $n \to \infty$ with respect to $\tau(p)$. Suppose that $T$ is continuous. Then

$$\upsilon = \lim_{n\to\infty} r_n = \lim_{n\to\infty} r_{2n+1} = \lim_{n\to\infty} r_{2n+2} = \lim_{n\to\infty} T(r_{2n+1}) = T(\lim_{n\to\infty} r_{2n+1}) = T(\upsilon).$$

Now we show that $\upsilon = S(\upsilon)$. Suppose on contrary that $p(\upsilon, S(\upsilon)) > 0$. Since, $(\upsilon, \upsilon) \in \gamma$, by (3.1), we obtain

$$\begin{aligned} \tau + F(p(\upsilon, S(\upsilon))) &= \tau + F(p(S(\upsilon), T(\upsilon))) \\ &\leq F(\mathscr{M}(\upsilon, \upsilon)) \\ F(p(\upsilon, S(\upsilon))) &< F(p(\upsilon, \upsilon)), \end{aligned}$$

that is a contradiction. Thus, $p(\upsilon, S(\upsilon)) = 0$. Due to $(p_1)$, $(p_2)$, we conclude that $\upsilon = S(\upsilon)$. Consequently, we have $S(\upsilon) = T(\upsilon) = \upsilon$, that is, $(S, T)$ have a common fixed point $\upsilon$.

(b) In the other case, using the assumption that $M$ is $\gamma$-regular, we have that $(r_n, \upsilon) \in \gamma$ for all $n \in \mathbb{N}$. To show that $\upsilon$ is a common fixed point of $S, T$, we split the proof into two parts.

(1) If $r_n = \upsilon$ for some $n$, then there exists $i_0 \in \mathbb{N}$ such that $r_{2i_0} = \upsilon$. Consider, $S(\upsilon) = S(r_{2i_0}) = r_{2i_0+1} \preceq \upsilon$ also $\upsilon = r_{2i_0} \preceq r_{2i_0+1} = S(\upsilon)$. Thus, $\upsilon = S(\upsilon)$ and by (3.1) we have $\upsilon = T(\upsilon)$

(2) If $r_n \neq \upsilon$ for all $n$, then $p(\upsilon, S(\upsilon)) > 0$. Since $\lim_{n\to\infty} r_{2i} = \upsilon$, there exists $\mathscr{N} \in \mathbb{N}$ such that

$$p(r_{2i+1}, S(\upsilon)) > 0 \text{ and } p(r_{2i}, \upsilon) < \frac{p(\upsilon, S(\upsilon))}{2} \text{ for all } i \geq \mathscr{N}.$$

$$\mathscr{M}(r_{2i}, \upsilon) = \max\left\{ p(r_{2i}, \upsilon), p(r_{2i}, S(r_{2i})), p(\upsilon, T(\upsilon)), \frac{p(\upsilon, S(r_{2i})) + p(r_{2i}, T(\upsilon))}{2} \right\},$$

$$\mathscr{M}(r_{2i}, \upsilon) \leq \frac{p(\upsilon, S(\upsilon))}{2} \text{ for all } i \geq \mathscr{N}.$$

As $(r_{2i}, \upsilon) \in \gamma$, by (3.1), we have

$$\begin{aligned}\tau + F(p(r_{2i+1}, S(\upsilon))) &= \tau + F(p(S(r_{2i}), T(\upsilon))) \\ &\leq F(\mathcal{M}(r_{2i}, \upsilon)),\end{aligned}$$

$$F(p(\upsilon, S(\upsilon))) < F(\frac{p(\upsilon, S(\upsilon))}{2}) \text{ as } i \to \infty,$$

that is a contradiction. Therefore, $p(\upsilon, S(\upsilon)) = 0$. Due to $(p_1)$, $(p_2)$, we conclude that $\upsilon = S(\upsilon)$. From (3.1), we have $\upsilon = T(\upsilon)$. Thus, $(S, T)$ have a common fixed point $\upsilon$. □

We denote set of common fixed points of $S, T$ by $\mathrm{Fix}(S, T)$.

**Remark 3.5.** If we assume that $\mathrm{Fix}(S, T)$ in Theorem 3.4 is a chain along with existing conditions, then it is singleton set (common fixed point is unique).

*Proof.* If $\omega$ is another common fixed point of $S, T$, then $\omega \preceq \upsilon$, also $p(S(\upsilon), T(\omega)) > 0$ (otherwise $\upsilon = \omega$) so, $(\upsilon, \omega) \in \gamma$. By the contractive condition (3.1), we have

$$\begin{aligned}\tau + F(p(\upsilon, \omega)) &= \tau + F(p(S(\upsilon), T(\omega))) \\ &\leq F\left(\mathcal{M}(\upsilon, \omega)\right), \end{aligned} \tag{3.13}$$

where

$$\begin{aligned}\mathcal{M}(\upsilon, \omega) &= \max\left\{p(\upsilon, \omega), p(\upsilon, S(\upsilon)), p(\omega, T(\omega)), \frac{p(\omega, S(\upsilon)) + p(\upsilon, T(\omega))}{2}\right\} \\ &= p(\upsilon, \omega)\end{aligned}$$

By (3.13), we have

$$F(p(\upsilon, \omega)) < F\left(p(\upsilon, \omega)\right), \tag{3.14}$$

which leads to a contradiction. Hence, $\upsilon = \omega$ and $\upsilon$ is a unique common fixed point of a pair $(S, T)$. □

**Remark 3.6.** If $\mathrm{Fix}(S, T)$ is not a chain and there exists $z$ in $M$ such that every element in the orbit $O_T(z) = \{z, T(z), T^2(z), \ldots\}$ is comparable to $\upsilon, \omega$. Then $\upsilon = \omega$ ($\upsilon$ is unique) provided that $S$ and $T$ are weakly increasing F-contractions.

*Proof.* Assume that $\upsilon, \omega$ are in $\mathrm{Fix}(S, T)$ and there exists an element $z \in M$ such that every element of $O_T(z) = \{z, T(z), T^2(z), \ldots\}$ is comparable to $\upsilon, \omega$. Hence $(T^{n-1}(z), S^{n-1}(\upsilon))$ and $(T^{n-1}(z), S^{n-1}(\omega))$ are elements of $\gamma$ for each $n \geq 1$. By (3.1), we have

$$\begin{aligned}\tau + F(p(\upsilon, T^n(z))) &= \tau + F(p(S^n(\upsilon), T^n(z)) \\ &\leq F(\mathcal{M}\left(S^{n-1}(\upsilon), T^{n-1}(z)\right)), \end{aligned} \tag{3.15}$$

where

$$\begin{aligned}\mathcal{M}\left(S^{n-1}(\upsilon), T^{n-1}(z)\right) &= \max\left\{\begin{array}{l} p\left(S^{n-1}(\upsilon), T^{n-1}(z)\right), p\left(S^{n-1}(\upsilon), S^{n}(\upsilon)\right), \\ p\left(T^{n-1}(z), T^{n}(z)\right), \\ \dfrac{p\left(T^{n-1}(z), S^{n}(\upsilon)\right) + p\left(S^{n-1}(\upsilon), T^{n}(z)\right)}{2} \end{array}\right\} \\ &= p\left(S^{n-1}(\upsilon), T^{n-1}(z)\right) = p\left(\upsilon, T^{n-1}(z)\right).\end{aligned}$$

Thus, by (3.15), we deduce that $\{p(\upsilon, T^n(z))\}$ is non-negative decreasing sequence which in turn converges to 0. Similarly, we can show that $\{p(\omega, T^n(z))\}$ is non-negative decreasing sequence converges to 0. Consequently, $\upsilon = \omega$. □

Below example illustrates Theorem 3.4 and shows that condition (3.1) is more general than contractivity condition used by Durmaz *et al.* ([4]).

**Example 3.7.** Let $M = [0,1]$ and define $p(x,y) = \max\{x,y\}$. Let $\prec_1$ be defined by $x \prec_1 y$ if and only if $y \leq x$ for all $x, y \in M$. Then $x \prec_1 y$ is a partial order on $M$ and $(M, \prec_1, p)$ is a complete ordered partial metric space. As before, define $d(x,y) = |x-y|$ so that $(M, \prec_1, d)$ is a complete ordered metric space. Define the mappings $S, T : M \to M$ as follows

$$T(r) = \begin{cases} \frac{r}{5}, & \text{if } r \in [0,1); \\ 0, & \text{if } r = 1, \end{cases} \quad \text{and } S(r) = \frac{3r}{7} \text{ for all } r \in M.$$

Define

$$\gamma_1 = \{(\alpha,\beta) \in M \times M : S(\alpha) \prec_1 TS(\alpha) \text{ and } T(\beta) \prec_1 ST(\beta)\}.$$

Then clearly $S, T$ are weakly increasing self-mappings with respect to $\prec_1$. Define the function $F : R^+ \to R$ by $F(r) = \ln(r)$, for all $r \in R^+ > 0$. Let $x, y \in M$ such that $p(S(x), T(y)) > 0$ and suppose that $y \prec_1 x$. Then

$$\mathscr{M}(x,y) = \max\left\{ y, \frac{xy}{1+x}, \frac{xy}{1+\max\left\{\frac{3x}{7}, \frac{y}{5}\right\}} \right\}.$$

Since $\frac{x}{1+x} < 1$ and

$$\frac{x}{1+\max\left\{\frac{3x}{7}, \frac{y}{5}\right\}} < 1,$$

we have that $\mathscr{M}(x,y) = y$. In a similar way, if $x \prec_1 y$, we obtain that $\mathscr{M}(x,y) = x$, i.e., $\mathscr{M}(x,y) = p(x,y)$. Let $\tau \leq \ln(\frac{7}{3})$. Since, $(x,y) \in \gamma_1$, it further implies that the contractive condition (3.1) is satisfied for all $x, y \in M$. Thus all the hypotheses of the Theorem 3.4 are satisfied. Pair of mapping $(S,T)$ have a unique common fixed point $r = 0$. Note that $T$ is not an $F_d$-contraction in $(M, \prec_1, d)$. Thus, we can not apply Theorem 2.11. Hence Theorem 2.12.

Let

$$\xi = \left\{(\alpha,\beta) \in M \times M : T(\alpha) \preceq T^2(\alpha) \text{ and } T(\beta) \preceq T^2(\beta)\right\}.$$

**Definition 3.8.** The space $(M, \preceq, p)$ is said to be a $\xi$-regular space, if the following condition hold:

$$\begin{cases} \text{If } \{r_n\} \subset M \text{ is a nondecreasing (nonincreasing) sequence with } r_n \to r, \\ \text{then } (r_n, r) \in \xi \ ((r, r_n) \in \xi) \text{ for all } n \end{cases}$$

**Definition 3.9.** Let $(M, \preceq, p)$ be an ordered partial metric space and $T : M \to M$ be a self-mapping. We say that $T$ is a weakly increasing $F$-contraction, if there exists $F \in \Delta_F$ and $\tau > 0$ such that,

$$\tau + F(p(T(\alpha), T(\beta)) \leq F(\mathscr{M}(\alpha,\beta)) \text{ for all } (\alpha,\beta) \in \xi,$$

where

$$\mathscr{M}(\alpha,\beta) = \max\left\{ p(\alpha,\beta), p(\alpha,T(\alpha)), p(\beta,T(\beta)), \quad \frac{p(\beta,T(\alpha)) + p(\alpha,T(\beta))}{2} \right\}.$$

The following Corollary generalizes Theorem 2.12 due to Durmaz, Minak and Altun [4]

**Corollary 3.10.** *Let $(M, \preceq, p)$ be an ordered complete partial metric space and $T : M \to M$ be a weakly increasing $F$-contraction. If there exists $r_0 \in M$ such that $r_0 \preceq T(r_0)$ and either*

(a) *$T$ is continuous or*
(b) *$M$ is $\xi$-regular,*

*then $T$ has a unique fixed point.*

*Proof.* Set $S = T$ in Theorem 3.4. From the proof of Theorem 3.4, we can obtain the desired conclusion immediately. □

## 4. An Application

This section contains an existence result which shows the usefulness of Theorem 3.4 in establishing the existence of solutions of implicit type integral equation:

$$\mathscr{A}(t, u(r,t)) = \int_0^a \int_0^a \mathscr{H}(t, \theta, \phi, u(\theta, \phi))\, d\theta d\phi, \tag{4.1}$$

where

$$u \in \mathscr{U} = \mathscr{L}[C([0,a]) \times [0,a]]$$

is a Lebesgue measurable space, $t, \theta, \phi \in I_a = [0,a]$. For $u \in \mathscr{U}$, define norm as: $\|u\| = \max_{t\in[0,a]} \{|u(t)|\}$. Let $\mathscr{U}$ be endowed with the partial metric $p : \mathscr{U} \times \mathscr{U} \to \mathbb{R}_0^+$ defined by

$$p(u,v) = d(u,v) + c = \max_{t\in[0,a]} |u(r,t) - v(r,t)| + c \text{ for all } u, v \in \mathscr{U}.$$

Also, $\mathscr{U}$ can be equipped with order $\prec_2$ defined by $u \prec_2 v$ if and only if $v(r,t) \le u(r,t)$. Obviously, $(\mathscr{U}, \|\cdot\|)$ is a Banach space and $(\mathscr{U}, \prec_2, p)$ is a complete ordered partial metric space.

**Theorem 4.1.** *Assume that:*

(a) *For all $u, v \in \mathscr{U}$ and $\kappa = |u(x,t) - v(x,t)| + c$,*

$$|\mathscr{A}(t, u(x,t)) - \mathscr{A}(t, v(x,t))| + c \le (\kappa)e^{-\tau} \text{ for each } t \in I_a,$$

(b) *$\mathscr{H}(t, \theta, \phi, u(\theta, \phi)) \le \frac{1}{a^2} u(x,t)$ for all $t \in I_a$,*
(c) *for all $t, \theta, \phi \in I_a$,*

$$\mathscr{A}(t, \int_0^a \int_0^a \mathscr{H}(t, \theta, \phi, u(\theta, \phi))\, d\theta d\phi) \le \int_0^a \int_0^a \mathscr{H}(t, \theta, \phi, u(\theta, \phi))\, d\theta d\phi,$$

(d) *$\mathscr{H}(t, \theta, \phi, \mathscr{A}(\theta, u(\theta, \phi))) \ge \dfrac{1}{a^2} \mathscr{A}(t, u(x,t))$,*

*then implicit integral equation (4.1) has a solution in $\mathscr{U}$.*

*Proof.* Define

$$S(u(x,t)) = \mathscr{A}(t, u(x,t))$$

and

$$T(u(x,t)) = \int_0^a \int_0^a \mathscr{H}(t, \theta, \phi, u(\theta, \phi))\, d\theta d\phi.$$

We show that $S$, $T$ are weakly increasing mappings. Consider

$$\begin{aligned}
S(T(u(x,t))) &= \mathscr{A}(t,T(u(x,t))) \\
&= \mathscr{A}\left(t,\int_0^a\int_0^a \mathscr{H}(t,\theta,\phi,u(\theta,\phi))\,d\theta d\phi\right) \\
&\leq \int_0^a\int_0^a \mathscr{H}(t,\theta,\phi,u(\theta,\phi))\,d\theta d\phi = T(u(x,t)) \\
&\quad \text{using (c)}
\end{aligned}$$

and

$$\begin{aligned}
T(S(u(x,t))) &= \int_0^a\int_0^a \mathscr{H}(t,\theta,\phi,S(u(\theta,\phi)))\,d\theta d\phi \\
&= \int_0^a\int_0^a \mathscr{H}(t,\theta,\phi,\mathscr{A}(\theta,u(\theta,\phi)))\,d\theta d\phi \\
&\leq \frac{1}{a^2}\int_0^a\int_0^a \mathscr{A}(t,u(x,t))\,d\theta d\phi = \mathscr{A}(t,u(x,t)) \\
&\quad \text{due to (b)}
\end{aligned}$$

Thus,

$$S(T(u(x,t))) \leq T(u(x,t)),$$

and

$$T(S(u(x,t))) \leq S(u(x,t)),$$

for all $t \in I_a$. These imply that $S,T$ are weakly increasing mappings with respect to $\prec_2$.

Now we consider

$$\begin{aligned}
p(S(u),T(v)) &= \max_{t\in I_a}|S(u(x,t)) - T(v(y,t))| + c \\
&= \max_{t\in I_a}\left|\mathscr{A}(t,u(x,t)) - \int_0^a\int_0^a \mathscr{H}(t,\theta,\phi,v(\theta,\phi))\,d\theta d\phi\right| + c \\
&\leq \max_{t\in I_a}\left|\mathscr{A}(t,u(x,t)) - \frac{1}{a^2}\int_0^a\int_0^a \mathscr{A}(t,v(x,t))\,d\theta d\phi\right| + c \\
&\quad \text{(using (d) )} \\
&= \max_{t\in I_a}|\mathscr{A}(t,u(x,t)) - \mathscr{A}(t,v(x,t))| + c \\
&\leq \max_{t\in I_a}(\kappa)e^{-\tau} \text{ using (a)} \\
&\leq e^{-\tau}p(u,v).
\end{aligned}$$

Therefore

$$\tau + \ln(p(S(u),T(v))) \leq \ln(p(u,v)) \leq \ln(\mathscr{M}(u,v)).$$

By taking $F(r) = \ln(r)$, we have

$$\tau + F(p(S(u),T(v))) \leq F(\mathscr{M}(u,v)).$$

Hence by Theorem 3.4 the integral equation (4.1) has a solution in $\mathscr{L}[C([0,a])\times[0,a]]$. □

## REFERENCES


[1] D. Wardowski, Fixed point theory of a new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl. 2012 (2012) Article ID 94.

[2] T. Abdeljawad, E. Karapinar, K. Tas, Existence and uniqueness of a common fixed point on partial metric spaces, Appl. Math. Lett. 24 (2011), 1900-1904.

[3] O. Acar, I. Altun, Multivalued $F$-contractive mappings with a graph and some fixed point results, Publicationes Math. 88 (2016), 305-317.

[4] G. Durmaz, G. Minak, I. Altun, Fixed points of ordered $F$-contractions, Hacettepe J. Math. Stat. 45 (2016), 15-21.

[5] H. Piri, P. Kumam, Some fixed point theorems concerning F-contraction in complete metric spaces, Fixed Point Theory Appl. 2014 (2014), Article ID 210.

[6] S. Shukla, S. Radenovic, Some common fixed point theorems for F-contraction type mappings on o-complete partial metric spaces, J. Math. 2013 (2013), Art. ID 878730.

[7] M. Cosentino, P. Vetro, Fixed point results for $F$-contractive mappings of Hardy-Rogers-type, Filomat 28 (2014), 715-722.

[8] G. Minak, A. Helvaci, I. Altun, Círíc type generalized F-contractions on complete metric spaces and fixed point results, Filomat 28 (2014), 1143-1151.

[9] S.G. Matthews, Partial metric topology, Ann. New York Acad. Sci. 728 (1994), 183-197.

[10] S. Romaguera, A Kirk type characterization of completeness for partial metric spaces, Fixed Point Theory Appl. 2010 (2010), Article ID 493298.

[11] H. Aydi, M. Jellali, E. Karapnar, Common fixed points for generalized $\alpha-$implicit contractions in partial metric s-paces: consequences and application, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 109 (2015), 367-384.

[12] R.H. Haghi, Sh. Rezapour, N. Shahzad, Be careful on partial metric fixed point results, Topology Appl. 160 (2013), 450-454

[13] M. Nazam, M. Arshad, C. Park, Fixed point theorems for improved $\alpha$-Geraghty contractions in partial metric spaces, J. Nonlinear Sci. Appl. 9 (2016), 4436-4449.

[14] R. Pant, R. Shukla, H. K. Nashine, R. Panicker, Some new fixed point theorems in partial metric spaces with applications, J. Funct. Spaces 2017 (2017), Article ID 1072750.

[15] O. Valero, On Banach fixed point theorems for partial metric spaces, Appl. General Topology 6 (2005) 229-240

[16] M. U. Ali, T. Kamran, Multivalued F-contractions and related fixed point theorems with an application, Filomat 30 (2016), 3779-3793.

[17] I. Altun, S. Romaguera, Characterizations of partial metric completeness in terms of weakly contractive mappings having fixed point, Appl. Anal. Discrete Math. 6 (2012), 247-256.

[18] I. Altun, F. Sola, H. Simsek, Generalized contractions on partial metric spaces, Topology Appl. 157 (2010), 2778-2785.

[19] I. Beg, M.A. Ahmed, H. A. Nafadi, Fixed points of £-fuzzy mappings in ordered b-metric spaces, J. Funct. Spaces, 2018 (2018), Article ID 5650242.

[20] I. Beg, H.K. Pathak, A variant of Nadler's theorem on weak partial metric spaces with application to a homotopy result, Vietnam J. Mathematics, 46 (2018), 693-706.

[21] N. Shobkolaei, S. Sedghi, J.R. Roshan, I. Altun, Common fixed point of mappings satisfying almost generalized $(S,T)$-contractive condition in partially ordered partial metric spaces, Appl. Math. Comput. 219 (2012), 443-452.