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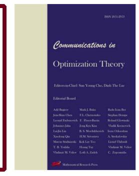

ON NABLA FRACTIONAL BOUNDARY VALUE PROBLEMS OF ORDER LESS THAN ONE

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Abstract. In this paper, we study a few qualitative properties of solutions to a nonlinear system of nabla fractional difference equations associated with periodic and nonperiodic boundary conditions. We present sufficient conditions under which solutions to the system will satisfy certain a priori bounds and admit at least one solution. The methods involve novel fractional difference inequalities and applications of the Scheafer's fixed point theorem. We illustrate the applicability of established results by example.

Keywords. A priori bound; Backward (nabla) difference; Boundary value problem; Fractional order; Existence of solution.

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1. Introduction

Over the past one decade, there has been a growing interest in the field of nabla fractional calculus. The combined efforts of a number of researchers laid a strong foundation for the theory of nabla fractional difference equations. We refer to a recent monograph [1] and the references therein for a detailed discussion on this topic. It should be noted that most of the literature on nabla fractional calculus devoted to the solvability of initial value problems. Recently, much attention has been paid to the study of boundary value problems for nabla fractional difference equations; see [2–8].

Tisdell [9–11] proposed a new approach which ensures *a priori* bounds on possible solutions to the boundary value problems (1.1), (1.2) and (1.3). These *a priori* bounds with fixed-point methods guarantee the existence of solutions to (1.1), (1.2) and (1.3).

(i) The first-order boundary value problem

$$\begin{cases} \mathbf{x}' = \mathbf{f}(t,\mathbf{x}), \quad a < t < b, \\ M\mathbf{x}(a) + R\mathbf{x}(b) = 0, \end{cases} \tag{1.1}$$

where $M, R \in \mathbb{R}$ and $\mathbf{f} \in C\big([a,b] \times \mathbb{R}^n, \mathbb{R}^n\big)$.

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(ii) The first-order discrete boundary value problem

$$\begin{cases} (\Delta \mathbf{x})(t) = \mathbf{f}(t, \mathbf{x}(t)), & t \in \mathbb{N}_0^T, \\ M\mathbf{x}(0) + R\mathbf{x}(T+1) = 0, & M + R \neq 0, \end{cases} \tag{1.2}$$

where Δ denotes the first-order forward difference operator, $T \in \mathbb{N}_1$ and $\mathbf{f} \in C\big(\mathbb{N}_0^T \times \mathbb{R}^n, \mathbb{R}^n\big)$.

(iii) The fractional boundary value problem

$$\begin{cases} D_0^\alpha\big[\mathbf{x} - \mathbf{x}(0)\big] = \mathbf{f}(t, \mathbf{x}), & 0 < t < b, \quad 0 < \alpha < 1, \\ \mathbf{M}\mathbf{x}(0) + \mathbf{R}\mathbf{x}(b) = \mathbf{c}, \end{cases} \tag{1.3}$$

where D_0^α denotes the α^{th}-order Riemann-Liouville type differential operator, $\mathbf{c}$ is a constant vector in $\mathbb{R}^n$, $\mathbf{f} \in C\big([0,b] \times \mathbb{R}^n, \mathbb{R}^n\big)$ and $\mathbf{M}$, $\mathbf{N}$ are constant matrices in $\mathbb{R}^{n \times n}$.

Also, Ferreira [12] provided criteria for the existence and uniqueness of solutions to a class of delta fractional boundary value problems of order $\alpha \in (0,1]$:

$$\begin{cases} \big({}_{\alpha-1}\Delta^\alpha u\big)(t) = g(t+\alpha-1, u(t+\alpha-1)), & t \in \mathbb{N}_0^T, \quad 0 < \alpha < 1, \\ Mu(\alpha-1) + Ru(\alpha+T) = c, \end{cases} \tag{1.4}$$

where ${}_{\alpha-1}\Delta^\alpha$ denotes the α^{th}-order Riemann-Liouville type delta difference operator, $S = [\alpha-1, \alpha-1+T] \cap \mathbb{N}_{\alpha-1}$, $g \in C\big(S \times \mathbb{R}, \mathbb{R}\big)$ and $c \in \mathbb{R}$.

Motivated by these results, we use Tisdell's approach to analyze a few qualitative properties of solutions of the following nabla fractional boundary value problem:

$$\begin{cases} \big(\nabla_{\rho(0)}^\alpha \mathbf{x}\big)(t) = \mathbf{f}(t, \mathbf{x}(t)), & t \in \mathbb{N}_1^T, \quad 0 < \alpha < 1, \\ M\mathbf{x}(0) + R\mathbf{x}(T) = 0, \end{cases} \tag{1.5}$$

where $\nabla_{\rho(0)}^\alpha$ denotes the α^{th}-order Riemann-Liouville type nabla difference operator, $T \in \mathbb{N}_1$ and $\mathbf{f} \in C\big(\mathbb{N}_0^T \times \mathbb{R}^n, \mathbb{R}^n\big)$. Observe that a solution to (1.5) is a $\mathbf{x} : \mathbb{N}_0^T \to \mathbb{R}^n$ that satisfies (1.5). It may also be equivalently denoted by $\{\mathbf{x}(t)\}_{t=0}^T \in \mathbb{R}^{(T+1)n}$.

Our article is organised as follows: In Section 2, we present preliminaries on nabla fractional calculus. Section 3 contains *a priori* bound results for solutions to (1.5). In Section 4, we apply the *a priori* bound results from Section 3 to provide new existence results for solutions to (1.5). Finally, Section 5 presents a few examples.

2. Preliminaries

Denote the set of all real numbers by $\mathbb{R}$. Define

$$\mathbb{N}_a := \{a, a+1, a+2, \dots\} \text{ and } \mathbb{N}_a^b := \{a, a+1, a+2, \dots, b\}$$

for any a, $b \in \mathbb{R}$ such that $b - a \in \mathbb{N}_1$. Assume that empty sums and products are taken to be 0 and 1, respectively.

Definition 2.1 (See [13])**.** The backward jump operator $\rho : \mathbb{N}_a \to \mathbb{N}_a$ is defined by

$$\rho(t) = \max\{a, (t-1)\}, \quad t \in \mathbb{N}_a.$$

Definition 2.2 (See [14, 15]). The Euler gamma function is defined by

$$\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}dt, \quad \Re(z) > 0.$$

Using the reduction formula

$$\Gamma(z+1) = z\Gamma(z), \quad \Re(z) > 0,$$

the Euler gamma function can be extended to the half-plane $\Re(z) \leq 0$ except for $z \neq 0, -1, -2, \ldots$

Definition 2.3 (See [1]). For $t \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$ and $r \in \mathbb{R}$ such that $(t+r) \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$, the generalized rising function is defined by

$$t^{\overline{r}} = \frac{\Gamma(t+r)}{\Gamma(t)}.$$

Also, we use the convention that if $t \in \{\ldots, -2, -1, 0\}$ and $r \in \mathbb{R}$ such that $(t+r) \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$. Then

$$t^{\overline{r}} := 0.$$

Definition 2.4 (See [13]). Let $u : \mathbb{N}_a \to \mathbb{R}$ and $N \in \mathbb{N}_1$. The first order backward (nabla) difference of u is defined by

$$\big(\nabla u\big)(t) := u(t) - u(t-1), \quad t \in \mathbb{N}_{a+1},$$

and the N^{th}-order nabla difference of u is defined recursively by

$$\big(\nabla^N u\big)(t) := \Big(\nabla\big(\nabla^{N-1}u\big)\Big)(t), \quad t \in \mathbb{N}_{a+N}.$$

Definition 2.5 (See [1]). Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $N \in \mathbb{N}_1$. The N^{th}-order nabla sum of u based at a is given by

$$\big(\nabla_a^{-N}u\big)(t) := \frac{1}{(N-1)!}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{N-1}}u(s), \quad t \in \mathbb{N}_a,$$

where by convention $\big(\nabla_a^{-N}u\big)(a) = 0$. We define $\big(\nabla_a^{-0}u\big)(t) = u(t)$ for all $t \in \mathbb{N}_{a+1}$.

Definition 2.6 (See [1]). Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $\nu > 0$. The ν^{th}-order nabla sum of u based at a is given by

$$\big(\nabla_a^{-\nu}u\big)(t) := \frac{1}{\Gamma(\nu)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\nu-1}}u(s), \quad t \in \mathbb{N}_a,$$

where by convention $\big(\nabla_a^{-\nu}u\big)(a) = 0$.

Definition 2.7 (See [1]). Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$, $\nu > 0$ and choose $N \in \mathbb{N}_1$ such that $N-1 < \nu \leq N$. The Riemann–Liouville type ν^{th}-order nabla difference of u is given by

$$\big(\nabla_a^{\nu}u\big)(t) := \Big(\nabla^N\big(\nabla_a^{-(N-\nu)}u\big)\Big)(t), \quad t \in \mathbb{N}_{a+N}.$$

Theorem 2.8 (See [1, 16]). *We observe the following properties of gamma and generalized rising functions.*

(1) $\Gamma(t) > 0$ *for all* $t > 0$.
(2) $t^{\overline{\alpha}}(t+\alpha)^{\overline{\beta}} = t^{\overline{\alpha+\beta}}$.
(3) If $t \leq r$, *then* $t^{\overline{\alpha}} \leq r^{\overline{\alpha}}$.
(4) If $\alpha < t \leq r$, *then* $r^{\overline{-\alpha}} \leq t^{\overline{-\alpha}}$.

Theorem 2.9 (See [1])**.** *Let $\nu \in \mathbb{R}^+$ and $\mu \in \mathbb{R}$ such that μ, $\mu+\nu$ and $\mu-\nu$ are nonnegative integers. Then,*

$$\nabla_a^{-\nu}(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}(t-a)^{\overline{\mu+\nu}}, \quad t \in \mathbb{N}_a,$$
$$\nabla_a^{\nu}(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\nu+1)}(t-a)^{\overline{\mu-\nu}}, \quad t \in \mathbb{N}_a.$$

Theorem 2.10 (See [17])**.** *u is a solution of the initial value problem*

$$\begin{cases} \left(\nabla_{\rho(a)}^{\nu}u\right)(t) = g(t,u(t)), \quad t \in \mathbb{N}_{a+1}, \quad 0<\nu<1, \\ \left[\left(\nabla_{\rho(a)}^{-(1-\nu)}u\right)(t)\right]_{t=a} = u(a) = u_0, \end{cases} \tag{2.1}$$

if and only if u has the representation

$$u(t) = \frac{(t-a+1)^{\overline{\nu-1}}}{\Gamma(\nu)}u_0 + \frac{1}{\Gamma(\nu)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\nu-1}}g(s,u(s)), \quad t \in \mathbb{N}_a. \tag{2.2}$$

For $\mathbf{c} \in \mathbb{R}^n$, we define the inner product as

$$\langle \mathbf{c}, \mathbf{c} \rangle := \|c\|^2,$$

where $\|\cdot\|$ is the usual Euclidean norm, that is, for $\mathbf{c} = (c_1, c_2, \cdots, c_n)^T \in R^n$,

$$\|\mathbf{c}\|^2 = c_1^2 + c_2^2 + \cdots + c_n^2.$$

For any matrix $A \in \mathbb{R}^{n\times n}$, $\|A\|$ represents any norm that is compatible with the above Euclidean norm.

Theorem 2.11 (See [18])**.** *For $\mathbf{x} : \mathbb{N}_{a+2} \to \mathbb{R}^n$ and $0<\nu<1$, the inequality involving the Riemann-Liouville type nabla difference operator holds:*

$$\nabla_a^{\nu}\left[\langle \mathbf{x}(t), \mathbf{x}(t) \rangle\right] \le 2\left\langle \mathbf{x}(t), \left(\nabla_a^{\nu}\mathbf{x}\right)(t)\right\rangle, \quad t \in \mathbb{N}_{a+2}. \tag{2.3}$$

3. A Priori Bound

First, we provide a theorem on the equivalence between (1.5) and a particular summation equation.

Theorem 3.1. *If*

$$\left[M + R\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right] \neq 0, \tag{3.1}$$

then the nabla fractional boundary value problem (1.5) is equivalent to the summation equation

$$\mathbf{x}(t) = \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{x}(s)) - \left[M + R\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right]^{-1}\frac{R(t+1)^{\overline{\alpha-1}}}{[\Gamma(\alpha)]^2}\sum_{s=1}^{T}(T-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{x}(s)), \tag{3.2}$$

for $t \in \mathbb{N}_0^T$.

Proof. An application of Theorem 2.10 to (1.5) yields

$$\mathbf{x}(t) = \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\mathbf{x}(0) + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{x}(s)), \quad t \in \mathbb{N}_0. \tag{3.3}$$

Substituting $t = T$ into (3.3), we obtain

$$\mathbf{x}(T) = \frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\mathbf{x}(0) + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{T}(T-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{x}(s)), \quad t \in \mathbb{N}_0. \tag{3.4}$$

Then, it follows from $M\mathbf{x}(0) + R\mathbf{x}(T) = 0$ that

$$M\mathbf{x}(0) + R\Big[\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\mathbf{x}(0) + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{T}(T-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{x}(s))\Big] = 0. \tag{3.5}$$

Rearranging the terms in (3.5), we obtain

$$\mathbf{x}(0) = -\Big[M + R\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\Big]^{-1}\frac{R}{\Gamma(\alpha)}\sum_{s=1}^{T}(T-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{x}(s)). \tag{3.6}$$

Substituting the value of $\mathbf{x}(0)$ from (3.6) into (3.3), we obtain (3.2). The proof is complete. □

Remark 3.2. Observe that

$$\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} = \frac{\Gamma(T+\alpha)}{\Gamma(T+1)\Gamma(\alpha)} > 0, \quad 0 < \alpha < 1, \quad T \in \mathbb{N}_1.$$

Also, from (4) of Theorem 2.8, we obtain

$$\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} < 1, \quad 0 < \alpha < 1, \quad T \in \mathbb{N}_1,$$

which implies that

$$1 + \frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} > 0, \quad 0 < \alpha < 1, \quad T \in \mathbb{N}_1,$$

and

$$1 - \frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} > 0, \quad 0 < \alpha < 1, \quad T \in \mathbb{N}_1.$$

This shows that for $(M,R) = (1,1)$ (i.e., anti-periodic boundary condition) and $(M,R) = (1,-1)$ (i.e., periodic boundary condition), (3.1) holds.

Now, we present our *a priori* bound result for (1.5).

Theorem 3.3. *Assume* (3.1) *holds. If there exist non-negative constants* β *and* L *such that*

$$\|\mathbf{f}(t,\mathbf{y})\| \le -2\beta\langle \mathbf{y}, \mathbf{f}(t,\mathbf{y})\rangle + L, \quad (t,\mathbf{y}) \in \mathbb{N}_0^T \times \mathbb{R}^n, \tag{3.7}$$

then all solutions $\mathbf{x}$ *to* (1.5) *satisfy the a priori bound*

$$\|\mathbf{x}(t)\| \le \beta\|\mathbf{x}(0)\|^2\Big[1 + |R|\Big|M + R\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\Big|^{-1}\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\Big] + \frac{LT^{\overline{\alpha}}}{\Gamma(\alpha+1)}\Big[1 + |R|\Big|M + R\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\Big|^{-1}\Big], \tag{3.8}$$

for all $t \in \mathbb{N}_1^T$.

Proof. Let $\mathbf{x}$ be a solution to (1.5) on $\mathbb{N}_0^T$ and define the function

$$r(t) := \|\mathbf{x}(t)\|^2, \quad t \in \mathbb{N}_0^T.$$

Now, by Theorem 2.11, we have

$$\nabla_{\rho(0)}^{\alpha} r(t) \leq 2\langle \mathbf{x}(t), \mathbf{f}(t,\mathbf{x}(t))\rangle, \quad t \in \mathbb{N}_1^T. \tag{3.9}$$

For $t \in \mathbb{N}_1^T$, we consider

$$\begin{aligned}
\|\mathbf{x}(t)\| &\leq \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{\alpha-1}} \|\mathbf{f}(s,\mathbf{x}(s))\| \\
&\quad + \left| M + R\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right|^{-1} \frac{|R|(t+1)^{\overline{\alpha-1}}}{[\Gamma(\alpha)]^2} \sum_{s=1}^{T} (T-\rho(s))^{\overline{\alpha-1}} \|\mathbf{f}(s,\mathbf{x}(s))\| \\
&\leq -\frac{2\beta}{\Gamma(\alpha)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{\alpha-1}} \langle \mathbf{x}(s), \mathbf{f}(s,\mathbf{x}(s))\rangle + \frac{L}{\Gamma(\alpha)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{\alpha-1}} \\
&\quad - \left| M + R\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right|^{-1} \frac{2\beta|R|(t+1)^{\overline{\alpha-1}}}{[\Gamma(\alpha)]^2} \sum_{s=1}^{T} (T-\rho(s))^{\overline{\alpha-1}} \langle \mathbf{x}(s), \mathbf{f}(s,\mathbf{x}(s))\rangle \\
&\quad + \left| M + R\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right|^{-1} \frac{L|R|(t+1)^{\overline{\alpha-1}}}{[\Gamma(\alpha)]^2} \sum_{s=1}^{T} (T-\rho(s))^{\overline{\alpha-1}} \\
&\leq -\frac{\beta}{\Gamma(\alpha)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{\alpha-1}} \nabla_{\rho(0)}^{\alpha} r(s) + \frac{L}{\Gamma(\alpha+1)} t^{\overline{\alpha}} \\
&\quad - \left| M + R\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right|^{-1} \frac{\beta|R|(t+1)^{\overline{\alpha-1}}}{[\Gamma(\alpha)]^2} \sum_{s=1}^{T} (T-\rho(s))^{\overline{\alpha-1}} \nabla_{\rho(0)}^{\alpha} r(s) \\
&\quad + \left| M + R\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right|^{-1} \frac{L|R|(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} \frac{1}{\Gamma(\alpha+1)} T^{\overline{\alpha}} \\
&= -\beta\left[r(t) - \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} r(0)\right] + \frac{L}{\Gamma(\alpha+1)} t^{\overline{\alpha}} \\
&\quad - \left| M + R\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right|^{-1} \frac{\beta|R|(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} \left[r(T) - \frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} r(0)\right] \\
&\quad + \left| M + R\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right|^{-1} \frac{L|R|(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} \frac{1}{\Gamma(\alpha+1)} T^{\overline{\alpha}} \\
&\leq \frac{\beta(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} r(0) + \frac{L}{\Gamma(\alpha+1)} t^{\overline{\alpha}} \\
&\quad + \left| M + R\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right|^{-1} \frac{\beta|R|(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} \frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} r(0) \\
&\quad + \left| M + R\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right|^{-1} \frac{L|R|(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} \frac{1}{\Gamma(\alpha+1)} T^{\overline{\alpha}} \\
&= \frac{\beta(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} \|\mathbf{x}(0)\|^2 \left[1 + \left| M + R\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right|^{-1} \frac{|R|(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right] \\
&\quad + \left| M + R\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right|^{-1} \frac{L|R|(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} \frac{1}{\Gamma(\alpha+1)} T^{\overline{\alpha}} + \frac{L}{\Gamma(\alpha+1)} t^{\overline{\alpha}}
\end{aligned}$$

$$\le \beta\|\mathbf{x}(0)\|^2\Big[1+|R|\Big|M+R\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\Big|^{-1}\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\Big]$$
$$+\frac{LT^{\overline{\alpha}}}{\Gamma(\alpha+1)}\Big[1+|R|\Big|M+R\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\Big|^{-1}\Big].$$

The proof is complete. □

4. Existence of Solutions

We now apply the results of Section 3 to obtain new existence results for solutions to (1.5). For this purpose, we need the Schaefer's fixed point theorem.

Theorem 4.1 (See [19]). *(Schaefer's fixed point theorem) Let F be a completely continuous mapping of a Banach space $\mathscr{B}$ into itself such that the set*

$$\Omega=\{x\in\mathscr{B}: x=\mu Fx \text{ for some } \mu\in[0,1]\}$$

is bounded. Then, F has a fixed point in $\mathscr{B}$.

Theorem 4.2. *If the conditions of Theorem 3.3 hold, then (1.5) has a solution.*

Proof. We shall use the fact that $\mathbb{R}^{(T+1)n}$ is a Banach space equipped with the maximum norm defined by

$$\|\mathbf{x}\|=\max_{t\in\mathbb{N}_0^T}\|\mathbf{x}(t)\|.$$

Define the operator $F:\mathbb{R}^{(T+1)n}\to\mathbb{R}^{(T+1)n}$ by

$$\begin{aligned}\big(F\mathbf{x}\big)(t)&=\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{x}(s))\\&\quad-\Big[M+R\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\Big]^{-1}\frac{R(t+1)^{\overline{\alpha-1}}}{[\Gamma(\alpha)]^2}\sum_{s=1}^{T}(T-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s,\mathbf{x}(s)),\end{aligned}\tag{4.1}$$

for $t\in\mathbb{N}_0^T$. Clearly, $\mathbf{x}$ is a fixed point of F if and only if $\mathbf{x}$ is a solution of (1.5). Since F is a summation operator on a discrete finite set, F is trivially continuous on $\mathbb{R}^{(T+1)n}$. To apply Schaefer's theorem, we show that the set

$$\Omega=\{x\in\mathbb{R}^{(T+1)n}: x=\mu Fx \text{ for some } \mu\in[0,1]\}$$

is bounded. This is equivalent to show that, for each $\mu\in[0,1]$, solutions to the following nabla fractional boundary value problem

$$\begin{cases}\big(\nabla^{\alpha}_{\rho(0)}\mathbf{x}\big)(t)=\mu\mathbf{f}(t,\mathbf{x}(t)), & t\in\mathbb{N}_1^T,\quad 0<\alpha<1,\\ M\mathbf{x}(0)+R\mathbf{x}(T)=0,\end{cases}\tag{4.2}$$

are bounded, with the bound independent of μ. Let $\mathbf{x}$ be a solution to (4.2) for a fixed $\mu\in[0,1]$. We show that $\mu\mathbf{f}$ satisfies the conditions of Theorem 3.3. Multiplying by μ on both sides of (3.7), we obtain

$$\|\mu\mathbf{f}(t,\mathbf{x})\|\le-2\beta\langle\mathbf{x},\mu\mathbf{f}(t,\mathbf{x})\rangle+\mu L\le-2\beta\langle\mathbf{x},\mu\mathbf{f}(t,\mathbf{x})\rangle+L,\quad(t,\mathbf{x})\in\mathbb{N}_0^T\times\mathbb{R}^{(T+1)n},$$

implying that $\mu\mathbf{f}$ satisfies the conditions of Theorem 3.3. Thus,

$$\|\mathbf{x}(t)\| \leq \beta\|\mathbf{x}(0)\|^2\Big[1+|R|\Big|M+R\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\Big|^{-1}\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\Big]$$
$$+\frac{LT^{\overline{\alpha}}}{\Gamma(\alpha+1)}\Big[1+|R|\Big|M+R\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\Big|^{-1}\Big],$$

for all $t \in \mathbb{N}_1^T$, with the bound independent of μ. So, it follows by Theorem 4.1 that the operator F has a fixed point. The proof is complete. □

5. Examples

In this section, we provide a few examples that satisfy the conditions of Theorem 3.3.

Example 5.1. Consider the scalar boundary value problem

$$\begin{cases} \big(\nabla^{\alpha}_{\rho(0)}x\big)(t) = c, \quad c\in\mathbb{R}, \quad t\in\mathbb{N}_1^T, \quad 0<\alpha<1, \\ x(0)+x(T)=0, \end{cases} \tag{5.1}$$

where $f(t,x) = c \in C\big(\mathbb{N}_0^T\times\mathbb{R},\mathbb{R}\big)$. From Remark 3.2, it follows that (3.1) holds. For $\beta = 0$ and $L \geq |c|$, we have

$$-2\beta x f(t,x)+L \geq |f(t,x)|,$$

for all $(t,x)\in\mathbb{N}_0^T\times\mathbb{R}$, implying that (3.7) holds. Thus, all the conditions of Theorem 3.3 hold. Therefore, by Theorem 4.2, (5.1) has a solution. In particular, one can verify that

$$x(t) = \frac{c}{\Gamma(\alpha+1)}\Big[t^{\overline{\alpha}} - \frac{T^{\overline{\alpha}}}{\Gamma(\alpha)}\Big[1+\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\Big]^{-1}(t+1)^{\overline{\alpha-1}}\Big], \quad t\in\mathbb{N}_0^T,$$

is a solution of (5.1).

Example 5.2. Consider the scalar boundary value problem

$$\begin{cases} \big(\nabla^{\alpha}_{\rho(0)}x\big)(t) = c, \quad c\in\mathbb{R}, \quad t\in\mathbb{N}_1^T, \quad 0<\alpha<1, \\ x(0)=x(T). \end{cases} \tag{5.2}$$

Here $f(t,x) = c \in C\big(\mathbb{N}_0^T\times\mathbb{R},\mathbb{R}\big)$. From Remark 3.2 it follows that (3.1) holds. For $\beta = 0$ and $L \geq |c|$, we have

$$-2\beta x f(t,x)+L \geq |f(t,x)|,$$

for all $(t,x)\in\mathbb{N}_0^T\times\mathbb{R}$. These imply that (3.7) holds. Thus, all the conditions of Theorem 3.3 hold. Therefore, by Theorem 4.2, (5.2) has a solution. In particular, one can verify that

$$x(t) = \frac{c}{\Gamma(\alpha+1)}\Big[t^{\overline{\alpha}} + \frac{T^{\overline{\alpha}}}{\Gamma(\alpha)}\Big[1-\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\Big]^{-1}(t+1)^{\overline{\alpha-1}}\Big], \quad t\in\mathbb{N}_0^T,$$

is a solution of (5.2).

Example 5.3. Consider the scalar boundary value problems

$$\begin{cases} \big(\nabla^{\alpha}_{\rho(0)}x\big)(t) = -x^{2m+1}(t), \quad m\in\mathbb{N}_0, \quad t\in\mathbb{N}_1^T, \quad 0<\alpha<1, \\ x(0)+x(T)=0, \end{cases} \tag{5.3}$$

and

$$\begin{cases} \left(\nabla^{\alpha}_{\rho(0)}x\right)(t) = -x^{2m+1}(t), \quad m \in \mathbb{N}_0, \quad t \in \mathbb{N}_1^T, \quad 0 < \alpha < 1, \\ x(0) = x(T). \end{cases} \tag{5.4}$$

Here $f(t,x) = -x^{2m+1} \in C\left(\mathbb{N}_0^T \times \mathbb{R}, \mathbb{R}\right)$. From Remark 3.2, it follows that (3.1) holds. For $\beta = \frac{1}{2}$ and $L = 1$, we have

$$-2\beta x f(t,x) + L = x^{2m+2} + 1 \geq |-x^{2m+1}|,$$

for all $(t,x) \in \mathbb{N}_0^T \times \mathbb{R}$. These imply that (3.7) holds. Thus, all the conditions of Theorem 3.3 hold. Therefore, by Theorem 4.2, (5.3) and (5.4) have at least one solution each. Such solution to (5.3) satisfy the a priori bound

$$|x(t)| \leq \frac{1}{2}|x(0)|^2\left[1 + \left|1 + \frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right|^{-1}\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right] + \frac{T^{\overline{\alpha}}}{\Gamma(\alpha+1)}\left[1 + \left|1 + \frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right|^{-1}\right],$$

for all $t \in \mathbb{N}_1^T$. Such solution to (5.4) satisfy the a priori bound

$$|x(t)| \leq \frac{1}{2}|x(0)|^2\left[1 + \left|1 - \frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right|^{-1}\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right] + \frac{T^{\overline{\alpha}}}{\Gamma(\alpha+1)}\left[1 + \left|1 - \frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right|^{-1}\right],$$

for all $t \in \mathbb{N}_1^T$.

Example 5.4. Assume $a > 0$, $d > 0$, $a = d$ and $b + c = 0$. Consider the boundary value problems

$$\begin{cases} \left(\nabla^{\alpha}_{\rho(0)}\mathbf{x}\right)(t) = \begin{pmatrix} -a & b \\ c & -d \end{pmatrix}\mathbf{x}(t), \quad t \in \mathbb{N}_1^T, \quad 0 < \alpha < 1, \\ \mathbf{x}(0) + \mathbf{x}(T) = 0, \end{cases} \tag{5.5}$$

and

$$\begin{cases} \left(\nabla^{\alpha}_{\rho(0)}\mathbf{x}\right)(t) = \begin{pmatrix} -a & b \\ c & -d \end{pmatrix}\mathbf{x}(t), \quad t \in \mathbb{N}_1^T, \quad 0 < \alpha < 1, \\ \mathbf{x}(0) = \mathbf{x}(T), \end{cases} \tag{5.6}$$

where

$$\mathbf{f}(t,x_1,x_2) = \begin{bmatrix} -ax_1 + bx_2 \\ cx_1 - dx_2 \end{bmatrix} \in C\left(\mathbb{N}_0^T \times \mathbb{R}^2, \mathbb{R}^2\right).$$

From Remark 3.2, it follows that (3.1) holds. We have

$$\|\mathbf{f}(t,x_1,x_2)\| = \sqrt{a^2x_1^2 + d^2x_2^2} \leq a\left[|x_1| + |x_2|\right].$$

For $\beta = \frac{1}{2}$ and $L = 2a$, we have

$$-2\beta\langle \mathbf{x}, \mathbf{f}(t,\mathbf{x})\rangle + L = a\left[x_1^2 + x_2^2\right] + 2a \geq a\left[|x_1| + |x_2|\right] \geq \|\mathbf{f}(t,x_1,x_2)\|,$$

for all $(t,\mathbf{x}) \in \mathbb{N}_0^T \times \mathbb{R}^2$. These imply that (3.7) holds. Thus, all the conditions of Theorem 3.3 hold. Therefore, by Theorem 4.2, (5.5) and (5.6) have at least one solution each. Such solution to (5.5) satisfy the a priori bound

$$\|\mathbf{x}(t)\| \leq \frac{1}{2}\|\mathbf{x}(0)\|^2\Big[1+\Big|1+\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\Big|^{-1}\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\Big] + \frac{2aT^{\overline{\alpha}}}{\Gamma(\alpha+1)}\Big[1+\Big|1+\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\Big|^{-1}\Big],$$

for all $t \in \mathbb{N}_1^T$. Such solution to (5.6) satisfy the a priori bound

$$\|\mathbf{x}(t)\| \leq \frac{1}{2}\|\mathbf{x}(0)\|^2\Big[1+\Big|1-\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\Big|^{-1}\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\Big] + \frac{2aT^{\overline{\alpha}}}{\Gamma(\alpha+1)}\Big[1+\Big|1-\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\Big|^{-1}\Big],$$

for all $t \in \mathbb{N}_1^T$.

Example 5.5. Consider the boundary value problem

$$\begin{cases} \big(\nabla^{\alpha}_{\rho(0)}\mathbf{x}\big)(t) = \begin{pmatrix} -x_1+x_2t \\ -x_1t-x_2 \end{pmatrix}, \quad t \in \mathbb{N}_1^T, \quad 0<\alpha<1, \\ \mathbf{x}(0)+\mathbf{x}(T)=0, \end{cases} \tag{5.7}$$

and

$$\begin{cases} \big(\nabla^{\alpha}_{\rho(0)}\mathbf{x}\big)(t) = \begin{pmatrix} -x_1+x_2t \\ -x_1t-x_2 \end{pmatrix}, \quad t \in \mathbb{N}_1^T, \quad 0<\alpha<1, \\ \mathbf{x}(0)=\mathbf{x}(T), \end{cases} \tag{5.8}$$

where

$$\mathbf{f}(t,x_1,x_2) = \begin{pmatrix} -x_1+x_2t \\ -x_1t-x_2 \end{pmatrix} \in C\big(\mathbb{N}_0^T \times \mathbb{R}^2, \mathbb{R}^2\big).$$

From Remark 3.2 it follows that (3.1) holds. We have

$$\|\mathbf{f}(t,x_1,x_2)\| = \sqrt{(1+t^2)\big[x_1^2+x_2^2\big]} \leq \sqrt{1+t^2}\big[|x_1|+|x_2|\big].$$

For $\beta = \frac{\sqrt{1+T^2}}{2}$ and $L = 2\sqrt{1+T^2}$, we have

$$\begin{aligned} -2\beta\langle \mathbf{x},\mathbf{f}(t,\mathbf{x})\rangle + L &= \sqrt{1+T^2}\big[x_1^2+x_2^2+2\big] \\ &\geq \sqrt{1+t^2}\big[x_1^2+x_2^2+2\big] \\ &\geq \sqrt{1+t^2}\big[|x_1|+|x_2|\big] \\ &\geq \|\mathbf{f}(t,x_1,x_2)\|, \end{aligned}$$

for all $(t,\mathbf{x}) \in \mathbb{N}_0^T \times \mathbb{R}^2$. These imply that (3.7) holds. Thus, all the conditions of Theorem 3.3 hold. Therefore, by Theorem 4.2, (5.7) and (5.8) have at least one solution each. Such solution to (5.7) satisfy the a priori bound

$$\|\mathbf{x}(t)\| \le \frac{\sqrt{1+T^2}}{2}\|\mathbf{x}(0)\|^2\Big[1+\Big|1+\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\Big|^{-1}\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\Big] + \frac{2T^{\overline{\alpha}}\sqrt{1+T^2}}{\Gamma(\alpha+1)}\Big[1+\Big|1+\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\Big|^{-1}\Big],$$

for all $t \in \mathbb{N}_1^T$. Such solution to (5.6) satisfy the a priori bound

$$\|\mathbf{x}(t)\| \le \frac{\sqrt{1+T^2}}{2}\|\mathbf{x}(0)\|^2\Big[1+\Big|1-\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\Big|^{-1}\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\Big] + \frac{2T^{\overline{\alpha}}\sqrt{1+T^2}}{\Gamma(\alpha+1)}\Big[1+\Big|1-\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\Big|^{-1}\Big],$$

for all $t \in \mathbb{N}_1^T$.

Example 5.6. Consider the boundary value problem

$$\begin{cases} \left(\nabla^{\alpha}_{\rho(0)}\mathbf{x}\right)(t) = \begin{pmatrix} \cos(x_1+x_2+t) \\ \sin(x_1+x_2+t) \end{pmatrix}, \quad t \in \mathbb{N}_1^T, \quad 0<\alpha<1, \\ \mathbf{x}(0)+\mathbf{x}(T) = 0, \end{cases} \tag{5.9}$$

and

$$\begin{cases} \left(\nabla^{\alpha}_{\rho(0)}\mathbf{x}\right)(t) = \begin{pmatrix} \cos(x_1+x_2+t) \\ \sin(x_1+x_2+t) \end{pmatrix}, \quad t \in \mathbb{N}_1^T, \quad 0<\alpha<1, \\ \mathbf{x}(0) = \mathbf{x}(T), \end{cases} \tag{5.10}$$

where

$$\mathbf{f}(t,x_1,x_2) = \begin{pmatrix} \cos(x_1+x_2+t) \\ \sin(x_1+x_2+t) \end{pmatrix} \in C\big(\mathbb{N}_0^T \times \mathbb{R}^2, \mathbb{R}^2\big).$$

From Remark 3.2, it follows that (3.1) holds. We have

$$\|\mathbf{f}(t,x_1,x_2)\| = \sqrt{\cos^2(x_1+x_2+t)+\sin^2(x_1+x_2+t)} = 1.$$

For $\beta = 0$ and $L \ge 1$, we have

$$-2\beta\langle \mathbf{x}, \mathbf{f}(t,\mathbf{x})\rangle + L = 2 \ge \|\mathbf{f}(t,x_1,x_2)\|,$$

for all $(t,\mathbf{x}) \in \mathbb{N}_0^T \times \mathbb{R}^2$. These imply that (3.7) holds. Thus, all the conditions of Theorem 3.3 hold. Therefore, by Theorem 4.2, (5.9) and (5.10) have at least one solution each. Such solution to (5.9) satisfy the a priori bound

$$\|\mathbf{x}(t)\| \le \frac{LT^{\overline{\alpha}}}{\Gamma(\alpha+1)}\Big[1+\Big|1+\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\Big|^{-1}\Big],$$

for all $t \in \mathbb{N}_1^T$. Such solution to (5.6) satisfy the a priori bound

$$\|\mathbf{x}(t)\| \le \frac{LT^{\overline{\alpha}}}{\Gamma(\alpha+1)}\Big[1+\Big|1-\frac{(T+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\Big|^{-1}\Big],$$

for all $t \in \mathbb{N}_1^T$.

Examples 5.4 and 5.6 are motivated from [9].

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