Commun. Optim. Theory 2019 (2019), Article ID 14 https://doi.org/10.23952/cot.2019.14

MATHRES Communications in Optimization Theory
Available online at http://cot.mathres.org

COMMON FIXED POINTS OF (R,α)-GENERALIZED RATIONAL MULTIVALUED CONTRACTIONS IN R-COMPLETE b-METRIC SPACES

MUJAHID ABBAS[1,2], ISMAT BEG[3,*], BAHRU TSEGAYE LEYEW[4]

[1]Department of Mathematics, Government College University, Lahore 54000, Pakistan
[2]Department of Mathematics and Applied Mathematics, University of Pretoria, Hatfield, Pretoria 002, South Africa
[3]Centre for Mathematics and Statistical Sciences, Lahore School of Economics, Lahore, Pakistan
[4]Department of Mathematics, College of Natural Sciences, Addis Ababa University, Addis Ababa, Ethiopia

Abstract. In this paper, we introduce the concept of the R-continuity and the R-closedness for a pair of multivalued mappings. A new class of (R,α)-generalized rational multivalued contraction mappings is defined. We establish the existence of common fixed points of such mappings in the framework of b-metric spaces endowed with an arbitrary binary relation R. Existence of fixed and common fixed points of single valued mappings is shown in the setting of R-complete b-metric spaces. Several examples are constructed in support of our new results.

Keywords. Common fixed point; (R,α)-generalized rational contraction; b-metric; multivalued contraction mapping.

2010 Mathematics Subject Classification. 47H04, 47H10.

1. Introduction and Preliminaries

We denote by the letters $\mathscr{R}$, $\mathbb{R}$ and $\mathbb{N}_0$, a nonempty arbitrary binary relation on a set X, the set of all real numbers and the set of whole numbers, respectively.

First, we present some important notations from [1–8].

Definition 1.1. [3, 6] Let X be a nonempty set and $s \geq 1$ a real number. A function $d : X \times X \to [0,\infty)$ is called a $b-$metric if for any $x,y,z \in X$, the following conditions hold:

(db$_1$): $d(x,y) = 0$ if and only if $x = y$;
(db$_2$): $d(x,y) = d(y,x)$;
(db$_3$): $d(x,y) \leq s\left[d(x,z) + d(z,y)\right]$.
: The pair (X,d) is called a b-metric space.

*Corresponding author.
E-mail addresses: mujahid.abbas@up.ac.za (M. Abbas), ibeg@lahoreschool.edu.pk (I.Beg), tsegayebah@gmail.com (B.T. Leyew).

Received May 30, 2019; Accepted July 23, 2019.

©2019 Communications in Optimization Theory

If $s = 1$, then a b-metric space is a metric space. But the converse does not hold in general. Therefore, the class of b-metric spaces is larger.

Example 1.2. [9] Let $X = \{0,1,2\}$ and $d : X \times X \to [0,\infty)$ be the mapping defined by

$$\begin{aligned} d(0,0) &= d(1,1) = d(2,2) = 0, \\ d(0,1) &= d(1,0) = d(1,2) = d(2,1) = 1, \text{ and} \\ d(2,0) &= d(0,2) = m, \end{aligned}$$

where m is a given real number such that $m \geq 2$. For any $x, y, z \in X$, we have

$$d(x,y) \leq \frac{m}{2}[d(x,z) + d(z,y)].$$

So, (X,d) is a b-metric space with constant $s = \frac{m}{2}$. However, (X,d) is not a metric space if $m > 2$.

Example 1.3. [7] Let (X,d) be a metric space and $\rho(x,y) = (d(x,y))^p$, where $p > 1$ is a real number. Then ρ is a b-metric with $s = 2^{p-1}$.

If $X = \mathbb{R}$ and $d(x,y) = |x-y|$ is the usual metric on $\mathbb{R}$, then $\rho(x,y) = |x-y|^3$ is a b-metric on $\mathbb{R}$ with $s = 4$.

Czerwik [5] proved Banach contraction theorem for single and multivalued mappings in b-metric spaces. Since then several results have been obtained for fixed points of various classes of single-valued and multi-valued operators in b-metric spaces (see [5, 6, 10, 11] and references mentioned therein).

Definition 1.4. [4] Let (X,d) be a b-metric space. A sequence $\{x_n\}$ in X is called (a) Cauchy sequence if for any arbitrary $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that for all $n, m \geq n_\varepsilon$, $d(x_n, x_m) < \varepsilon$ holds; (b) convergent sequence if there exists $x \in X$ such that for any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$ $d(x_n, x) < \varepsilon$. In this case, we write $\lim_{n\to\infty} x_n = x$.

A b-metric space X is complete if every Cauchy sequence $\{x_n\}$ in X is convergent to some point $x \in X$.

In a b-metric space (X,d), every convergent sequence is a Cauchy sequence and has a unique limit. However, a b-metric $d : X \times X \to [0,\infty)$ is not necessarily continuous in each variable. If b-metric d is continuous in one variable, then it is continuous in the other variable (see [2]).

Definition 1.5. [12] Let (X,d) be a b-metric space. A subset $Y \subseteq X$ is called:

- **(a):** closed if and only if $Y = \overline{Y}$, where $\overline{Y} := \{x \in X :$ there exists a sequence $\{x_n\}$ in Y such that $\lim_{n\to\infty} x_n = x\}$;
- **(b):** compact if and only if for every sequence of elements in Y, there exists a subsequence that converges to an element in Y;
- **(c):** bounded if and only if $\Delta(Y) := \sup\{d(a,b) | a, b \in Y\} < \infty$.

Nadler [13] introduced the concept of a set-valued contraction mapping and generalized the Banach contraction principle to set-valued mappings using the concept of a Hausdorff metric. Since then, a lot of research activity has been carried out to obtain fixed points for multivalued mappings satisfying certain contractive conditions via Hausdorff metric (see [6, 14–18] and the references mentioned therein).

From now on, unless stated otherwise, X will denote a b-metric space equipped with a b-metric d. Also

$$\begin{aligned} P(X) &= \{A : A \text{ is a subset of } X\}, \\ N(X) &= \{A : A \text{ is a nonempty subset of } X\}, \\ B(X) &= \{A : A \text{ is a nonempty bounded subset of } X\}, \\ CL(X) &= \{A : A \text{ is a nonempty closed subset of } X\}, \\ C(X) &= \{A : A \text{ is a nonempty compact subset of } X\}, \\ CB(X) &= \{A : A \text{ is a nonempty closed and bounded subset of } X\}. \end{aligned}$$

Let $T, S : X \to N(X)$ and $f, g : X \to X$. A point $x^* \in X$ is called:

(i): a fixed point of f if $x^* = fx^*$ (a fixed point of T if $x^* \in Tx^*$).

(iii): a common fixed point of f and g if $x^* = fx^* = gx^*$ (a common fixed point of T and S if $x^* \in Tx^* \cap Sx^*$).

We denote by $F(f)$ (or $F(T)$) the set of fixed points of f (T).

For $A, B \in CB(X)$ and $x \in X$, define,

$$\begin{aligned} \delta(A,B) &= \sup\{d(x,B) : x \in A\} \text{ and} \\ H(A,B) &= \max\{\delta(A,B), \delta(B,A)\}, \end{aligned}$$

where,

$$d(x,B) = \inf\{d(x,y) : y \in B\}.$$

The function H is called the Hausdorff b-metric on $CB(X)$ induced by the b-metric d. Furthermore, $(CB(X), H)$ is complete if (X,d) is complete. A b-metric d is bounded if $\Delta = \sup\{d(x,y) : x, y \in X\} < \infty$. In a b-metric space (X,d), for $B \in P(X)$, we obtain $d(x,B) \leq d(x,b)$ for all $b \in B$.

Throughout this paper, we assume that a b-metric d is continuous in one variable. Next we state the following important Lemmas from [6].

Lemma 1.6. *[6] For any $x \in X$ and $A, B \in P(X)$, the following inequality holds:*

$$d(x,A) \leq s\,(d(x,B) + H(B,A))\,.$$

In particular, for any $x, y \in X$ and $A \in P(X)$, we have

$$d(x,A) \leq s\,(d(x,y) + d(y,A))\,.$$

Lemma 1.7. *[6] For any $A, B, C \in P(X)$, we have*

$$H(A,C) \leq s\,(H(A,B) + H(B,C))\,.$$

Lemma 1.8. *[6] Let A, $B \in CB(X)$. If $a \in A$, then $d(a,B) \leq H(A,B)$.*

Lemma 1.9. *[6] Let A, $B \in CB(X)$. Then, for any $\varepsilon > 0$ and $a \in A$, there exists $b \in B$ such that $d(a,b) \leq H(A,B) + \varepsilon$.*

Lemma 1.10. *[6] For $A \in CB(X)$ and $x \in X$, $d(x,A) = 0$ implies that $x \in A$.*

Definition 1.11. [19] Let X be a nonempty set. A subset $\mathscr{R}$ of X^2 is called a binary relation on X. For each pair $x,y \in X$, we say that "x is $\mathscr{R}$-related to y" or "x relates to y under $\mathscr{R}$" if and only if $(x,y) \in \mathscr{R}$ and $(x,y) \notin \mathscr{R}$ means that "x is not $\mathscr{R}$-related to y" or '"x doesn't relate to y under $\mathscr{R}$".

Notice that X^2 and $\emptyset$ being subsets of X^2 are binary relations on X, which are called universal relation (or full relation) and empty relation respectively.

Recently, Alam and Imdad [1] proved a fixed point theorem in the framework of a complete metric space endowed with a binary relation which under a universal relation reduces to a well known Banach contraction principle. Further, they derived the results of Ran and Reurings [20], Nieto, R. Rodríguez-López [21,22], Turinici [23,24] from the relation-theoretic contraction principle.

Definition 1.12. [1] Any pair of points x,y in X is said to be $\mathscr{R}$-comparative if either $(x,y) \in \mathscr{R}$ or $(y,x) \in \mathscr{R}$, denoted by $[x,y] \in \mathscr{R}$.

Definition 1.13. [25] A binary relation $\mathscr{R}$ defined on a nonempty set X is called (i) reflexive if $(x,x) \in \mathscr{R}$ $\forall x \in X$, (ii) irreflexive if $(x,x) \notin \mathscr{R}$ for some $x \in X$; (iii) symmetric if $(x,y) \in \mathscr{R}$ implies $(y,x) \in \mathscr{R}$ $\forall x,y \in X$; (iv) antisymmetric if $(x,y) \in \mathscr{R}$ and $(y,x) \in \mathscr{R}$ imply $x = y$, $\forall x,y \in X$; (v) transitive if $(x,y) \in \mathscr{R}$ and $(y,z) \in \mathscr{R}$ imply $(x,z) \in \mathscr{R}$ $\forall x,y,z \in X$; (vi) preorder if $\mathscr{R}$ is reflexive and transitive; (vii) partial order if $\mathscr{R}$ is reflexive, antisymmetric and transitive; (viii) equivalence if $\mathscr{R}$ is reflexive, symmetric and transitive.

Definition 1.14. Let X be a nonempty set and $\mathscr{R}$ a binary relation on X.

(1) The inverse, transpose or dual relation of $\mathscr{R}$, denoted by $\mathscr{R}^{-1}$, is defined by

$$\mathscr{R}^{-1} = \{(x,y) \in X^2 : (y,x) \in \mathscr{R}\}.$$

(2) The symmetric closure of $\mathscr{R}$, denoted by $\mathscr{R}^s$, is defined as a set $\mathscr{R} \cup \mathscr{R}^{-1}$, that is, $\mathscr{R}^s := \mathscr{R} \cup \mathscr{R}^{-1}$. Indeed, $\mathscr{R}^s$ is the smallest symmetric relation on X containing $\mathscr{R}$.

Proposition 1.15. *[1] For a binary relation $\mathscr{R}$ defined on a nonempty set X,*

$$(x,y) \in \mathscr{R}^s \text{ if and only if } [x,y] \in \mathscr{R}.$$

Definition 1.16. [1] Let X be a nonempty set and $\mathscr{R}$ a binary relation on X. A sequence $\{x_n\} \subset X$ is called $\mathscr{R}$-preserving if $(x_n, x_{n+1}) \in \mathscr{R}$ for all $n \in \mathbb{N}_0$.

We now present the notion of a $\mathscr{R}$ b-metric space which is more general than that of a b-metric space.

Definition 1.17. A nonempty set X is called $\mathscr{R}$ b-metric space if there exists a b-metric d on X and $\mathscr{R}$ is a binary relation on X. We denote the $\mathscr{R}$ b-metric space by $(X,d,\mathscr{R})$.

Definition 1.18. Let $(X,d,\mathscr{R})$ be a $\mathscr{R}$ b-metric space. We say that $(X,d,\mathscr{R})$ is $\mathscr{R}$-complete b-metric space if every $\mathscr{R}$-preserving Cauchy sequence in X converges to a point in X.

Remark 1.19. For any binary relation $\mathscr{R}$, every complete b-metric space is $\mathscr{R}$-complete b-metric space. Particularly, under the universal relation the notion of $\mathscr{R}$-completeness coincides with usual completeness.

Definition 1.20. Let $(X,d,\mathscr{R})$ be a $\mathscr{R}$ b-metric space and $T,S : X \to N(X)$. A pair (T,S) is called:

(a): $\mathscr{R}$-continuous at x if for any $\mathscr{R}$-preserving sequence $\{x_n\}$ with $x_n \xrightarrow{d} x$, we have $Tx_n \xrightarrow{H} Sx$.
(b): $\mathscr{R}$-closed if for any $x, y \in X$, $(x,y) \in \mathscr{R}$ implies that $(u,v) \in \mathscr{R}$ for any $u \in Tx$ and $v \in Sy$.

Remark 1.21. A pair (T,S) is called $\mathscr{R}$-continuous on X if it is $\mathscr{R}$-continuous at each point of X. Every continuous mapping is $\mathscr{R}$-continuous. Particularly, under the universal relation the notion of $\mathscr{R}$-continuity coincides with the usual continuity.

Definition 1.22. Let $(X,d,\mathscr{R})$ be a $\mathscr{R}$ b-metric space. A binary relation $\mathscr{R}$ is called d-self-closed if for any $\mathscr{R}$-preserving sequence $\{x_n\}$ with $x_n \xrightarrow{d} x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $[x_{n_k}, x] \in \mathscr{R}$ for all $k \in \mathbb{N}_0$.

Next a new class of multivalued mappings called $(\mathscr{R},\alpha)$-generalized rational contraction, in the setup of a $\mathscr{R}$ b-metric space is introduced.

Definition 1.23. Let $(X,d,\mathscr{R})$ be a $\mathscr{R}$ b-metric space with constant $s \geq 1$ and $\alpha : X \times X \to [0,\infty)$. Mappings $T, S : X \to CB(X)$ are called:

(a): $(\mathscr{R},\alpha)$-generalized rational contraction if for any $(x,y) \in \mathscr{R}$, $\alpha(x,y) \geq 1$,

$$\begin{aligned} \alpha(x,y)H(Tx,Sy) &\leq a_1 d(x,Tx) + a_2 d(y,Sy) + a_3 d(x,Sy) + a_4 d(y,Tx) + a_5 d(x,y) \\ &\quad + a_6 \frac{d(x,Tx)(1+d(x,Tx))}{1+d(x,y)} \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} \alpha(x,y)H(Sx,Ty) &\leq a_1 d(x,Sx) + a_2 d(y,Ty) + a_3 d(x,Ty) + a_4 d(y,Sx) + a_5 d(x,y) \\ &\quad + a_6 \frac{d(x,Sx)(1+d(x,Sx))}{1+d(x,y)}; \end{aligned} \tag{1.2}$$

(b): $\mathscr{R}$ generalized rational contraction if, for any $(x,y) \in \mathscr{R}$,

$$\begin{aligned} H(Tx,Sy) &\leq a_1 d(x,Tx) + a_2 d(y,Sy) + a_3 d(x,Sy) + a_4 d(y,Tx) + a_5 d(x,y) \\ &\quad + a_6 \frac{d(x,Tx)(1+d(x,Tx))}{1+d(x,y)} \end{aligned}$$

and

$$\begin{aligned} H(Sx,Ty) &\leq a_1 d(x,Sx) + a_2 d(y,Ty) + a_3 d(x,Ty) + a_4 d(y,Sx) + a_5 d(x,y) \\ &\quad + a_6 \frac{d(x,Sx)(1+d(x,Sx))}{1+d(x,y)} \end{aligned}$$

where $a_i \geq 0$, $i = 1,2,\ldots,6$, $a_1 + a_2 + 2sa_3 + a_4 + a_5 + a_6 < 1$ and $a_1 + sa_3 + a_5 + a_6 > 0$.

Definition 1.24. Let $(X,d,\mathscr{R})$ be a $\mathscr{R}$ b-metric space with constant $s \geq 1$ and $\alpha : X \times X \to [0,\infty)$. A mapping $T : X \to CB(X)$ is called:

(c): $(\mathscr{R},\alpha)$-generalized rational contraction if, for any $(x,y) \in \mathscr{R}$, $\alpha(x,y) \geq 1$ and

$$\begin{aligned} \alpha(x,y)H(Tx,Ty) &\leq a_1 d(x,Tx) + a_2 d(y,Ty) + a_3 d(x,Ty) + a_4 d(y,Tx) + a_5 d(x,y) \\ &\quad + a_6 \frac{d(x,Tx)(1+d(x,Tx))}{1+d(x,y)}; \end{aligned} \tag{1.3}$$

(d): $\mathscr{R}$ generalized rational contraction if for any $(x,y)\in\mathscr{R}$,

$$\begin{aligned} H(Tx,Ty) &\leq a_1d(x,Tx)+a_2d(y,Ty)+a_3d(x,Ty)+a_4d(y,Tx)+a_5d(x,y) \\ &\quad +a_6\frac{d(x,Tx)(1+d(x,Tx))}{1+d(x,y)} \end{aligned} \tag{1.4}$$

where $a_i\geq 0$, $i=1,2,\ldots,6$, $a_1+a_2+2sa_3+a_4+a_5+a_6<1$ and $a_1+sa_3+a_5+a_6>0$.

We set

$$X(T,\mathscr{R})=\{x\in X:(x,y)\in\mathscr{R}\text{ for some }y\in Tx\}.$$

Our results extend, unify and generalize the main result in [1] and several other recent results in the existing literature.

2. Main results

Theorem 2.1. *Let $(X,d,\mathscr{R})$ be $\mathscr{R}$-complete b-metric space and $T,S:X\to CB(X)$ be $(\mathscr{R},\alpha)$-generalized rational contraction. If*

(A_1): $X(T,\mathscr{R})$ is nonempty;
(A_2): (T,S) and (S,T) are $\mathscr{R}$-closed;
(A_3): either (T,S) and (S,T) are $\mathscr{R}$-continuous or $\mathscr{R}$ is d-self-closed,
: then there exists a point $x^*\in X$ such that $x^*\in Tx^*\cap Sx^*$.

Proof. As $X(T,\mathscr{R})$ is nonempty, we choose a point x_0 in X such that $(x_0,x_1)\in\mathscr{R}$ for some $x_1\in Tx_0$. This further implies that $\alpha(x_0,x_1)\geq 1$. If $x_0=x_1$, then the result follows. Assume that $x_0\neq x_1$. By Lemma 1.9, there exists $x_2\in Sx_1$ such that

$$\begin{aligned} d(x_1,x_2) &\leq H(Tx_0,Sx_1)+(a_1+sa_3+a_5+a_6) \\ &\leq \alpha(x_0,x_1)H(Tx_0,Sx_1)+(a_1+sa_3+a_5+a_6) \\ &\leq a_1d(x_0,Tx_0)+a_2d(x_1,Sx_1)+a_3d(x_0,Sx_1)+a_4d(x_1,Tx_0)+a_5d(x_0,x_1) \\ &\quad +a_6\frac{d(x_0,Tx_0)(1+d(x_0,Tx_0))}{1+d(x_0,x_1)}+(a_1+sa_3+a_5+a_6) \\ &\leq a_1d(x_0,x_1)+a_2d(x_1,x_2)+a_3d(x_0,x_2)+a_4d(x_1,x_1)+a_5d(x_0,x_1) \\ &\quad +a_6\frac{d(x_0,x_1)(1+d(x_0,x_1))}{1+d(x_0,x_1)}+(a_1+sa_3+a_5+a_6) \\ &\leq a_1d(x_0,x_1)+a_2d(x_1,x_2)+a_3s\left[d(x_0,x_1)+d(x_1,x_2)\right]+a_5d(x_0,x_1) \\ &\quad +a_6d(x_0,x_1)+(a_1+sa_3+a_5+a_6) \\ &= (a_1+sa_3+a_5+a_6)d(x_0,x_1)+(a_2+sa_3)d(x_1,x_2)+(a_1+sa_3+a_5+a_6). \end{aligned}$$

Thus,

$$d(x_1,x_2)\leq\frac{(a_1+sa_3+a_5+a_6)}{1-(a_2+sa_3)}d(x_0,x_1)+\frac{(a_1+sa_3+a_5+a_6)}{1-(a_2+sa_3)}. \tag{2.1}$$

As $(x_0,x_1)\in\mathscr{R}$, $x_1\in Tx_0$, $x_2\in Sx_1$ and (T,S) is $\mathscr{R}$-closed, it follows that $(x_1,x_2)\in\mathscr{R}$ and hence we have $\alpha(x_1,x_2)\geq 1$. If $x_1=x_2$, then the result follows. Assume that $x_1\neq x_2$. Using Lemma 1.9, there exists

$x_3 \in Tx_2$ and by (1.2), we have

$$\begin{aligned} d(x_2,x_3) &\leq H(Sx_1,Tx_2)+\frac{(a_1+sa_3+a_5+a_6)^2}{1-(a_2+sa_3)} \\ &\leq \alpha(x_1,x_2)H(Sx_1,Tx_2)+\frac{(a_1+sa_3+a_5+a_6)^2}{1-(a_2+sa_3)} \\ &\leq a_1d(x_1,Sx_1)+a_2d(x_2,Tx_2)+a_3d(x_1,Tx_2)+a_4d(x_2,Sx_1)+a_5d(x_1,x_2) \\ &\quad +a_6\frac{d(x_1,Sx_1)(1+d(x_1,Sx_1))}{1+d(x_1,x_2)}+\frac{(a_1+sa_3+a_5+a_6)^2}{1-(a_2+sa_3)} \\ &\leq a_1d(x_1,x_2)+a_2d(x_2,x_3)+a_3d(x_1,x_3)+a_4d(x_2,x_2)+a_5d(x_1,x_2) \\ &\quad +a_6\frac{d(x_1,x_2)(1+d(x_1,x_2))}{1+d(x_1,x_2)}+\frac{(a_1+sa_3+a_5+a_6)^2}{1-(a_2+sa_3)} \\ &\leq a_1d(x_1,x_2)+a_2d(x_2,x_3)+a_3s\left[d(x_1,x_2)+d(x_2,x_3)\right]+a_5d(x_1,x_2) \\ &\quad +a_6d(x_1,x_2)+\frac{(a_1+sa_3+a_5+a_6)^2}{1-(a_2+sa_3)} \\ &= (a_1+sa_3+a_5+a_6)d(x_1,x_2)+(a_2+sa_3)d(x_2,x_3)+\frac{(a_1+sa_3+a_5+a_6)^2}{1-(a_2+sa_3)}. \end{aligned}$$

Thus

$$d(x_2,x_3) \leq \frac{(a_1+sa_3+a_5+a_6)}{1-(a_2+sa_3)}d(x_1,x_2)+\left[\frac{a_1+sa_3+a_5+a_6}{1-(a_2+sa_3)}\right]^2. \tag{2.2}$$

By (2.1) and (2.2), we obtain that

$$d(x_2,x_3) \leq \left[\frac{a_1+sa_3+a_5+a_6}{1-(a_2+sa_3)}\right]^2 d(x_0,x_1)+2\left[\frac{a_1+sa_3+a_5+a_6}{1-(a_2+sa_3)}\right]^2. \tag{2.3}$$

As $(x_1,x_2) \in \mathscr{R}$, $x_2 \in Sx_1$, $x_3 \in Tx_2$ and (S,T) is $\mathscr{R}$-closed, it follows that $(x_2,x_3) \in \mathscr{R}$. Also, $\alpha(x_2,x_3) \geq 1$. If $x_2 = x_3$, then the result follows. Assume that $x_2 \neq x_3$. Using Lemma 1.9, there exists $x_4 \in Sx_3$, we have

$$\begin{aligned} d(x_3,x_4) &\leq H(Tx_2,Sx_3)+\frac{(a_1+sa_3+a_5+a_6)^3}{1-(a_2+sa_3)} \\ &\leq \alpha(x_2,x_3)H(Tx_2,Sx_3)+\frac{(a_1+sa_3+a_5+a_6)^3}{1-(a_2+sa_3)}. \end{aligned}$$

By applying (1.1) in the above inequality, we obtain that

$$d(x_3,x_4) \leq \frac{(a_1+sa_3+a_5+a_6)}{1-(a_2+sa_3)}d(x_2,x_3)+\left[\frac{a_1+sa_3+a_5+a_6}{1-(a_2+sa_3)}\right]^3. \tag{2.4}$$

By (2.3) and (2.4), we obtain that

$$d(x_3,x_4) \leq \left[\frac{a_1+sa_3+a_5+a_6}{1-(a_2+sa_3)}\right]^3 d(x_0,x_1)+3\left[\frac{a_1+sa_3+a_5+a_6}{1-(a_2+sa_3)}\right]^3. \tag{2.5}$$

Continuing this way, we construct a sequence $\{x_k\}$ of distinct points in X such that $x_{2k+1} \in Tx_{2k}$, $(x_{2k},x_{2k+1}) \in \mathscr{R}$ with $\alpha(x_{2k},x_{2k+1}) \geq 1$ and

$$d(x_{2k+1},x_{2k+2}) \leq \left[\frac{a_1+sa_3+a_5+a_6}{1-(a_2+sa_3)}\right]^{2k+1} d(x_0,x_1)+(_{2k+1})\left[\frac{a_1+sa_3+a_5+a_6}{1-(a_2+sa_3)}\right]^{2k+1} \tag{2.6}$$

for all $k \in \mathbb{N}_0$. As $(x_{2k}, x_{2k+1}) \in \mathscr{R}$, $x_{2k+1} \in Tx_{2k}$, $x_{2k+2} \in Sx_{2k+1}$ and (T,S) is $\mathscr{R}$-closed, it follows that $(x_{2k+1}, x_{2k+2}) \in \mathscr{R}$. Also, $\alpha(x_{2k+1}, x_{2k+2}) \geq 1$. Using Lemma 1.9, there exists $x_{2k+3} \in Tx_{2k+2}$ such that

$$\begin{aligned} d(x_{2k+2}, x_{2k+3}) &\leq H(Sx_{2k+1}, Tx_{2k+2}) + \frac{(a_1 + sa_3 + a_5 + a_6)^{2k+2}}{1 - (a_2 + sa_3)} \\ &\leq \alpha(x_{2k+1}, x_{2k+2}) H(Sx_{2k+1}, Tx_{2k+2}) + \frac{(a_1 + sa_3 + a_5 + a_6)^{2k+2}}{1 - (a_2 + sa_3)}. \end{aligned}$$

By applying (1.2) in the above inequality, we obtain that

$$d(x_{2k+2}, x_{2k+3}) \leq \frac{a_1 + sa_3 + a_5 + a_6}{1 - (a_2 + sa_3)} d(x_{2k+1}, x_{2k+2}) + \left[\frac{a_1 + sa_3 + a_5 + a_6}{1 - (a_2 + sa_3)}\right]^{2k+2}. \tag{2.7}$$

By (2.6) and (2.7), we obtain that

$$d(x_{2k+2}, x_{2k+3}) \leq \left[\frac{a_1 + sa_3 + a_5 + a_6}{1 - (a_2 + sa_3)}\right]^{2k+2} d(x_0, x_1) + (2k+2)\left[\frac{a_1 + sa_3 + a_5 + a_6}{1 - (a_2 + sa_3)}\right]^{2k+2}. \tag{2.8}$$

Then, by induction we conclude that for any sequence $\{x_n\}$, we obtain that $(x_{n-1}, x_n) \in \mathscr{R}$ for all $n \in \mathbb{N}$. Also $\alpha(x_{n-1}, x_n) \geq 1$ and

$$d(x_n, x_{n+1}) \leq \left[\frac{a_1 + sa_3 + a_5 + a_6}{1 - (a_2 + sa_3)}\right]^{n} d(x_0, x_1) + n\left[\frac{a_1 + sa_3 + a_5 + a_6}{1 - (a_2 + sa_3)}\right]^{n}. \tag{2.9}$$

Therefore, the sequence $\{x_n\}$ is $\mathscr{R}$-preserving as $(x_n, x_{n+1}) \in \mathscr{R}$ for all $n \in \mathbb{N}_0$. We can write (2.9) as:

$$d(x_n, x_{n+1}) \leq \eta^n d(x_0, x_1) + n\eta^n$$

where

$$\eta = \frac{a_1 + sa_3 + a_5 + a_6}{1 - (a_2 + sa_3)}.$$

As $\eta < 1$, the series $\sum \eta^n$ and $\sum n\eta^n$ have the same radius of convergence, the sequence $\{x_n\}$ is Cauchy sequence in X. By $\mathscr{R}$-completeness of X, there exists $x^* \in X$ such that $\lim_{n\to\infty} d(x_n, x^*) = 0$. If (T,S) is $\mathscr{R}$-continuous. Then $\lim_{n\to\infty} H(Tx_{2n}, Sx^*) = 0$. By Lemma 1.8, we have

$$\begin{aligned} d(x^*, Sx^*) &\leq s\left[d(x^*, x_{2n+1}) + d(x_{2n+1}, Sx^*)\right] \\ &\leq s\left[d(x^*, x_{2n+1}) + H(Tx_{2n}, Sx^*)\right]. \end{aligned}$$

On taking limit as $n \to \infty$ on both sides of the above inequality, we obtain that $d(x^*, Sx^*) = 0$ and hence $x^* \in Sx^*$. If (S,T) is $\mathscr{R}$-continuous, then $\lim_{n\to\infty} H(Sx_{2n+1}, Tx^*) = 0$. By Lemma 1.8, we have

$$\begin{aligned} d(x^*, Tx^*) &\leq s\left[d(x^*, x_{2n+2}) + d(x_{2n+2}, Tx^*)\right] \\ &\leq s\left[d(x^*, x_{2n+2}) + H(Sx_{2n+1}, Tx^*)\right]. \end{aligned}$$

On taking limit as $n \to \infty$ on both sides of the above inequality, we obtain that $d(x^*, Tx^*) = 0$ and hence $x^* \in Tx^*$. This shows that there exists $x^* \in X$ such that $x^* \in Tx^* \cap Sx^*$. Now, if $\mathscr{R}$ is d-self-closed, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $[x_{n_k}, x^*] \in \mathscr{R}$, for all $k \in \mathbb{N}_0$. That is, $(x_{n_k}, x^*) \in \mathscr{R}$ or $(x^*, x_{n_k}) \in \mathscr{R}$,

for all $k \in \mathbb{N}_0$. Assume that $(x_{n_k}, x^*) \in \mathscr{R}$ for all $k \in \mathbb{N}_0$. Thus, $\alpha(x_{n_k}, x^*) \geq 1$ for all $k \in \mathbb{N}_0$. By using (1.1), Lemmas 1.6, and 1.8, we obtain that

$$\begin{aligned}
d(x^*, Sx^*) &\leq s\left[d(x^*, x_{2n_k+1}) + d(x_{2n_k+1}, Sx^*)\right] \\
&\leq s\left[d(x^*, x_{2n_k+1}) + H(Tx_{2n_k}, Sx^*)\right] \\
&\leq s\left[d(x^*, x_{2n_k+1}) + \alpha(x_{2n_k}, x^*)H(Tx_{2n_k}, Sx^*)\right] \\
&\leq s\left[\begin{array}{l} d(x^*, x_{2n_k+1}) + a_1 d(x_{2n_k}, Tx_{2n_k}) + a_2 d(x^*, Sx^*) + a_3 d(x_{2n_k}, Sx^*) \\ +a_4 d(x^*, Tx_{2n_k}) + a_5 d(x_{2n_k}, x^*) + a_6 \dfrac{d(x_{2n_k}, Tx_{2n_k})(1 + d(x_{2n_k}, Tx_{2n_k}))}{1 + d(x_{2n_k}, x^*)} \end{array}\right] \\
&\leq s\left[\begin{array}{l} d(x^*, x_{2n_k+1}) + a_1 d(x_{2n_k}, x_{2n_k+1}) + a_2 d(x^*, Sx^*) + a_3 d(x_{2n_k}, Sx^*) \\ +a_4 d(x^*, x_{2n_k+1}) + a_5 d(x_{2n_k}, x^*) + a_6 \dfrac{d(x_{2n_k}, x_{2n_k+1})(1 + d(x_{2n_k}, x_{2n_k+1}))}{1 + d(x_{2n_k}, x^*)} \end{array}\right].
\end{aligned}$$

Taking limit as $n \to \infty$,on both sides of the above inequality, we have

$$d(x^*, Sx^*) \leq s\left[a_2 d(x^*, Sx^*) + a_3 d(x^*, Sx^*)\right]$$

and hence $(1 - s(a_2 + a_3))d(x^*, Sx^*) \leq 0$, which gives that $d(x^*, Sx^*) = 0$, that is, $x^* \in S$. By using (1.2), Lemmas 1.6, and 1.8, we obtain that

$$\begin{aligned}
& d(x^*, Tx^*) \\
\leq\ & s\left[d(x^*, x_{2n_k+2}) + d(x_{2n_k+2}, Tx^*)\right] \\
\leq\ & s\left[d(x^*, x_{2n_k+2}) + H(Sx_{2n_k+1}, Tx^*)\right] \\
\leq\ & s\left[d(x^*, x_{2n_k+2}) + \alpha(x_{2n_k+1}, x^*)H(Sx_{2n_k+1}, Tx^*)\right] \\
\leq\ & s\left[\begin{array}{l} d(x^*, x_{2n_k+2}) + a_1 d(x_{2n_k+1}, Sx_{2n_k+1}) + a_2 d(x^*, Tx^*) + a_3 d(x_{2n_k+1}, Tx^*) \\ +a_4 d(x^*, Sx_{2n_k+1}) + a_5 d(x_{2n_k+1}, x^*) + a_6 \dfrac{d(x_{2n_k+1}, Sx_{2n_k+1})(1 + d(x_{2n_k+1}, Sx_{2n_k+1}))}{1 + d(x_{2n_k+1}, x^*)} \end{array}\right] \\
\leq\ & s\left[\begin{array}{l} d(x^*, x_{2n_k+2}) + a_1 d(x_{2n_k+1}, x_{2n_k+2}) + a_2 d(x^*, Tx^*) + a_3 d(x_{2n_k+1}, Tx^*) \\ +a_4 d(x^*, x_{2n_k+2}) + a_5 d(x_{2n_k+1}, x^*) + a_6 \dfrac{d(x_{2n_k+1}, x_{2n_k+2})(1 + d(x_{2n_k+1}, x_{2n_k+2}))}{1 + d(x_{2n_k+1}, x^*)} \end{array}\right].
\end{aligned}$$

Taking limit as $n \to \infty$ on both sides of the above inequality, we have

$$d(x^*, Tx^*) \leq s\left[a_2 d(x^*, Tx^*) + a_3 d(x^*, Tx^*)\right],$$

Hence $(1 - s(a_2 + a_3))d(x^*, Tx^*) \leq 0$. This gives that $d(x^*, Tx^*) = 0$, that is, $x^* \in T$. This shows that there exists $x^* \in X$ such that $x^* \in Tx^* \cap Sx^*$. Following arguments similar to those given above, the same result follows if we take $(x^*, x_{n_k}) \in \mathscr{R}$ for all $k \in \mathbb{N}_0$. This completes the proof. □

Theorem 2.2. *Let $(X, d, \mathscr{R})$ be $\mathscr{R}$-complete b-metric space and $T, S : X \to CB(X)$ be $(\mathscr{R}, \alpha)$-generalized rational contraction. If the following conditions hold:*

- **($\mathbf{A}_1$):** $X(T, \mathscr{R})$ is nonempty;
- **($\mathbf{A}_2$):** (T, S) and (S, T) are $\mathscr{R}$-closed;
- **($\mathbf{A}_3$):** for an $\mathscr{R}$-preserving sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$ for some $x \in X$, we have $(x_n, x) \in \mathscr{R}$,
- **:** then there exists a point $x^* \in X$ such that $x^* \in Tx^* \cap Sx^*$.

Proof. Following arguments similar to those in proof of Theorem 2.1, we obtain that $\{x_n\}$ is a Cauchy sequence in X. By $\mathscr{R}$-completeness of X, there exists $x^* \in X$ such that $\lim_{n\to\infty} d(x_n, x^*) = 0$. By condition (A_3), we have $(x_n, x^*) \in \mathscr{R}$. Also $\alpha(x_n, x^*) \geq 1$. By Lemma 1.8 and (1.1), we obtain that

$$\begin{aligned}
d(x^*, Sx^*) &\leq s\left[d(x^*, x_{2n+1}) + d(x_{2n+1}, Sx^*)\right] \\
&\leq s\left[d(x^*, x_{2n+1}) + H(Tx_{2n}, Sx^*)\right] \\
&\leq s\left[d(x^*, x_{2n+1}) + \alpha(x_{2n}, x^*)H(Tx_{2n}, Sx^*)\right] \\
&\leq s\left[\begin{array}{l} d(x^*, x_{2n+1}) + a_1 d(x_{2n}, Tx_{2n}) + a_2 d(x^*, Sx^*) + a_3 d(x_{2n}, Sx^*) \\ +a_4 d(x^*, Tx_{2n}) + a_5 d(x_{2n}, x^*) + a_6 \dfrac{d(x_{2n}, Tx_{2n})(1 + d(x_n, Tx_{2n}))}{1 + d(x_{2n}, x^*)} \end{array}\right] \\
&\leq s\left[\begin{array}{l} d(x^*, x_{2n+1}) + a_1 d(x_{2n}, x_{2n+1}) + a_2 d(x^*, Sx^*) + a_3 d(x_{2n}, Sx^*) \\ +a_4 d(x^*, x_{2n+1}) + a_5 d(x_{2n}, x^*) + a_6 \dfrac{d(x_{2n}, x_{2n+1})(1 + d(x_{2n}, x_{2n+1}))}{1 + d(x_{2n}, x^*)} \end{array}\right].
\end{aligned}$$

Taking limit as $n \to \infty$ on both sides of the above inequality, we obtain that

$$d(x^*, Sx^*) \leq s\left[a_2 d(x^*, Sx^*) + a_3 d(x^*, Sx^*)\right].$$

Hence $(1 - s(a_2 + a_3))d(x^*, Sx^*) \leq 0$ which implies that $d(x^*, Sx^*) = 0$, that is, $x^* \in S$. By following the similar argument as above and using Lemma 1.8 and (1.2), we have $x^* \in T$. This shows that there exists a point $x^* \in X$ such that $x^* \in Tx^* \cap Sx^*$. □

Remark 2.3. In Theorem 2.2 a pair (T, S) and (S, T) are not necessarily $\mathscr{R}$-continuous or $\mathscr{R}$ is not necessarily d-self-closed.

Corollary 2.4. *Let $(X, d, \mathscr{R})$ be $\mathscr{R}$-complete b-metric space and $T, S : X \to CB(X)$ be $\mathscr{R}$ generalized rational contraction. Suppose that the following conditions hold:*

- **(A_1):** $X(T, \mathscr{R})$ is nonempty;
- **(A_2):** (T, S) and (S, T) are $\mathscr{R}$-closed;
- **(A_3):** either (T, S) and (S, T) are $\mathscr{R}$-continuous or $\mathscr{R}$ is d-self-closed or for an $\mathscr{R}$-preserving sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$ for some $x \in X$, we have $(x_n, x) \in \mathscr{R}$.
- **:** Then there exists a point $x^* \in X$ such that $x^* \in Tx^* \cap Sx^*$.

Proof. Define the mapping $\alpha : X \times X \to [0, \infty)$ by $\alpha(x, y) = \begin{cases} 1 \text{ if } (x, y) \in \mathscr{R}, \\ 0 \text{ if } (x, y) \notin \mathscr{R}. \end{cases}$ for all $x, y \in X$. The result follows from Theorem 2.1 and Theorem 2.2. □

Theorem 2.5. *Let $(X, d, \mathscr{R})$ be $\mathscr{R}$-complete b-metric space and $T : X \to CB(X)$ an $(\mathscr{R}, \alpha)$-generalized rational contraction. Suppose that the following conditions hold:*

- **(A_1):** $X(T, \mathscr{R})$ is nonempty;
- **(A_2):** T is $\mathscr{R}$-closed;
- **(A_3):** either T is $\mathscr{R}$-continuous or $\mathscr{R}$ is d-self-closed or for an $\mathscr{R}$-preserving sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$ for some $x \in X$, we have $(x_n, x) \in \mathscr{R}$.
- **:** Then there exists a point $x^* \in X$ such that $x^* \in Tx^*$.

Proof. The result follows from Theorem 2.1 and Theorem 2.2 by choosing $S = T$. □

Corollary 2.6. *Let $(X,d,\mathscr{R})$ be $\mathscr{R}$-complete b-metric space and $T : X \to CB(X)$ an $\mathscr{R}$ generalized rational contraction. Suppose that the following conditions hold:*

(A_1): $X(T,\mathscr{R})$ is nonempty;
(A_2): T is $\mathscr{R}$-closed;
(A_3): either T is $\mathscr{R}$-continuous or $\mathscr{R}$ is d-self-closed or for an $\mathscr{R}$-preserving sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$ for some $x \in X$, we have $(x_n,x) \in \mathscr{R}$.
: Then there exists a point $x^* \in X$ such that $x^* \in Tx^*$.

Proof. Define the mapping $\alpha : X \times X \to [0,\infty)$ as $\alpha(x,y) = \begin{cases} 1 \text{ if } (x,y) \in \mathscr{R}, \\ 0 \text{ if } (x,y) \notin \mathscr{R}. \end{cases}$ for all $x,y \in X$. The result follows from Theorem 2.5. □

Example 2.7. Let $X = \{0,1,2\}$. Define a binary relation $\mathscr{R}$ on X by

$$\mathscr{R} = \{(0,0),(0,1),(1,0)\}.$$

Let $d : X \times X \to [0,\infty)$ be defined by

$$\begin{aligned} d(0,0) &= d(1,1) = d(2,2) = 0, \\ d(0,1) &= d(1,0) = d(0,2) = d(2,0) = 1 \\ d(1,2) &= d(2,1) = 4. \end{aligned}$$

Since $4 = d(1,2) \not\leq d(1,0) + d(0,2) = 2$, we have that d is a b-metric with $s = 2$. Define the mapping $T : X \to 2^X$ and $\alpha : X \times X \to [0,\infty)$ by

$$Tx = \begin{cases} \{0, \frac{x}{2}\}, \text{ if } x \in \{0,2\}, \\ \{2x\}, \text{ otherwise} \end{cases}$$

and

$$\alpha(x,y) = \begin{cases} 1, \text{ if } (x,y) \in \mathscr{R}, \\ 1/4, \text{ if } (x,y) \notin \mathscr{R}. \end{cases}$$

Consider $a_1 = a_2 = \frac{1}{4}$, $a_3 = \frac{1}{12}$, $a_4 = a_6 = 0$ and $a_5 = \frac{1}{8}$. Now, if $(x,y) \in \mathscr{R}$, then we have

$$\begin{aligned} H(T0,T0) &= 0 \leq a_1 d(0,T0), \\ H(T0,T1) &= 1 \leq \frac{1}{4} d(1,T1) = a_2 d(1,T1), \\ H(T1,T0) &= 1 \leq \frac{1}{4} d(1,T0) = a_1 d(1,T0). \end{aligned}$$

Thus for any $(x,y) \in \mathscr{R}$ the contraction (1.3) holds. Hence T is $(\mathscr{R},\alpha)$-generalized rational contraction on X. The mapping T is not $\mathscr{R}$-closed. Indeed, $(0,1) \in \mathscr{R}$ but $(0,2) \notin \mathscr{R}$, where $0 \in T0 = \{0\}$ and $2 \in T1 = \{2\}$. Therefore, Theorem 2.5 is not applicable but 0 is a fixed point of T.

Example 2.8. Let $X = [0,\infty)$ be equipped with a b-metric d given by $d(x,y) = |x-y|^2$ with $s = 2$. Define a binary relation $\mathscr{R}$ on X by

$$\mathscr{R} = \{(x,y) \in X^2 : x \in [0, \frac{1}{2}] \text{ and } y = 0\}.$$

Hence $(X,d,\mathscr{R})$ is $\mathscr{R}$-complete b-metric space. Define the mappings $T: X \to CB(X)$ and $\alpha: X \times X \to [0,\infty)$ as:

$$Tx = \begin{cases} [0,\frac{x}{4}], \text{ if } x \in [0,\frac{1}{2}], \\ [\frac{x}{2},x], \text{ if } x \in (\frac{1}{2},1], \\ \{x\}, \text{ otherwise} \end{cases}$$

and

$$\alpha(x,y) = \begin{cases} 1, \text{ if } (x,y) \in \mathscr{R}, \\ 0, \text{ if } (x,y) \notin \mathscr{R}. \end{cases}$$

Consider $a_1 = a_2 = a_5 = \frac{1}{6}$ and $a_3 = a_4 = a_6 = \frac{1}{12}$. For $(x,y) \in \mathscr{R}$, the mapping T satisfies (1.3). Indeed, $(x,y) \in \mathscr{R}$ implies $x \in [0,\frac{1}{2}]$ and $y = 0$. We have

$$\begin{aligned} &H(Tx,Ty) = H([0,\tfrac{x}{4}],T0) = H([0,\tfrac{x}{4}],\{0\}) \\ &= \left|\tfrac{x}{4} - 0\right|^2 = \tfrac{1}{16}x^2 \leq \tfrac{1}{6}\left|x - \tfrac{x}{4}\right|^2 = a_1 d(x,Tx). \end{aligned}$$

Thus, T is $(\mathscr{R},\alpha)$-generalized rational contraction on X. Take $x_0 = 0$ and $x_1 = 0 \in T0 = \{0\}$, which implies that $(x_0,x_1) = (0,0) \in \mathscr{R}$. Hence $X(T,\mathscr{R})$ is nonempty. If $(x,y) \in \mathscr{R}$, then $x \in [0,\frac{1}{2}]$ and $y = 0$. Hence $Tx = [0,\frac{x}{4}] \subseteq [0,\frac{1}{8}] \subseteq [0,\frac{1}{2}]$ and $T0 = \{0\}$. As a result $(u,0) \in \mathscr{R}$ for $u \in Tx$ and $0 \in T0$. This shows that T is $\mathscr{R}$-closed. Let $\{x_n\}$ be $\mathscr{R}$-preserving sequence with $x_n \to x \in X$ as $n \to \infty$. This implies that $(x_n,x_{n+1}) \in \mathscr{R}$, that is, $\{x_n\} \subset [0,\frac{1}{2}]$ and $x_{n+1} = 0$. As $x_{n+1} \to x$, we have $x = 0$. It follows that

$$H(Tx_n,T0) = H([0,\frac{x_n}{4}],\{0\}) = \left|\frac{x_n}{4} - 0\right|^2 \to 0$$

as $n \to \infty$. Hence T is $\mathscr{R}$-continuous. Let $\{x_n\}$ be $\mathscr{R}$-preserving sequence in X with $x_n \to x \in X$ as $n \to \infty$. We have $(x_n,x_{n+1}) \in \mathscr{R}$, that is, $\{x_n\} \subset [0,\frac{1}{2}]$ and $x_{n+1} = 0$. As $[0,\frac{1}{2}]$ is closed, we have $[x_n,x] \in \mathscr{R}$. Hence $\mathscr{R}$ is d-self-closed. Also as $x_{n+1} \to x$, we have $x = 0$. It follows that $(x_n,0) = (x_n,x) \in \mathscr{R}$ for all $n \in \mathbb{N}_0$. Thus, all the conditions of Theorem 2.5 are satisfied. However, T has many fixed points in X.

Corollary 2.9. *Let (X,d) be complete b-metric space and $T: X \to CB(X)$. If for any $x,y \in X$, then the following condition holds:*

$$\begin{aligned} H(Tx,Ty) \quad \leq \quad & a_1 d(x,Tx) + a_2 d(y,Ty) + a_3 d(x,Ty) + a_4 d(y,Tx) + a_5 d(x,y) \\ & + a_6 \frac{d(x,Tx)(1+d(x,Tx))}{1+d(x,y)}, \end{aligned} \tag{2.10}$$

where $a_i \geq 0$, $i = 1,2,\ldots,6$ with $a_1 + a_2 + 2sa_3 + a_4 + a_5 + a_6 < 1$ and $a_1 + sa_3 + a_5 + a_6 > 0$. Then there exists a point $x^* \in X$ such that $x^* \in Tx^*$.

Proof. Define the relation $\mathscr{R}$ on X by $\mathscr{R} = X \times X$, that is, the universal relation on X and $\alpha(x,y) = 1$ for all $x,y \in X$. Then, all the conditions of Theorem 2.5 are satisfied and hence the mapping T has a fixed point in X. □

Remark 2.10. Let $\lambda,\beta,\gamma,\delta$, and μ be nonnegative constants and $x,y \in X$, where X is a complete b-metric space. We set

$$\begin{aligned} C \quad &= \quad \{T: X \to CB(X): \ T \text{ satisfied (2.10)}\} \\ &= \quad \text{The class of generalized rational contractions} \end{aligned}$$

and

$$\begin{aligned} C_1 &= \{T : X \to CB(X) : H(Tx,Ty) \leq \lambda d(x,y) \text{ with } \lambda \in [0,1)\ \} \\ &= \text{The class of Nadler contraction mappings,} \\ C_2 &= \{T : X \to CB(X) : H(Tx,Ty) \leq \beta\,[d(x,Tx)+d(y,Ty)] \text{ with } \beta \in [0,\frac{1}{2})\ \} \\ &= \text{The class of multivalued Kannan mappings,} \\ C_3 &= \{T : X \to CB(X) : H(Tx,Ty) \leq \beta\,[d(x,Ty)+d(y,Tx)] \text{ with } \beta \in [0,\frac{1}{2})\} \\ &= \text{The class of multivalued Chatterjea mappings,} \end{aligned}$$

$$\begin{aligned} C_4 &= \{T : X \to CB(X) : H(Tx,Ty) \\ &\leq \lambda d(x,y)+\beta d(x,Tx)+\gamma d(y,Ty) \text{ with } \lambda+\beta+\gamma<1\ \} \\ &= \text{The class of multivalued Reich mappings,} \\ C_5 &= \{T : X \to CB(X) : H(Tx,Ty) \\ &\leq \lambda d(x,y)+\beta d(x,Tx)+\gamma d(y,Ty)+\delta\,[d(x,Ty)+d(y,Tx)] \text{ with } \lambda+\beta+\gamma+\delta<1\} \\ &= \text{The class of multivalued Ciric mappings,} \\ C_6 &= \{T : X \to CB(X) : H(Tx,Ty) \\ &\leq \lambda d(x,y)+\beta d(x,Tx)+\gamma d(y,Ty)+\delta d(x,Ty)+\mu d(y,Tx) \text{ with } \lambda+\beta+\gamma+\delta+\mu<1\} \\ &= \text{The class of multivalued Hardy and Rogers mappings,} \\ C_7 &= \{T : X \to CB(X) : H(Tx,Ty) \leq \lambda \max\{d(x,Tx),d(y,Ty)\} \text{ with } \lambda \in [0,1)\ \} \\ &= \text{The class of multivalued Bianchini mappings.} \end{aligned}$$

The class C of mappings considered in Corollary 2.9 contains $C_1 - C_7$ as proper subclasses.

Remark 2.11. Relation theoretic version of Corollary 2.9 generalizes and extends relation theoretic versions of (i) Banach contraction principle (ii) Kannan fixed point theorem in [26]; (iii) Chatterjea fixed point theorem in [27]; (iv) Reich fixed point theorem in [28]; (v) Ćirić fixed point theorem in [29]; (vi) Hardy and Rogers fixed point theorem in [30]; (vii) Bianchini fixed point theorem in [30], from single valued to multivalued mappings. Note that, all these special cases of Corollary 2.9 are itself new results in the existing literature.

Example 2.12. Consider $X = [0,3]$ equipped with b-metric $d(x,y) = |x-y|^2$ and

$$\mathscr{R} = \{(0,0),(0,1),(0,\frac{1}{2}),(0,3)\}$$

a binary relation on X. Define mappings $T : X \to CB(X)$ and $\alpha : X \times X \to [0,\infty)$ as:

$$Tx = \begin{cases} \{x\}, \text{ if } x \in [0,1), \\ \{\frac{x}{2}\}, \text{ if } x \in [1,3), \\ \{1\}, \text{ if } x = 3. \end{cases}$$

and

$$\alpha(x,y) = \begin{cases} 1, \text{ if } (x,y) \in \mathscr{R}, \\ 0, \text{ if } (x,y) \notin \mathscr{R}. \end{cases}$$

Let $a_2 = a_3 = a_4 = a_5 = \frac{1}{8}$ and $a_1 = a_6 = 0$. For $(x,y) \in \mathscr{R}$, one can easily show that T is $(\mathscr{R},\alpha)$-generalized rational contraction. Note that (X,d) is $\mathscr{R}$-complete b-metric space with $s = 2$. Clearly, T is not $\mathscr{R}$-continuous. Take an $\mathscr{R}$-preserving sequence $\{x_n\}$ in X such that $x_n \xrightarrow{d} x$ and $(x_n, x_{n+1}) \in \mathscr{R}$ for all $n \in \mathbb{N}_0$. As $(x_n, x_{n+1}) \notin \{(0,3)\}$, $(x_n, x_{n+1}) \in \{(0,0),(0,1),(0,\frac{1}{2})\}$ for all $n \in \mathbb{N}_0$ which implies that $\{x_n\} \subset \{0,1,\frac{1}{2}\}$. Since $\{0,1,\frac{1}{2}\}$ is closed, we have $[x_n, x] \in \mathscr{R}$. Hence $\mathscr{R}$ is d-self-closed. Thus all the conditions (A_1)-(A_3) of Theorem 2.5 are satisfied. Moreover $x = 0$ is a fixed point of T.

Remark 2.13. (1) Note that in the above example 2.12, there is no $\beta \in [0, \frac{1}{2})$ satisfying

$$H(T0,T1) = H(\{0\},\{\frac{1}{2}\}) = \frac{1}{4} \leq \beta\,[d(0,T0) + d(1,T1)]\,.$$

Therefore, T is not Kannan contraction mapping and hence Kannan contraction theorem is not applicable.

(2) The relation $\mathscr{R}$ is not reflexive, irreflexive, symmetry, transitive, preordered, partial order. This shows that our result is more general than other recent and classical results in the literature.

(3) If $\mathscr{R}$ is a universal relation, then, all the conditions (A_1), (A_2) and (A_3) of Theorem 2.1, Theorem 2.2, Theorem 2.5, Corollary 2.4 and Corollary 2.6 trivially hold and hence the results generalize and extend several comparable results in the set up of b-metric spaces which itself are new in the literature.

Next, we give some special cases of Theorem 2.1, Theorem 2.2 and Theorem 2.5 in partially ordered b-metric spaces.

Definition 2.14. Let X be a nonempty set. Then $(X,d,\preceq)$ is called a partially ordered b-metric space if (X,d) is a b-metric space and $(X,\preceq)$ a partially ordered set.

Definition 2.15. Let $(X,d,\preceq)$ be a partially ordered b-metric space. Then the sequence $\{x_n\}$ in X is called $\preceq$-preserving if $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}_0$.

Corollary 2.16. *Let $(X,d,\preceq)$ be complete partially ordered b-metric space and $T,S : X \to CB(X)$ be $(\preceq,\alpha)$-generalized rational contraction. Suppose the following conditions hold:*

(A_1): there exists $x_0 \in X$ such that $x_0 \preceq x_1$ for some $x_1 \in Tx_0$;
(A_2): (T,S) and (S,T) are $\preceq$-closed;
(A_3): (T,S) and (S,T) are $\preceq$-continuous or $\preceq$ is d-self-closed or for any $\preceq$-preserving sequence $\{x_n\}$ in X and $x_n \to x$ for some $x \in X$, we have $x_n \preceq x$. Then there exists $x^* \in X$ such that $x^* \in Tx^* \cap Sx^*$.

Proof. Define the relation $\mathscr{R}$ on X by $\mathscr{R} = \{(x,y) \in X \times X : x \preceq y\}$. Condition (A_1) implies that, there exists $x_0 \in X$ such that $(x_0, x_1) \in \mathscr{R}$. Therefore $X(T,\mathscr{R})$ is nonempty. Thus all the conditions of Theorem 2.1 or Theorem 2.2 are satisfied. Hence the result. □

Corollary 2.17. *Let $(X,d,\preceq)$ be complete partially ordered b-metric space and $T : X \to CB(X)$ a $(\preceq,\alpha)$-generalized rational contraction. Suppose the following conditions hold:*

(A_1): there exists $x_0 \in X$ such that $x_0 \preceq x_1$ for some $x_1 \in Tx_0$;
(A_2): T is $\preceq$-closed;

(A_3): T is $\preceq$-continuous or $\preceq$ is d-self-closed or for any $\preceq$-preserving sequence $\{x_n\}$ in X and $x_n \to x$ for some $x \in X$, we have $x_n \preceq x$. Then there exists $x^* \in X$ such that $x^* \in Tx^*$.

Proof. Define the relation $\mathscr{R}$ on X by $\mathscr{R} = \{(x,y) \in X \times X : x \preceq y\}$. Hypothesis ($A_1$) implies that, there exists $x_0 \in X$ such that $(x_0, x_1) \in \mathscr{R}$. Therefore $X(T,\mathscr{R})$ is nonempty. Thus all the conditions of Theorem 2.5 are satisfied. Hence the result. □

Example 2.18. Let $X = [0,\infty)$. Define a partial order " $\preceq$ " as follows:

$$\preceq := \{(x,y) \in [0,1]^2 : x \leq y\}.$$

Let $d : X \times X \to [0,\infty)$ be the b-metric defined by

$$d(x,y) = |x-y|^3 \text{ for all } x,y \in X.$$

Then $(X,d,\preceq)$ is complete partially ordered b-metric space with $s = 4$. Define the mapping $T : X \to CB(X)$ and $\alpha : X \times X \to [0,\infty)$ as:

$$Tx = \begin{cases} \{\frac{x}{3}\}, \text{ if } x \in [0,1], \\ [x, 2x], \text{ if } x \in (1,\infty) \end{cases}$$

and

$$\alpha(x,y) = \begin{cases} 1, \text{ if } x \preceq y, \\ 0, \text{ otherwise.} \end{cases}$$

Clearly, T is $\preceq$-continuous and $(\preceq,\alpha)$-generalized rational contraction for the constants $a_1 = \frac{1}{8}$, $a_2 = \frac{1}{3}$, $a_3 = \frac{1}{32}$, $a_4 = a_6 = 0$ and $a_5 = \frac{1}{27}$. For $x_0 = 0$, we have $0 = x_1 \in Tx_0$ such that $x_0 \preceq x_1$. Thus, all the conditions (A_1)-(A_3) of Corollary 2.17 are satisfied. Moreover, T has infinitely many fixed points on X.

Remark 2.19. If we take $s = 1$, then all our results generalize several corresponding results in metric spaces which itself are new in the literature.

Corollary 2.20. *Let $(X,d,\mathscr{R})$ be a $\mathscr{R}$-complete b-metric space and $f,g : X \to X$ be $(\mathscr{R},\alpha)$-generalized rational contraction. Suppose that the following conditions hold:*

(A_1): $X(f,\mathscr{R})$ is nonempty;
(A_2): (f,g) and (g,f) are $\mathscr{R}$-closed;
(A_3): either (f,g) and (g,f) are $\mathscr{R}$-continuous or $\mathscr{R}$ is d-self-closed or for an $\mathscr{R}$-preserving sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$ for some $x \in X$, we have $(x_n, x) \in \mathscr{R}$.
: Then there exists a point $x^* \in X$ such that $x^* = fx^* = gx^*$.

Proof. Define the mapping $T,S : X \to CB(X)$ as $Tx = \{fx\}$ and $Sx = \{gx\}$ for $x \in X$. Hence T,S are $(\mathscr{R},\alpha)$-generalized rational contractions. Thus, all the conditions of Theorem 2.1 or Theorem 2.2 are satisfied and hence T and S have a common fixed point x^* in X. Hence $x^* \in Tx^* \cap Sx^* = \{fx^*\} \cap \{gx^*\}$, that is, $x^* = fx^* = gx^*$. □

Corollary 2.21. *Let $(X,d,\mathscr{R})$ be a $\mathscr{R}$-complete b-metric space and $f,g : X \to X$ be $\mathscr{R}$ generalized rational contraction. Suppose that the following conditions hold:*

(A_1): $X(f,\mathscr{R})$ is nonempty;
(A_2): (f,g) and (g,f) are $\mathscr{R}$-closed;

(A_3): either (f,g) and (g,f) are $\mathscr{R}$-continuous or $\mathscr{R}$ is d-self-closed or for an$\mathscr{R}$-preserving sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$ for some $x \in X$, we have $(x_n, x) \in \mathscr{R}$.

: Then there exists a point $x^* \in X$ such that $x^* = fx^* = gx^*$.

Proof. Define the mapping $T, S : X \to CB(X)$ as $Tx = \{fx\}$ and $Sx = \{gx\}$ for $x \in X$. Hence the result follows from Corollary 2.4. □

Corollary 2.22. *Let $(X,d,\mathscr{R})$ be a $\mathscr{R}$-complete b-metric space and $f : X \to X$ an $(\mathscr{R},\alpha)$-generalized rational contraction. Suppose that the following conditions hold:*

(A_1): $X(f,\mathscr{R})$ is nonempty;

(A_2): f is $\mathscr{R}$-closed;

(A_3): either f is $\mathscr{R}$-continuous or $\mathscr{R}$ is d-self-closed or for an $\mathscr{R}$-preserving sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$ for some $x \in X$, we have $(x_n, x) \in \mathscr{R}$.

: Then there exists $x^* \in X$ such that $x^* = fx^*$.

Proof. Hence the result follows from Corollary 2.20 by taking $f = g$. □

Remark 2.23. In the framework of a complete metric space, Corollary 2.22 extends and generalizes result of Alam et. al. [1, Theorem 3.1].

Corollary 2.24. *Let $(X,d,\preceq)$ be complete partially ordered b-metric space and $f,g : X \to X$ be $(\preceq,\alpha)$-generalized rational contraction. Suppose that the following conditions hold:*

(A_1): there exists $x_0 \in X$ such that $x_0 \preceq fx_0$;

(A_2): (f,g) and (g,f) are $\preceq$-closed;

(A_3): either (f,g) and (f,g) are $\preceq$-continuous or $\preceq$ is d-self-closed or for any $\preceq$-preserving sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$ for some $x \in X$, we have $x_n \preceq x$.

: Then there exists a point $x^* \in X$ such that $x^* = fx^* = gx^*$.

Proof. Define the relation $\mathscr{R}$ on X by $\mathscr{R} = \{(x,y) \in X \times X : x \preceq y\}$. The condition ($A_1$) implies that, there exists $x_0 \in X$ such that $(x_0, fx_0) \in \mathscr{R}$. Therefore $X(f,\mathscr{R})$ is nonempty. Hence the result follows from Corollary 2.20. □

Corollary 2.25. *Let $(X,d,\preceq)$ be a complete partially ordered b-metric space and $f : X \to X$ an $(\preceq,\alpha)$-generalized rational contraction. Suppose the following conditions hold:*

(A_1): there exists $x_0 \in X$ such that $x_0 \preceq fx_0$;

(A_2): f is $\preceq$-closed;

(A_3): either f is $\preceq$-continuous or $\preceq$ is d-self-closed or for any $\preceq$-preserving sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$ for some $x \in X$, we have $x_n \preceq x$.

: Then there exists $x^* \in X$ such that $x^* = fx^*$.

Proof. Define the relation $\mathscr{R}$ on X by $\mathscr{R} = \{(x,y) \in X \times X : x \preceq y\}$. Condition ($A_1$) implies that there exists $x_0 \in X$ such that $(x_0, fx_0) \in \mathscr{R}$. Therefore $X(f,\mathscr{R})$ is nonempty. Hence the result follows from Corollary 2.22. □

REFERENCES

[1] A. Alam and M. Imdad, Relation-Theoretic contraction principle, J. Fixed Point Theory Appl. 17 (2015), 693-702.

[2] T. V. An, L. Q. Tuyen and N.V. Dung, Stone-type theorem on b-metric spaces and applications, Topology Appl. 185 (2015), 50-64

[3] I. A. Bakhtin,The contraction principle in quasimetric spaces, Func. An., Gos. Ped. Inst. Unianowsk 30 (1989), 26-37.

[4] M. Boriceanu, Fixed point theory for multivalued generalized contraction on a set with two b-metrics, Stud. Univ. Babes-Bolyai, Math. 54 (2009), 3-14.

[5] S. Czerwik, Contraction mappings in b-metric spaces, Acta Mathematica et Informatica Universitatis Ostraviensis, 1 (1993), 5-11.

[6] S. Czerwik, Nonlinear set-valued contraction mappings in b-metric spaces, Atti del Seminario Matematico e Fisico dell' Università di Modena e Reggio Emilia 46 (1998), 263-276

[7] J. R. Roshan, V. Parvaneh and I. Altun, Some coincidence point results in ordered b-metric spaces and applications in a system of integral equations, Appl. Math. Comput. 226 (2014), 725-737.

[8] M. H. Shah and N. Hussain, Nonlinear contractions in partially ordered quasi b-metric spaces, Commun. Korean Math. Soc. 27 (2012), 117-128.

[9] C. X. Zhu, C. F. Chen and X. Zhang, Some results in quasi-b-metric-like spaces, J. Inequal. Appl. 2014 (2014), 437.

[10] I. Beg, M.A. Ahmed and H. A. Nafadi, Fixed points of £-fuzzy mappings in ordered b-metric spaces, J. Funct. Spaces, 2018 (2018), Article ID 5650242.

[11] N. V. Dung, N. T. T. Ly, V. D. Thinh and N. T. Hieu, Suzuki-type fixed point theorems for two maps in metric-type spaces, J. Nonlinear Anal. Optim. 4 (2013), 17-29.

[12] M. Boriceanu, M. Bota and A. Petrusel, Multivalued fractals in b-metric spaces, Cent. Eur. J. Math. 8 (2010), 367-377.

[13] S.B. Nadler Jr., Multivalued contraction mappings, Pac. J. Math. 30 (1969), 475-488.

[14] I. Beg and M. Abbas, Common fixed point of multivalued mappings satisfying generalized $\phi-$ contractive condition in ordered G- metric spaces. Indian J. Math. 56 (2014), 77—95.

[15] I. Beg and A. Azam, Fixed points of asymptotically regular multivalued mappings, J. Austral. Math. Soc. 53 (1992), 313-326.

[16] L. B. Ćirić, M. Abbas, M. Rajović and B. Ali, Suzuki type fixed point theorems for generalized multi-valued mappings on a set endowed with two b-metrics, Appl. Math. Comput. 219 (2012), 1712-1723.

[17] A. Latif, T. Husain, I. Beg, Fixed point of nonexpansive type and K-multivalued maps, Int. J. Math. Math. Sci. 17 (1994), 429-436.

[18] A. Latif, D.T. Luc, Variational relation problems: existence of solutions and fixed points of set-valued contraction mappings, Fixed Point Theory Appl. 2013 (2013), 315

[19] B. Samet, M. Turinici, Fixed point theorems on a metric space endowed with an arbitrary binary relation and applications, Commun. Math. Anal. 13 (2012), 82-97.

[20] A. C. M. Ran, M. C. B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc. 132 (2003), 1435-1443

[21] J. J. Nieto, R. Rodríguez-López, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, Order 22 (2005), 223-239.

[22] J. J. Nieto, R. Rodríguez-López, Existence and uniqueness of fixed point in partially oredered sets and applications to ordinary differential equations, Acta Math. Sin. (Engl.Ser.) 23 (2007), 2205-2212.

[23] M. Turinici, Ran-Reurings fixed point results in ordered metric space, Libertas Math. 31 (2011), 49-55.

[24] M. Turinici, Nieto-Lopez theorems in ordered metric space, Math. Student, 81 (2012), 219-229.

[25] R. D. Maddux, Relation algebras, Stud. Logic Found. Math. Elsevier B. V. Amsterdam, (2006).

[26] R. Kannan, Some results on fixed points, Bull. Calcutta Math. Soc. 60 (1968), 71-76.

[27] S. K. Chatterjea, Fixed-point theorems, Comptes Rendus de l'Académie Bulgare des Sciences 25 (1972), 727-730..

[28] S. Reich, Some remarks concerning contraction mappings, Canadian Math. Bull. 14 (1971), 121-124.

[29] L. B. Ćirić, Generalized contractions and fixed-point theorems, Publ. Inst. Math. (Beograd)(NS) 12 (1971), 19-26

[30] G. E. Hardy and T. D. Rogers, A generalization of a fixed point theorem of Reich, Canadian Math. Bull. 16 (1973), 201-206.