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THE UNIFORM GLOBAL WELL-POSEDNESS AND THE STABILITY OF THE 3D GENERALIZED MAGNETOHYDRODYNAMIC EQUATIONS WITH THE CORIOLIS FORCE

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Abstract. This paper deals with the Cauchy problem of the 3D generalized magnetohydrodynamic equations with the Coriolis force (GMHDC). By using the Fourier localization argument and the Littlewood-Paley theory, we obtain the uniform global well-posedness results with small initial data (u_0, b_0) belonging to the critical Fourier-Besov-Morrey spaces $\mathscr{F}\mathscr{N}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}(\mathbb{R}^3)$. Moreover, the stability of global solutions is also discussed.

Keywords. Magnetohydrodynamic equations; Fourier-Besov-Morrey space; Stability; Global solution.

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1. Introduction

We consider the generalized magnetohydrodynamic equations with the Coriolis force in the whole space $\mathbb{R}^3$

$$\begin{cases} u_t + u\cdot\nabla u + \mu(-\Delta)^\alpha u + \Omega e_3 \times u - b\cdot\nabla b + \nabla\pi = 0 \quad \text{in } (0,+\infty)\times\mathbb{R}^3, \\ \nabla\cdot u = 0, \qquad \nabla\cdot b = 0, \\ b_t + u\cdot\nabla b + \nu(-\Delta)^\alpha b - b\cdot\nabla u = 0 \quad \text{in } (0,+\infty)\times\mathbb{R}^3, \\ (u,b)|_{t=0} = (u_0,b_0), \end{cases} \tag{1.1}$$

where u denotes the velocity field of the fluid, π denotes the pressure function, b is the magnetic field, $\nu > 0$ is the magnetic diffusivity, $\mu > 0$ is the viscosity, $\nabla\cdot u = 0$ and $\nabla\cdot b = 0$ is the incompressible conditions, $\Omega \in \mathbb{R}$ denotes twice the speed of rotation around the vertical unit vector $e_3 = (0,0,1)$, and u_0 and b_0 denote the initial velocity and the initial magnetic field with $\nabla\cdot u_0 = 0$ and $\nabla\cdot b_0 = 0$, respectively. $(-\Delta)^\alpha$ is the pseudo-differential operator with symbol $|\xi|^{2\alpha}$. Mathematically, the equation (1.1) is used to explain why the earth has a non-zero large-scale magnetic field whose polarity turns out to invert over several hundred centuries when $\alpha = 1$. For this concept, we refer readers to [1] and the references therein.

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If $\Omega = 0$, then equation (1.1) becomes the generalized Magnetohydrodynamics equations, which consist to examine the magnetic properties of electrically conducting incompressible fluids. Let us take a time to briefly cite some recent results. Duvaut and Lions [2] developed a global Leray-Hopf weak solution to MHD system. Cao and Wu [3] showed global regularity of classical solutions for the MHD equations with magnetic diffusion and mixed partial dissipation. Besides, numerous notable successes have been achieved on the fundamental issues of the regularity criteria or blow-up criteria to system (1.1). For more results in this direction, see [4, 5, 6, 7, 8, 9, 10, 11, 12] and references therein.

When $b = 0$ and $\Omega \neq 0$, Wang and Wu [13] investigated the global well-posedness and the Gevrey class regularity of mild solutions to the 3D incompressible generalized Navier-Stokes equations with the Coriolis force in the Lei-Lin space $\mathscr{X}^{1-2\alpha}$ defined by

$$\mathscr{X}^{1-2\alpha} = \{u \in \mathscr{D}'(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\xi|^{1-2\alpha} |\hat{u}(\xi)| d\xi < +\infty\}.$$

Recently, Wang and Wu [14] established the global well-posedness of the generalized rotating magnetohydrodynamic equations in the Lei-Lin space $\mathscr{X}^{1-2\alpha}$.

The goal of this paper is to show the existence and the stability of an uniform global solution to the 3D generalized rotating magnetohydrodynamic equations (1.1) in the critical Fourier-Besov-Morrey spaces $\mathscr{F}\dot{\mathscr{N}}^{s}_{p,\lambda,q}(\mathbb{R}^3)$ with $s = 4 - 2\alpha + \frac{\lambda-3}{p}$. In fact, this space contains many classical spaces, e.g., the Fourier-Herz space $\dot{\mathscr{B}}^{s}_{q} = \mathscr{F}\dot{\mathscr{N}}^{s}_{1,0,q}$, the Fourier-Besov-Lebesgue space $\mathrm{F}\dot{\mathrm{B}}^{s}_{p,q} = \mathscr{F}\dot{\mathscr{N}}^{s}_{p,0,q}$ and the Lei-Lin's space $\mathscr{X}^{s} = \mathscr{F}\dot{\mathscr{N}}^{s}_{1,0,1}$. Motivated by the results [13, 14], we prove the uniform global existence in the sense that the smallness condition on the initial data is independent of the size of the speed of rotation Ω. Specifically, this paper generalizes the results of existence given in [14] from Lei-Lin's space $\mathscr{X}^{1-2\alpha}$ to Fourier Besov-Morrey spaces $\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}(\mathbb{R}^3)$.

2. Preliminaries and main results

In this section, we briefly mention some notations and give the fundamental properties of the Fourier-Besov-Morrey spaces that will be used in the rest of the paper. In the following, we first introduce the homogeneous decomposition of Littlewood-Paley decomposition.

Let χ, φ be two nonnegative smooth radial functions satisfying

$$\operatorname{supp}\varphi \subset \{\xi \in \mathbb{R}^n : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}, \quad \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^n \backslash \{0\},$$

$$\operatorname{supp}\chi \subset \{\xi \in \mathbb{R}^n : |\xi| \leq \frac{4}{3}\}, \quad \chi(\xi) + \sum_{j\geq 0} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^n.$$

We denote the space of tempered distributions by S' and we designate $\varphi_j(\xi) = \varphi(2^{-j}\xi)$. $\mathscr{P}$ is the set of all polynomials.

We introduce some frequency localization operators, such as,

$$\dot{\Delta}_j u = \varphi(2^{-j}D)u = 2^{jn} \int h(2^j y) u(x-y)\, dy,$$

$$\dot{S}_j u = \sum_{k\leq j-1} \dot{\Delta}_k u = \chi(2^{-j}D)u = 2^{jn} \int \tilde{h}(2^j y) u(x-y)\, dy,$$

where $h = \mathscr{F}^{-1}\varphi$ and $\tilde{h} = \mathscr{F}^{-1}\chi$.

First, we define an appropriate version of the classical Morrey spaces which are a complement to L^p spaces.

Definition 2.1. For $1 \le p < \infty$, $0 \le \lambda < n$, the Morrey spaces $\mathrm{M}_p^\lambda = \mathrm{M}_p^\lambda(\mathbb{R}^n)$ is the set of functions $f \in L^p_{loc}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathrm{M}_p^\lambda} = \sup_{x_0\in\mathbb{R}^n} \sup_{r>0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x_0,r))} < \infty, \tag{2.1}$$

where $B(x_0,r)$ is the ball in $\mathbb{R}^n$ with center x_0 and radius r.

The following lemma is devoted to some basic properties related to Morrey spaces, see [15].

Lemma 2.2. *Let $1 \le p_1, p_2, p_3 < \infty$ and $0 \le \lambda_1,\ \lambda_2,\ \lambda_3 < n$.*

(1) (Hölder's inequality) Let $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{\lambda_3}{p_3} = \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2}$. If $f_i \in \mathrm{M}_{p_i}^{\lambda_i}$ for $i = 1,2$, then $f_1 f_2 \in \mathrm{M}_{p_3}^{\lambda_3}$ and

$$\|f_1 f_2\|_{\mathrm{M}_{p_3}^{\lambda_3}} \le \|f_1\|_{\mathrm{M}_{p_1}^{\lambda_1}} \|f_2\|_{\mathrm{M}_{p_2}^{\lambda_2}}.$$

(2) (Young's inequality) Assume that $\varphi \in L^1$ and $g \in \mathrm{M}_{p_1}^{\lambda_1}$, then

$$\|\varphi * g\|_{\mathrm{M}_{p_1}^{\lambda_1}} \le \|\varphi\|_{L^1} \|g\|_{\mathrm{M}_{p_1}^{\lambda_1}}, \tag{2.2}$$

(3) (Bernstein-type inequality) Let $1 \le p_2 \le p_1 < \infty$ such that

$$\frac{n-\lambda_1}{p_1} \le \frac{n-\lambda_2}{p_2}.$$

If $\operatorname{supp}(\widehat{f}) \subset \{\xi \in \mathbb{R}^n : |\xi| \le A2^j\}$ then there is a constant $C > 0$ independent of f and j such that

$$\|(i\xi)^\gamma \widehat{f}\|_{\mathrm{M}_{p_2}^{\lambda_2}} \le C 2^{j|\gamma| + j(\frac{n-\lambda_2}{p_2} - \frac{n-\lambda_1}{p_1})} \|\widehat{f}\|_{\mathrm{M}_{p_1}^{\lambda_1}}, \tag{2.3}$$

where γ is a multi-index and $j \in \mathbb{Z}$.

Now, let us define the homogeneous Fourier-Besov-Morrey spaces.

Definition 2.3. Let $s \in \mathbb{R}$, $1 \le p < +\infty$, $1 \le q \le +\infty$, and $0 \le \lambda < n$, the space $\mathscr{F}\dot{\mathscr{N}}^s_{p,\lambda,q}(\mathbb{R}^n)$ is defined by

$$\mathscr{F}\dot{\mathscr{N}}^s_{p,\lambda,q}(\mathbb{R}^n) = \Big\{ u \in \mathscr{S}'(\mathbb{R}^n)/\mathscr{P};\quad \|u\|_{\mathscr{F}\dot{\mathscr{N}}^s_{p,\lambda,q}(\mathbb{R}^n)} < \infty \Big\},$$

where

$$\|u\|_{\mathscr{F}\dot{\mathscr{N}}^s_{p,\lambda,q}(\mathbb{R}^n)} = \begin{cases} \Big\{ \sum_{j\in\mathbb{Z}} 2^{jqs} \|\widehat{\dot{\Delta}_j u}\|^q_{\mathrm{M}_p^\lambda} \Big\}^{1/q} & for\ \ q < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js} \|\widehat{\dot{\Delta}_j u}\|_{\mathrm{M}_p^\lambda} & for\ \ q = \infty, \end{cases}$$

where $\mathscr{P}$ is the set of all polynomials on $\mathbb{R}^n$.

We notice that the homogeneous Fourier-Besov-Morrey spaces are some refined functional spaces which cover many classical spaces, such as; Fourier-Herz space $\dot{\mathscr{B}}^s_q$, Fourier-Besov-Lebesgue space $F\dot{B}^s_{p,q}$ and Lei-Lin's space χ^{-1}, thus such spaces are more suitable and more adapted for studying certain equations of fluid mechanics.

Now, we recall the definition of the mixed space-time spaces.

Definition 2.4. Let $s\in\mathbb{R}$, $1\le p<\infty$, $1\le q,\rho\le\infty$, $0\le\lambda<n$, and $I=[0,T)$, $T\in(0,\infty]$. The space-time norm is defined on $u(t,x)$ by

$$\|u(t,x)\|_{\mathscr{L}^\rho(I,\mathscr{F}\dot{\mathscr{N}}^s_{p,\lambda,q})}=\Big\{\sum_{j\in\mathbb{Z}}2^{jqs}\|\widehat{\dot{\Delta}_j u}\|^q_{L^\rho(I,\mathrm{M}_p^\lambda)}\Big\}^{1/q},$$

and denote by $\mathscr{L}^\rho(I,\mathscr{F}\dot{\mathscr{N}}^s_{p,\lambda,q})$ the set of distributions in $S'(\mathbb{R}\times\mathbb{R}^n)/\mathscr{P}$ with finite $\|.\|_{\mathscr{L}^\rho(I,\mathscr{F}\dot{\mathscr{N}}^s_{p,\lambda,q})}$ norm. Due to the Minkowski inequality, we have

$$\|u\|_{\mathscr{L}^\rho(I;\mathscr{F}\dot{\mathscr{N}}^s_{p,\lambda,q})}\le\|u\|_{L^\rho(I;\mathscr{F}\dot{\mathscr{N}}^s_{p,\lambda,q})},\ \text{if }\rho\le q\quad\text{and}\quad\|u\|_{L^\rho(I;\mathscr{F}\dot{\mathscr{N}}^s_{p,\lambda,q})}\le\|u\|_{\mathscr{L}^\rho(I;\mathscr{F}\dot{\mathscr{N}}^s_{p,\lambda,q})},\ \text{if }\rho\ge q,$$

where $\|u(t,x)\|_{L^\rho(I;\mathscr{F}\dot{\mathscr{N}}^s_{p,\lambda,q})}:=\Big(\int_I\|u(\tau,\cdot)\|^\rho_{\mathscr{F}\dot{\mathscr{N}}^s_{p,\lambda,q}}d\tau\Big)^{1/\rho}$.

Our first main result is on the global well-posedness result.

Theorem 2.5. *Let $1\le p<\infty$, $1\le q\le 2$, $0\le\lambda<3$, and $\frac12<\alpha\le\frac52+\frac{\lambda-3}{2p}$. Then there exists a constant $C_0(\alpha,p,q)$ such that, for any $(u_0,b_0)\in\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}$ satisfying $\nabla\cdot u_0=\nabla\cdot b_0=0$ and*

$$\|(u_0,b_0)\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}}\le C_0\min\{\mu,\nu\},$$

and the Cauchy problem (1.1) admits a unique global solution (u,b),

$$(u,b)\in\mathscr{C}\big([0,\infty);\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}\big)\cap\mathscr{L}^1\big([0,\infty),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q}\big),$$

and it satisfies

$$\begin{aligned}\Big\|(u,b)\Big\|_{\mathscr{L}^\infty\big([0,\infty);\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}\big)\cap\mathscr{L}^1\big([0,\infty),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q}\big)}&\\ \le 2\Big(1+\Big(\frac{16}{9}\Big)^\alpha\Big)\Big\|(u_0,b_0)\Big\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}}&.\end{aligned}$$

Now, we give some remarks about this result.

Remark 2.6. We not that for $\frac12\le\alpha\le 1$ and $b=0$, the system (1.1) becomes the fractional Navier-Stokes equations with the Coriolis force studied by Wang and Gang [13] in the Lei-Lin space $\mathscr{X}^{1-2\alpha}$. The proof of Theorem 2.5 is based on two methods, that is, the use of the dissipative equation and the classical semigroup approach.

Theorem 2.5 extends the result of [14] from Lei-Lin spaces to Fourier-Besov-Morrey spaces, since for $(\lambda=0,\,q=1,\,p=1)$, $\mathscr{F}\dot{\mathscr{N}}^s_{1,0,1}=\mathscr{X}^s$. In addition, Theorem 2.5 also holds in the Fourier-Besov-Lebesgue spaces since, for $\lambda=0$, $\mathscr{F}\dot{\mathscr{N}}^s_{p,0,q}=\mathrm{F}\dot{\mathrm{B}}^s_{p,q}$.

We also remark that the fractional magnetohydrodynamic equations with the Coriolis force is uniformly globally well-posed in the sense that the smallness conditions are independent of Ω.

The second result of this paper is to give the stability of global solutions.

Theorem 2.7. *Let T^* denote the maximal time of existence of a solution (u,b) in*

$$\mathscr{L}^\infty\Big([0,T^*);\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}\Big)\cap\mathscr{L}^1\Big([0,T^*),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q}\Big).$$

If $T^* < \infty$, then

$$\|(u,b)\|_{\mathscr{L}^1([0,T^*),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})} = \infty.$$

Besides, if $(u,b) \in C(\mathbb{R}^+, \mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})$ is a global solution of (1.1), and for all

$$(\tilde{u}_0, \tilde{b}_0) \in \mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}$$

such that

$$\begin{aligned} &\|\tilde{u}_0 - u_0\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}} + \|\tilde{b}_0 - b_0\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}} \\ &< C_0 \frac{\min\{\mu,\nu\}}{8} \exp\Big\{\int_0^\infty -\frac{1}{C_0}(|\Omega| + \|u\|_{\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q}} + \|b\|_{\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q}})\Big\}, \end{aligned} \tag{2.4}$$

for some constant C_0 sufficiently small, then

$$\begin{aligned} &\|\tilde{u}(t) - u(t)\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}} + \frac{\mu}{4}\|\tilde{u} - u\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})} \\ &\quad + \|\tilde{b}(t) - b(t)\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}} + \frac{\nu}{4}\|\tilde{b} - b\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})} \\ &\le C(\|\tilde{u}_0 - u_0\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}} + \|\tilde{b}_0 - b_0\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}}) \\ &\quad \times \exp\Big\{\int_0^\infty C(|\Omega| + \|u\|_{\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q}} + \|b\|_{\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q}})\Big\}, \end{aligned}$$

where C is a positive constant.

Remark 2.8. When $\alpha = 1$ and $\Omega = 0$, Wang [16] obtained the same result in the space χ^{-1}. Theorem 2.7 generalizes the stability of global solution of (1.1) to the Fourier-Besov-Morrey space $\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}$.

In what follows, we consider the linear dissipative equation

$$\begin{aligned} u_t + \mu(-\Delta)^\alpha u = f(t,x) \quad & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3 \\ u(0,x) = u_0(x) \quad & x \in \mathbb{R}^3, \end{aligned} \tag{2.5}$$

for which we give the following result.

Lemma 2.9 ([17])**.** *Let $I = [0,T)$, $0 < T \le \infty$, $s \in \mathbb{R}$, $0 \le \lambda < n$, $1 \le p < \infty$, and $1 \le q \le \infty$. If $u_0 \in \mathscr{F}\dot{\mathscr{N}}^s_{p,\lambda,q}$ and $f \in \mathscr{L}^1(I, \mathscr{F}\dot{\mathscr{N}}^s_{p,\lambda,q})$, then the solution $u(t,x)$ to the Cauchy problem (2.5) satisfies*

$$\begin{aligned} &\|u\|_{\mathscr{L}^\infty(I,\mathscr{F}\dot{\mathscr{N}}^s_{p,\lambda,q})} + \mu\|u\|_{\mathscr{L}^1(I,\mathscr{F}\dot{\mathscr{N}}^{s+2\alpha}_{p,\lambda,q})} \\ &\le (1 + (\frac{16}{9})^\alpha)(\|u_0\|_{\mathscr{F}\dot{\mathscr{N}}^s_{p,\lambda,q}} + \|f\|_{\mathscr{L}^1(I,\mathscr{F}\dot{\mathscr{N}}^s_{p,\lambda,q})}). \end{aligned} \tag{2.6}$$

If q is finite, then u belongs to $\mathscr{C}(I, \mathscr{F}\dot{\mathscr{N}}^s_{p,\lambda,q})$.

Lemma 2.10 ([17])**.** *Let $1 \le p < \infty$, $1 \le \rho \le \infty$ $1 \le q \le 2$, $0 \le \lambda < 3$, $I = [0,T)$, $0 < T \le +\infty$, $\frac{1}{2} < \alpha < \frac{5+\frac{\lambda-3}{p}}{4-\frac{2}{\rho}}$ and set*

$$X = \mathscr{L}^\infty(I, \mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}) \cap \mathscr{L}^\rho(I, \mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{2\alpha}{\rho}+\frac{\lambda-3}{p}}_{p,\lambda,q}),$$

with the norm

$$\|u\|_X=\|u\|_{\mathscr{L}^\infty(I,\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}+\min\{\mu,\nu\}\|u\|_{\mathscr{L}^\rho(I,\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{2\alpha}{\rho}+\frac{\lambda-3}{p}}_{p,\lambda,q})}.$$

There exists a constant $C=C(\alpha,p,q)>0$ depending on α,p,q such that

$$\|\nabla.(u\otimes v)\|_{\mathscr{L}^\rho(I,\mathscr{F}\dot{\mathscr{N}}^{4-4\alpha+\frac{2\alpha}{\rho}+\frac{\lambda-3}{p}}_{p,\lambda,q})}\leq C(\min\{\mu,\nu\})^{-1}\|u\|_X\|v\|_X.$$

3. The Well-Posedness

First, we observe that the equation (1.1) reduces to the fractional Navier-Stokes equations with the Coriolis force when the magnetic field $b=0$. We therefore introduce the corresponding generalized Stokes-Coriolis semigroup. More precisely, we investigate the following linear Stokes problem with the Coriolis force

$$\begin{cases} u_t+\mu(-\Delta)^\alpha u+\Omega e_3\times u+\nabla\pi=0 \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3,\\ \nabla.u=0,\\ u(0,x)=u_0(x) \;\; x\in\mathbb{R}^3.\end{cases} \tag{3.1}$$

The solution of equation (3.1) can be obtained from the generalized Stokes-Coriolis semigroup $T_{\Omega,\alpha}$ [18, 19, 20], which is represented as

$$T_{\Omega,\alpha}(t)u=\mathscr{F}^{-1}[\cos(\Omega\frac{\xi_3}{|\xi|}t)I+\sin(\Omega\frac{\xi_3}{|\xi|}t)R(\xi)]*(e^{-\mu(-\Delta)^\alpha t}u),$$

for divergence-free vector fields u and $t\geq 0$. Here, I represents the identity matrix in $\mathbb{R}^3$ and $R(\xi)$ is the skew-symmetric matrix symbol for the Riesz transform, which is expressed by

$$R(\xi):=\frac{1}{|\xi|}\begin{pmatrix}0&\xi_3&-\xi_2\\-\xi_3&0&\xi_1\\\xi_2&-\xi_1&0\end{pmatrix}.$$

Thus, we are easy to get a semigroup

$$\mathscr{A}_{\Omega,\alpha}(t)=\begin{pmatrix}T_{\Omega,\alpha}(t)&0\\0&S_\alpha(t)\end{pmatrix},$$

where $S_\alpha(t):=e^{-\mu(-\Delta)^\alpha t}=\mathscr{F}^{-1}(e^{-\mu|\xi|^{2\alpha}t})$.

Now we can rewrite GMHDC equation (1.1) in the form of the integral

$$\begin{pmatrix}u\\b\end{pmatrix}=\mathscr{A}_{\Omega,\alpha}(t)\begin{pmatrix}u_0\\b_0\end{pmatrix}-\int_0^t\mathscr{A}_{\Omega,\alpha}(t-\tau)\mathbb{P}\begin{pmatrix}\nabla\cdot(u\otimes u-b\otimes b)\\\nabla\cdot(u\otimes b-b\otimes u)\end{pmatrix}(\cdot,\tau)\,d\tau,$$

where $\mathbb{P}=Id-\nabla\Delta^{-1}div$ is the Leray-Hopf projector.

The following estimate corresponds to the Stokes-Coriolis semigroup $T_{\Omega,\alpha}$.

Lemma 3.1. *Let $0<T\leq\infty$, $s\in\mathbb{R}$, $0\leq\lambda<3, 1\leq p<\infty$, $1\leq q,\rho,r\leq\infty$ and*

$$f\in\mathscr{L}^r([0,T),\mathscr{F}\dot{\mathscr{N}}^s_{p,\lambda,q}(\mathbb{R}^3)).$$

There exists a constant $C>0$ such that

$$\|\textstyle\int_0^t T_{\Omega,\alpha}(t-\tau)f(\tau)d\tau\|_{\mathscr{L}^\rho([0,T),\mathscr{F}\dot{\mathscr{N}}^s_{p,\lambda,q})}\leq C\|f\|_{\mathscr{L}^r([0,T),\mathscr{F}\dot{\mathscr{N}}^{s-2\alpha-\frac{2\alpha}{\rho}+\frac{2\alpha}{r}}_{p,\lambda,q})}.$$

Besides, if $u_0 \in \mathscr{F}\mathscr{N}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}(\mathbb{R}^3)$, *then there exists a constant* $C > 0$ *such that*

$$\|T_{\Omega,\alpha}(t)u_0\|_{\mathscr{L}^\infty([0,T),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})} \le C\|u_0\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}}, \tag{3.2}$$

$$\|T_{\Omega,\alpha}(t)u_0\|_{\mathscr{L}^1([0,T),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})} \le C\|u_0\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}},. \tag{3.3}$$

Proof. The definition of the space-time norm of $\mathscr{L}^\rho([0,T),\mathscr{F}\dot{\mathscr{N}}^s_{p,\lambda,q})$ and Young's inequality give

$$\begin{aligned}
&\Big\|\int_0^t T_{\Omega,\alpha}(t-\tau)f(\tau)d\tau\Big\|_{\mathscr{L}^\rho([0,T),\mathscr{F}\dot{\mathscr{N}}^s_{p,\lambda,q})}\\
&\le \Big\{\sum_{j\in\mathbb{Z}} 2^{jqs}\Big(\int_0^T \|\varphi_j\int_0^t e^{-\mu|\xi|^{2\alpha}(t-\tau)}\hat{f}(\tau)d\tau\|^\rho_{\mathrm{M}^\lambda_p}dt\Big)^{\frac{q}{\rho}}\Big\}^{1/q}\\
&\le \Big\{\sum_{j\in\mathbb{Z}} 2^{jqs}\Big(\int_0^T \|\varphi_j\int_0^t e^{-\mu 2^{2\alpha j}(t-\tau)}\hat{f}(\tau)d\tau\|^\rho_{\mathrm{M}^\lambda_p}dt\Big)^{\frac{q}{\rho}}\Big\}^{1/q}\\
&\le C\Big\{\sum_{j\in\mathbb{Z}} 2^{jq(s-2\alpha-\frac{2\alpha}{\rho}+\frac{2\alpha}{r})}\|\varphi_j\hat{f}(\tau)\|^q_{L^r([0,T),\mathrm{M}^\lambda_p)}\Big\}^{1/q}\\
&\le C\|f\|_{\mathscr{L}^r([0,T),\mathscr{F}\dot{\mathscr{N}}^{s-2\alpha-\frac{2\alpha}{\rho}+\frac{2\alpha}{r}}_{p,\lambda,q})},
\end{aligned}$$

where we have used $1+\frac{1}{\rho} = \frac{1}{\bar{\rho}}+\frac{1}{r}$. To show inequality (3.2), it is sufficient to write that

$$\begin{aligned}
&\|T_{\Omega,\alpha}(t)u_0\|_{\mathscr{L}^\infty([0,T),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}\\
&\le C\Big(\sum_{j\in\mathbb{Z}} 2^{j(4-2\alpha+\frac{\lambda-3}{p})q}\|\varphi_j\hat{u}_0\|^q_{\mathrm{M}^\lambda_p}\Big)^{\frac{1}{q}}\\
&\le C\|u_0\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}}.
\end{aligned}$$

Similarly, for proving (3.3), we write

$$\begin{aligned}
&\|T_{\Omega,\alpha}(t)u_0\|_{\mathscr{L}^1([0,T),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})}\\
&\le \Big(\sum_{j\in\mathbb{Z}} 2^{j(4+\frac{\lambda-3}{p})q}\Big(\int_0^T e^{-t\mu 2^{2\alpha j}}\|\varphi_j\hat{u}_0\|_{\mathrm{M}^\lambda_p}dt\Big)^q\Big)^{\frac{1}{q}}\\
&\le C\|u_0\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}}.
\end{aligned}$$

□

Proof of Theorem 2.5. In usual practice, the mild solution (u,b) for GMHDC equation (1.1) can be rewritten as follows

$$\begin{aligned}
u &= T_{\Omega,\alpha}(t)u_0 - \int_0^t T_{\Omega,\alpha}(t-\tau)\mathbb{P}\nabla\cdot(u\otimes u - b\otimes b)(\cdot,\tau)\,d\tau := \mathscr{T}_1(u,b),\\
b &= e^{-t\nu(-\Delta)^\alpha}b_0 - \int_0^t e^{-\nu(t-\tau)(-\Delta)^\alpha}\mathbb{P}\nabla\cdot(u\otimes b - b\otimes u)(\cdot,\tau)\,d\tau := \mathscr{T}_2(u,b),
\end{aligned} \tag{3.4}$$

where $\mathbb{P} = Id - \nabla\Delta^{-1}div$ is the Leray-Hopf projector.

Let

$$X = \mathscr{L}^\infty\Big([0,\infty);\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}\Big)\cap\mathscr{L}^1\Big([0,\infty),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q}\Big).$$

We define the norm of the vector (u,b) for $u,b\in X$, as follows

$$\|(u,b)\|_X=\|u\|_X+\|b\|_X.$$

Let

$$L_\mu(u,v):=\int_0^t T_{\Omega,\alpha}(t-\tau)\mathbb{P}\nabla\cdot(u\otimes v)(\cdot,\tau)\,d\tau.$$

It is easy to rewrite the system (3.4) as follows

$$(u,b)=(\mathscr{T}_1(u,b),\mathscr{T}_2(u,b)):=\mathscr{T}(u,b).$$

Lemma 3.1 and Lemma 2.10 lead to

$$\begin{aligned}
&\|L_\mu(u,u)-L_\mu(b,b)\|_{\mathscr{L}^1([0,\infty),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})}\\
&=\Big\|\int_0^t T_{\Omega,\alpha}(t-\tau)\mathbb{P}\nabla.(u\otimes u-b\otimes b)(\tau)d\tau\Big\|_{\mathscr{L}^1([0,\infty),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})}\\
&\le C\|\nabla.(u\otimes u-b\otimes b)\|_{\mathscr{L}^1([0,\infty),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}\\
&\le C(\min\{\mu,\nu\})^{-1}(\|u\|_X^2+\|b\|_X^2).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\|L_\mu(u,u)-L_\mu(b,b)\|_{\mathscr{L}^\infty([0,\infty),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}\\
&=\Big\|\int_0^t T_{\Omega,\alpha}(t-\tau)\mathbb{P}\nabla.(u\otimes u-b\otimes b)(\tau)d\tau\Big\|_{\mathscr{L}^\infty([0,\infty),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}\\
&\le C\|\nabla.(u\otimes u-b\otimes b)\|_{\mathscr{L}^1([0,\infty),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}\\
&\le C(\min\{\mu,\nu\})^{-1}(\|u\|_X^2+\|b\|_X^2).
\end{aligned}$$

Finally,

$$\|L_\mu(u,u)-L_\mu(b,b)\|_X\le C(\min\{\mu,\nu\})^{-1}(\|u\|_X^2+\|b\|_X^2). \tag{3.5}$$

Lemma 3.1 yields

$$\|T_{\Omega,\alpha}(t)u_0\|_X\le C\|u_0\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}}. \tag{3.6}$$

The estimates (3.5) and (3.6) allow us to obtain

$$\begin{aligned}\|\mathscr{T}_1(u,b)\|_X\le C\|u_0\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}}&\\
+C(\min\{\mu,\nu\})^{-1}\big(\|u\|_X^2+\|b\|_X^2\big).&\end{aligned} \tag{3.7}$$

Similarly, let

$$L_\nu(u,v):=\int_0^t e^{-\nu(t-\tau)(-\Delta)^\alpha}\mathbb{P}\nabla.(u\otimes v)(\tau,x)d\tau,$$

For the second equation, we remark that $L_\nu(u,b)$ can be considered as the solution to the heat equation (2.5) with $u_0=0$ and force $f=\mathbb{P}\nabla.(u\otimes v)$. According to Lemma 2.9 with $s=4-2\alpha+\frac{\lambda-3}{p}$ and Lemma

2.10 with $\rho = 1$, we get

$$\begin{aligned} \|L_\nu(u,b)\|_X &\le C\|\mathbb{P}\nabla.(u\otimes b)\|_{\mathscr{L}^1(I,\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})} \\ &\le C(\min\{\mu,\nu\})^{-1}\|u\|_X\|b\|_X. \end{aligned} \tag{3.8}$$

We also notice that $e^{-\mu t(-\Delta)^\alpha}b_0$ is the solution to the dissipative equation with $f = 0$ and $b_0 = b_0$. As a result, Lemma 2.9 gives

$$\|e^{-\mu t(-\Delta)^\alpha}b_0\|_X \le C\|b_0\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}}. \tag{3.9}$$

Through the estimates (3.8) and (3.9), we arrive at

$$\begin{aligned} \|\mathscr{T}_2(u,b)\|_X &\le C\|b_0\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}} \\ &\quad + 2C(\min\{\mu,\nu\})^{-1}\|u\|_X\|b\|_X. \end{aligned} \tag{3.10}$$

Since $\|(u_0,b_0)\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}} \le C_0\min\{\mu,\nu\}$, we define

$$E = \Big\{(u,b)|(u,b)\in X, \|(u,b)\|_X \le 2CC_0\min\{\mu,\nu\}\Big\},$$

where C_0 is a constant that can be specified later. Combining (3.6), (3.7), and (3.10), it follows that

$$\begin{aligned} &\|\mathscr{T}(u,b)\|_X \\ &\le C\|(u_0,b_0)\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}} + C(\min\{\mu,\nu\})^{-1}\|(u,b)\|_X^2 \\ &\le CC_0\min\{\mu,\nu\} + 4C^3C_0^2\min\{\mu,\nu\}, \end{aligned}$$

which implies that $\mathscr{T}(u,b)\in E$ when we choose C_0 small enough such that $C_0 < \frac{1}{16C^2}$.

On the other hand, for any (u_1,b_1), $(u_2,b_2)\in E$, we have

$$\begin{aligned} &\|\mathscr{T}_1(u_1,b_1) - \mathscr{T}_1(u_2,b_2)\|_X \\ &\le \|L_\mu(u_1,u_1) - L_\mu(u_2,u_2)\|_X + \|L_\mu(b_1,b_1) - L_\mu(b_2,b_2)\|_X \\ &\le \|L_\mu(u_1,u_1-u_2) + L_\mu(u_1-u_2,u_2)\|_X + \|B_\mu(b_1-b_2,b_2) + L_\mu(b_1,b_1-b_2)\|_X \\ &\le C(\min\{\mu,\nu\})^{-1}((\|u_1\|_X + \|u_2\|_X)\|u_1-u_2\|_X \\ &\quad + (\|b_1\|_X + \|b_2\|_X)\|b_1-b_2\|_X) \\ &\le 4C^2C_0(\|u_1-u_2\|_X + \|b_1-b_2\|_X) \\ &\le \frac{1}{4}(\|u_1-u_2\|_X + \|b_1-b_2\|_X). \end{aligned}$$

Similarly,

$$\begin{aligned} &\|\mathscr{T}_2(u_1,b_1) - \mathscr{T}_2(u_2,b_2)\|_X \\ &\le \|L_\nu(u_2,b_2) - L_\nu(u_1,b_1)\|_X + \|L_\nu(b_2,u_2) - L_\nu(b_1,u_1)\|_X \\ &\le 4C^2C_0(\|u_1-u_2\|_X + \|b_1-b_2\|_X) \\ &\le \frac{1}{4}(\|u_1-u_2\|_X + \|b_1-b_2\|_X). \end{aligned}$$

Consequently,

$$\|\mathscr{T}(u_1,b_1)-\mathscr{T}(u_2,b_2)\|_X \le \frac{1}{2}(\|u_1-u_2\|_X+\|b_1-b_2\|_X).$$

Based on the above estimate, we obtain that $\mathscr{T}$ is a contraction mapping from E to E. From the Banach fixed point theorem, we deduce that $\mathscr{T}$ has a unique fixed point $(u,b)\in E$ which is the solution of system (1.1). The proof is finished. □

4. Stability of global solutions

In this section, we prove the result of the stability for the global solutions and the blow up criteria when the maximal time of existence is finite.

Let T^* be the maximal existence time of solutions of (1.1) in

$$\mathscr{L}^\infty\left([0,T^*);\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}\right)\cap\mathscr{L}^1\left([0,T^*),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q}\right).$$

To prove Theorem 2.7, we assume that $T^*<\infty$ and

$$\|(u,b)\|_{\mathscr{L}^1([0,T^*),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})}<\infty.$$

Then, we can find $0<T_0<T^*$ satisfying

$$\|(u,b)\|_{\mathscr{L}^1([T_0,T^*),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})}<\frac{1}{4}.$$

For $t\in[T_0,T^*)$, we expressly consider the integral equation

$$\begin{cases} u=T_{\Omega,\alpha}(t)u(T_0)-\int_{T_0}^t T_{\Omega,\alpha}(t-\tau)\mathbb{P}\nabla\cdot(u\otimes u-b\otimes b)(\tau,\cdot)\,d\tau,\\ \\ b=e^{-t\nu(-\Delta)^\alpha}b(T_0)-\int_{T_0}^t e^{-\nu(t-\tau)(-\Delta)^\alpha}\mathbb{P}\nabla\cdot(u\otimes b-b\otimes u)(\tau,\cdot)\,d\tau.\end{cases} \tag{4.1}$$

The same method as for Lemma 2.10 gives

$$\begin{aligned}\|u\|_{\mathscr{L}^\infty([T_0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})} \lesssim{}& \|u(T_0)\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}}\\ &+\|u\|_{\mathscr{L}^\infty([T_0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}\|u\|_{\mathscr{L}^1([T_0,t),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})}\\ &+\|b\|_{\mathscr{L}^\infty([T_0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}\|b\|_{\mathscr{L}^1([T_0,t),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})},\end{aligned}$$

and

$$\begin{aligned}\|b\|_{\mathscr{L}^\infty([T_0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})} \lesssim{}& \|b(T_0)\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}}\\ &+\|u\|_{\mathscr{L}^\infty([T_0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}\|b\|_{\mathscr{L}^1([T_0,t),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})}\\ &+\|b\|_{\mathscr{L}^\infty([T_0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}\|u\|_{\mathscr{L}^1([T_0,t),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})},\end{aligned}$$

As a result, we have

$$\begin{aligned}\|(u,b)\|_{\mathscr{L}^\infty([T_0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})} \lesssim{}& \|(u(T_0),b(T_0))\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}}\\ &+\frac{1}{4}\|(u,b)\|_{\mathscr{L}^\infty([T_0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}.\end{aligned}$$

We can deduce from this that

$$\sup_{T_0\le s\le t}\|(u,b)(s)\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}}\lesssim 2\|(u(T_0),b(T_0))\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}},\forall t\in[T_0,T^*).$$

Putting

$$\mathrm{M}=\max(2\|(u(T_0),b(T_0))\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}},\max_{t\in[0,T_0]}\|(u,b)\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}}),$$

we have

$$\|(u(t),b(t))\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}}\lesssim \mathrm{M},\ \forall t\in[0,T^*).$$

From (1.1), we obtain

$$\begin{cases} u(t')-u(t)=-\mu\int_t^{t'}(-\Delta)^\alpha u(\tau)\,d\tau-\int_t^{t'}\mathbb{P}\nabla\cdot(u\otimes u-b\otimes b)\,d\tau-\Omega\int_t^{t'}\mathbb{P}(e_3\times u)\,d\tau,\\ b(t')-b(t)=-\nu\int_t^{t'}(-\Delta)^\alpha b(\tau)\,d\tau-\int_t^{t'}\nabla\cdot(u\otimes b-b\otimes u)\,d\tau.\end{cases}\tag{4.2}$$

From (4.2), we obtain that

$$\begin{aligned}
&\|u(t')-u(t)\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}}+\|b(t')-b(t)\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}}\\
&\le \mu\|u\|_{\mathscr{L}^1([t,t'),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})}+\|\mathbb{P}\nabla\cdot(u\otimes u-b\otimes b)\|_{\mathscr{L}^1([t,t'),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}\\
&\quad+\nu\|b\|_{\mathscr{L}^1([t,t'),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})}+\|\nabla\cdot(u\otimes b-b\otimes u)\|_{\mathscr{L}^1([t,t'),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}\\
&\quad+|\Omega|\|\mathbb{P}e_3\times u\|_{\mathscr{L}^1([t,t'),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}\\
&\lesssim \mu\|u\|_{\mathscr{L}^1([t,t'),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})}+\nu\|b\|_{\mathscr{L}^1([t,t'),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})}+|\Omega|\|e_3\times u\|_{\mathscr{L}^1([t,t'),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}\\
&\quad+\|(u,b)\|_{\mathscr{L}^\infty([t,t'),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}\|(u,b)\|_{\mathscr{L}^1([t,t'),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})}\\
&\lesssim(\mu+\nu+\mathrm{M}+|\Omega|)\|(u,b)\|_{\mathscr{L}^1([t,t'),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})}.
\end{aligned}$$

The dominated convergence theorem yields

$$\limsup_{t,t'\nearrow T^*,t\le t'}(\|u(t)-u(t')\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}}+\|b(t)-b(t')\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}})=0.$$

This implies that $u(t)$ and $b(t)$ satisfies the Cauchy criterion at T^*. Since $\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}$ is a Banach space, there exists an element u^*,b^* in $\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}$ such that $u(t)\to u^*,b(t)\to b^*$ in $\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}$ as $t\to T^*$. Now set $u(T^*)=u^*,b(T^*)=b^*$ and consider the generalized magnetohydrodynamic equations with the Coriolis force starting by u^*,b^*. Using the well-posedness, we obtain a solution existing on a larger time interval than $[0,T^*)$, which is a contradiction.

Now, let $\tilde u,\tilde b\in\mathscr{C}\big([0,T^*);\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}\big)\cap\mathscr{L}^1\big([0,T^*),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q}\big)$ be the maximal solution of (1.1) corresponding to the initial condition $\tilde u_0,\tilde b_0$. We need to prove $T^*=\infty$. Set $w=\tilde u-u$ and $d=\tilde b-b$, which satisfies

$$\begin{cases}\partial_t w+\mu(-\Delta)^\alpha w+\Omega e_3\times w+(\tilde u\cdot\nabla)w+(w\cdot\nabla)u+\nabla\pi=(\tilde b\cdot\nabla)d+(d\cdot\nabla)b,\\ \partial_t d+\nu(-\Delta)^\alpha d+(\tilde u\cdot\nabla)d+(w\cdot\nabla)b=(\tilde b\cdot\nabla)w+(d\cdot\nabla)u,\\ \nabla\cdot w=0,\qquad \nabla\cdot d=0.\end{cases}$$

We apply $\mathbb{P}$ to the above system. Then

$$\begin{cases} \partial_t w+\mu(-\Delta)^{\alpha}w=-\Omega\mathbb{P}e_3\times w-\mathbb{P}\nabla.(\tilde{u}\otimes w)-\mathbb{P}\nabla.(w\otimes u)+\mathbb{P}\nabla.(\tilde{b}\otimes d)+\mathbb{P}\nabla.(d\otimes b),\\ \partial_t d+\nu(-\Delta)^{\alpha}d=-\mathbb{P}\nabla.(\tilde{u}\otimes d)-\mathbb{P}\nabla.(w\otimes b)+\mathbb{P}\nabla.(\tilde{b}\otimes w)+\mathbb{P}\nabla.(d\otimes u). \end{cases}$$

For $t\in[0,T^*)$, we get

$$\begin{aligned}
&\|w(t)\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}}+\mu\|w\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})}\\
&\quad+\|d(t)\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}}+\nu\|d\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})}\\
&\le C\Big\{\|\nabla.(b\otimes w)\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}+\|\nabla.(w\otimes b)\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}\\
&\quad+\|\nabla.(b\otimes d)\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}+\|\nabla.(d\otimes b)\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}\\
&\quad+\|\nabla.(u\otimes w)\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}+\|\nabla.(w\otimes u)\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}\\
&\quad+\|\nabla.(u\otimes d)\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}+\|\nabla.(d\otimes u)\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}\\
&\quad+\|\nabla.(w\otimes w)\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}+\|\nabla.(d\otimes w)\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}\\
&\quad+\|\nabla.(d\otimes d)\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}+\|\nabla.(w\otimes d)\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}\\
&\quad+|\Omega|\|e_3\times w\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}+\|(w_0,d_0)\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}}\Big\}\\
&\le C\{L_1+L_2+L_3+L_4+L_5+L_6+|\Omega|\|e_3\times w\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}\\
&\quad+\|(w_0,d_0)\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}}\},
\end{aligned}$$

where

$$\begin{aligned}
L_1&=\|\nabla.(b\otimes w)\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}+\|\nabla.(w\otimes b)\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})},\\
L_2&=\|\nabla.(b\otimes d)\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}+\|\nabla.(d\otimes b)\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})},\\
L_3&=\|\nabla.(u\otimes w)\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}+\|\nabla.(w\otimes u)\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})},\\
L_4&=\|\nabla.(u\otimes d)\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}+\|\nabla.(d\otimes u)\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})},\\
L_5&=\|\nabla.(w\otimes w)\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}+\|\nabla.(d\otimes w)\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})},\\
L_6&=\|\nabla.(d\otimes d)\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}+\|\nabla.(w\otimes d)\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}.
\end{aligned}$$

The same calculus of the proof of Lemma 2.10 lead to

$$
\begin{aligned}
L_1+L_2 &\lesssim 2\int_0^t (\|w\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}} + \|d\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}})\|b\|_{\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q}}, \\
L_3+L_4 &\leq 2\int_0^t (\|w\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}} + \|d\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}})\|u\|_{\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q}}, \\
L_5 &\lesssim \|w\|_{\mathscr{L}^\infty([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}\|w\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})} \\
&\quad + \|d\|_{\mathscr{L}^\infty([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}\|w\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})}, \\
L_6 &\lesssim \|d\|_{\mathscr{L}^\infty([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}\|d\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})} \\
&\quad + \|w\|_{\mathscr{L}^\infty([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}\|d\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\|w(t)\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}} + \mu\|w\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})} \\
&\quad + \|d(t)\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}} + \nu\|d\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})} \\
&\leq C\Big\{\|(w_0,d_0)\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}} + (\|w\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})} + \|d\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})}) \\
&\quad \times (\|w\|_{\mathscr{L}^\infty([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})} + \|d\|_{\mathscr{L}^\infty([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}) \\
&\quad + \int_0^t (\|w\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}} + \|d\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}})(\|b\|_{\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q}} + \|u\|_{\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q}}) \\
&\quad + |\Omega|\|w\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})}\Big\}.
\end{aligned}
$$

Let

$$
T = \sup\Big\{t\in[0,T^*), \|(w,d)\|_{\mathscr{L}^\infty([0,t),\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q})} < \frac{\min\{\mu,\nu\}}{4C}\Big\}. \tag{4.3}
$$

For $t\in[0,T)$, we have

$$
\begin{aligned}
&\|w(t)\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}} + \frac{\mu}{4}\|w\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})} \\
&\quad + \|d(t)\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}} + \frac{\nu}{4}\|d\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})} \\
&\leq C\|(w_0,d_0)\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}} + \int_0^t C\|(w,d)\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}}(|\Omega| + \|b\|_{\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q}} + \|u\|_{\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q}}).
\end{aligned}
$$

Gronwall's Lemma gives

$$\begin{aligned}
&\|w(t)\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}} + \frac{\mu}{4}\|w\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})} \\
&\quad + \|d(t)\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}} + \frac{\nu}{4}\|d\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})} \\
&\le C\|(w_0,d_0)\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}} \exp\left\{\int_0^t C(|\Omega| + \|u\|_{\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q}} + \|b\|_{\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q}})\right\} \\
&\le C\|(w_0,d_0)\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}} \exp\left\{\int_0^\infty C(|\Omega| + \|u\|_{\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q}} + \|b\|_{\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q}})\right\}.
\end{aligned}$$

Taking C_0 sufficiently small in (2.4), we have

$$\begin{aligned}
&\|w(t)\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}} + \frac{\mu}{4}\|w\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})} \\
&\quad + \|d(t)\|_{\mathscr{F}\dot{\mathscr{N}}^{4-2\alpha+\frac{\lambda-3}{p}}_{p,\lambda,q}} + \frac{\nu}{4}\|d\|_{\mathscr{L}^1([0,t),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})} \\
&< \frac{\min\{\mu,\nu\}}{8C},
\end{aligned}$$

which contradicts the definition (4.3). Then $T = T^*$ and

$$\|(u,b)\|_{\mathscr{L}^1([0,T^*),\mathscr{F}\dot{\mathscr{N}}^{4+\frac{\lambda-3}{p}}_{p,\lambda,q})} < \infty.$$

Therefore $T^* = \infty$. This completes the proof of Theorem 2.7.

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