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A GENERALIZATION OF NADLER THEOREM IN CONE b-METRIC SPACES OVER BANACH ALGEBRAS

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Abstract. In this paper, we prove some coincidence point and common fixed point results for hybrid pairs of mappings defined on a cone b-metric space. Our results generalize the famous Nadler fixed point theorem and some recent results on cone b-metric spaces. An example is presented to illustrate our main result.

Keywords. Cone b-metric space; H-cone b-metric space; Coincidence point; Common fixed point.

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1. Introduction

The Banach contraction principle is one of the most interesting and applicable result in mathematics. It says that:

If (X,d) is a complete metric space and $T\colon X\to X$ is a contraction mapping on X, i.e., T satisfies the following property: there exists $\lambda\in[0,1)$ such that

$$d(Tx,Ty)\le\lambda d(x,y)\ \textit{for all}\ x,y\in X.$$

Then mapping T has a unique fixed point in X, i.e., there exists a unique $x^*\in X$ such that $Tx^*=x^*$.

The mapping considered by Banach was a single-valued mapping, i.e., for every $x\in X$ the image Tx of x under T is a unique point in X.

Let (X,d) be a metric space. Denote by $CB(X)$, the set of all closed and bounded subsets of the space X. Pompeiu and Hausdorff introduced a function $d_H\colon CB(X)\times CB(X)\to\mathbb{R}$ such that the pair

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$(CB(X), d_H)$ forms a metric space itself (for details, see, [1, 2]). The function d_H is called the Pompeiu-Hausdorff metric on $CB(X)$ and it is defined as follows:

$$d_H(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(a,b)\right\}, \quad \forall A,B \in CB(X).$$

A point $x \in X$ is called a fixed point of the mapping T if $x \in Tx$.

In 1969, Nadler [3] considered the mappings $T: X \to 2^X$ defined on a metric space such that for each $x \in X$ the image Tx of x under T is a unique subset of X. Such mappings are called a set-valued or multi-valued mappings. We assume that the values of a set-valued mapping are nonempty. Nadler obtained an analogue of Banach contraction principle on complete metric spaces. Precisely, he proved the following theorem:

Theorem 1.1 (Nadler [3])**.** *Let (X,d) be a complete metric space and $T: X \to CB(X)$ (where $CB(X)$ denotes the set of all closed and bounded subsets of X) be a set valued mapping. If there exists $\lambda \in (0,1)$ such that*

$$d_H(T(x), T(y)) \leq \lambda d(x,y), \ \forall \ x,y \in X.$$

Then T has a fixed point in X.

The concept of metric spaces has several generalizations. One such generalization of metric spaces is K-metric spaces which was introduced in the mid-20th century [4, 5] (see also, [6], [7]). In such spaces the metric function takes values into an ordered set (Banach space) instead of reals. In the year 2007, Huang and Zhang [8] initiate the study of fixed points of mappings in such spaces and named such spaces as cone metric spaces. Although, a criticism of fixed point results in cone metric spaces can be seen, e.g., in [9, 10, 11, 12]. Liu and Xu [13] introduced the notion of cone metric spaces over Banach algebras and prove some fixed point results in this new setting, in such a way that the contractive condition on mappings uses the vector constants and cannot be derived from their usual metric analogue. In 2014, Xu and Radenovic [14] showed that the results of Liu and Xu [15] remains valid if we use only solid cones instead of normal cones. The works of Bourbaki [16] and Bakhtin [17] were the initiation of the b-metric spaces. Czerwik [18] formally defined the b-metric spaces in which the triangular inequality of metric function was improved by introducing a generalized condition involving a constant.

Huang and Radenović [19] introduced the notion of cone b-metric spaces over Banach algebra and unified and generalized the concepts of cone metric spaces and b-metric spaces. They extended the Banach contraction principle and several other results in this new setting. Wardowski [20], introduced a new generalization of Pompeiu-Hausdorff metric on cone metric spaces. He introduced the concept of set-valued contraction of Nadler type in cone metric spaces and prove a fixed point theorem (see also, [21, 22]). Inspired by Wardowski [20], recently, Özavşar [23] introduced the concept of H-cone b-metric on b-metric spaces and proved a new cone b-metric version of theorem of Nadler [3].

Commutativity of two mappings defined on metric spaces and several of its weaker forms, e.g., compatibility, weak compatibility, R-weak commutativity etc., have been extended in their corresponding set-valued forms. For the pair of a single-valued and set-valued mappings (hybrid pair), similar concepts have been introduced and studied by several authors, see, e.g., [24, 25, 26, 27, 28].

Inspired by the results of Nadler [3] and Özavşar [23], in this paper, we prove some coincidence and common fixed point results of a pair of a single-valued and a set-valued mapping on cone b-metric spaces over Banach algebras. The results of this paper are coincidence point and common fixed point results which extend and generalize the results of Nadler [3], Wardowski [20] and Özavşar [23] for two mappings. An example is presented which illustrate our result.

2. Preliminaries

In this section, we recall some well-known definitions which will be needed in the sequel and can be found in [13, 15, 20, 29, 30].

Definition 2.1. Let $\mathfrak{B}$ be a real Banach algebra, i.e., $\mathfrak{B}$ is a real Banach space with a product that satisfies

(1) $x(yz) = (xy)z$;
(2) $x(y+z) = xy + xz$;
(3) $\alpha(xy) = (\alpha x)y = x(\alpha y)$;
(4) $\|xy\| \leq \|x\|\|y\|$,

for all $x, y, z \in \mathfrak{B}$, $\alpha \in \mathbb{R}$.

The Banach algebra $\mathfrak{B}$ is said to be unital if there exists an element $e \in \mathfrak{B}$ such that $ex = xe = x$ for all $x \in \mathfrak{B}$. The element e is called the unit. An $x \in \mathfrak{B}$ is said to be invertible if there is a $y \in \mathfrak{B}$ such that $xy = yx = e$. The inverse of x, if it exists, is unique and will be denoted by x^{-1}. For more details, see [30].

Proposition 2.2. [30] Let $\mathfrak{B}$ be a Banach algebra with unit e and $x \in \mathfrak{B}$. If the spectral radius $\rho(x)$ of x is less than 1, i.e.,

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|x^n\|^{1/n} < 1,$$

then $e - x$ is invertible and $(e-x)^{-1} = \sum_{i=0}^{\infty} x^i$.

Let $\mathfrak{B}$ be a unital Banach algebra. A non-empty closed set $P \subset \mathfrak{B}$ is said to be a cone (see, [13, 15]) if

(1) $e \in P$
(2) $P + P \subset P$,
(3) $\alpha P \subset P$ for all $\alpha \geq 0$,
(4) $P^2 \subset P$
(5) $P \cap (-P) = \{\theta\}$, where θ is the null (zero) vector of $\mathfrak{B}$.

Given a cone $P \subset \mathfrak{B}$ one can define a partial order $\preceq$ on $\mathfrak{B}$ by $x \preceq y$ if and only if $y - x \in P$. The notation $x \ll y$ will stand for $y - x \in P^\circ$, where P° denotes the interior of P. In rest of the discussion by an ordered Banach algebra with a given cone we mean a Banach algebra with partial order generated by that cone.

The cone P is said to be normal if there exists a number $K > 0$ such that, for all $a, b \in \mathfrak{B}$,

$$a \preceq b \quad \text{implies} \quad \|a\| \leq K\|b\|.$$

The least positive value of K satisfying the above inequality is called the normal constant (see [8]). Note that, for any normal cone P we have $K \geq 1$ (see [31]).

Definition 2.3. [19] Let $\mathfrak{B}$ be a an ordered Banach algebra with cone P, $s \geq 1$ be a constant and $X \neq \emptyset$. A cone b-metric space over $\mathfrak{B}$ is given by a pair (X, d), where d is a mapping from $X \times X$ into $\mathfrak{B}$ satisfying:

(cbm1) $\theta \preceq d(x,y)$ and $d(x,y) = \theta$ if and only if $x = y$;
(cbm2) $d(x,y) = d(y,x)$;

(cbm3) $d(x,y) \preceq s[d(x,z)+d(z,y)]$

for all $x,y,z \in X$ and for null vector $\theta \in \mathfrak{B}$.

If one takes $s=1$ in the above definition, the above definition reduces into the definition of a cone metric space (see, [8] and [13]). Hence, the cone b-metric spaces are a generalization of cone metric spaces. Also, if one takes $\mathfrak{B}=\mathbb{R}$ and $P=[0,\infty)$ then the above definition reduces into the definition of a b-metric space (see, [17] and [18]). Hence, the cone b-metric spaces generalize the b-metric spaces as well.

Example 2.4. [23] Let $\mathfrak{B}$ be the usual Banach algebra of all real-valued continuous functions on $X=[0,1]$ which also have continuous derivatives on X. If $\mathfrak{B}$ is equipped with the norm $\|f\|=\|f\|_\infty+\|f'\|_\infty$, then $\mathfrak{B}$ becomes a Banach algebra with unit $e=1$. Moreover, $P=\{f \in \mathfrak{B} \colon f(t) \geq 0 \text{ for all } t \in X\}$ is a nonnormal cone. Consider a mapping $d \colon X \times X \to \mathfrak{B}$ defined by $d(x,y)(t)=\frac{1}{2}|x-y|^2 e^t$ for all $x,y \in X$. It is easy to see that (x,d) is a cone b-metric space with $s=2$ on the Banach algebra $\mathfrak{B}$.

Several more examples of cone b-metric spaces can be found in [19] and [23].

Lemma 2.5. [32] *Let $P \subset \mathfrak{B}$ be a solid cone and $a,b,c \in P$.*

(a) If $a \preceq b$ and $b \ll c$, then $a \ll c$.
(b) If $a \ll b$ and $b \ll c$, then $a \ll c$.
(c) If $\theta \preceq u \ll c$ for each $c \in P^\circ$, then $u=\theta$.
(d) If $c \in P^\circ$ and $a_n \to \theta$, then there exist $n_0 \in \mathbb{N}$ such that, for all $n>n_0$ we have $a_n \ll c$.
(e) If $\theta \preceq a_n \preceq b_n$ for each n and $a_n \to a,\ b_n \to b$, then $a \preceq b$.

Lemma 2.6. [29] *Let $\mathfrak{B}$ be a Banach algebra with a unit e, P be a cone in $\mathfrak{B}$ and $a,b,c \in P$.*

(i) If $\rho(a)<1$, then $\rho(a^m) \leq \rho(a) < 1$ for each $m \in \mathbb{N}$.
(ii) If $\rho(a)<1$ and $b \preceq ac$, then $b \preceq c$.

Remark 2.7 (Xu and Radenović [14])**.** If $\rho(a)<1$ then $\|a^n\| \to 0$ as $n \to \infty$.

Lemma 2.8. [19] *Let P be a cone and $k \in P$ with $\rho(k)<1$. Then, for every $c \in P^\circ$ there exists $n_0 \in \mathbb{N}$ such that $k^n \ll c$ for all $n>n_0$.*

Henceforth, we will assume that the real Banach algebra $\mathfrak{B}$ is unital and that the cone $P \subset \mathfrak{B}$ is a solid cone, i.e., $P^\circ \neq \emptyset$.

Definition 2.9. [19] Let (X,d) be a cone b-metric space over the Banach algebra $\mathfrak{B}$, $x \in X$ and $\{x_n\}$ be a sequence in X. Then:

(i) $\{x_n\}$ is said to be convergent to x if for every $c \in \mathfrak{B}$ with $\theta \ll c$ there exists a natural number n_0 such that $d(x_n,x) \ll c$ for all $n>n_0$. We denote this fact by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
(ii) $\{x_n\}$ is called a Cauchy sequence if for every $c \in \mathfrak{B}$ with $\theta \ll c$ there exists a natural number n_0 such that $d(x_n,x_m) \ll c$ for all $n,m>n_0$.
(iii) (X,d) is called complete if every Cauchy sequence in X converges to some point in X.
(iv) A subset $A \subseteq X$ is called closed if every sequence $\{x_n\}$ in A such that $x_n \to x \in X$ as $n \to \infty$ we have $x \in A$.

Definition 2.10. [20, 29] Let (X,d) be a cone metric space over a Banach algebra $\mathfrak{B}$ and let $\mathscr{A}$ be a collection of nonempty subsets of X. A map $H\colon \mathscr{A}\times\mathscr{A}\to\mathfrak{B}$ is called a H-cone metric with respect to d if for any $A_1,A_2\in\mathscr{A}$ the following conditions hold:

(H1) $H(A_1,A_2)=0 \implies A_1=A_2$;
(H2) $H(A_1,A_2)=H(A_2,A_1)$;
(H3) $\forall_{c\in E,\theta\ll c}\,\forall_{x\in A_1}\,\exists_{y\in A_2}\quad d(x,y)\preceq H(A_1,A_2)+c$;
(H4) One of the following is satisfied:
  (i) $\forall_{c\in E,\theta\ll c}\,\exists_{x\in A_1}\,\forall_{y\in A_2}\,H(A_1,A_2)\preceq d(x,y)+c$;
  (ii) $\forall_{c\in E,\theta\ll c}\,\exists_{x\in A_2}\,\forall_{y\in A_1}\,H(A_1,A_2)\preceq d(x,y)+c$.

It is obvious that each H-cone metric depends on the choice of the collection $\mathscr{A}$. For the examples of H-cone metrics on cone metric spaces, see [20] and [29].

Definition 2.11. [23] Let (X,d) be a cone b-metric space over $\mathfrak{B}$. A mapping $H\colon\mathscr{A}\times\mathscr{A}\to\mathfrak{B}$ is an H-cone b-metric with respect to the cone b-metric d if it satisfies the conditions $(H1)-(H4)$ given above.

Proposition 2.12. [23] *If a mapping $H\colon\mathscr{A}\times\mathscr{A}\to\mathfrak{B}$ is an H-cone b-metric with respect to the cone b-metric space (X,d) over $\mathfrak{B}$, then $(\mathscr{A},H)$ is a cone b-metric space over $\mathfrak{B}$.*

Example 2.13. Let (X,d) be a cone b-metric space over a Banach algebra $\mathfrak{B}$ and let $\mathscr{A}=\{\{x\}\colon x\in X\}$. Then the mapping $H\colon\mathscr{A}\times\mathscr{A}\to E$ given by the formula

$$H(\{x\},\{y\})=d(x,y)\text{ for all }x,y\in X,$$

is a H-cone b-metric with respect to d.

Example 2.14. Let (X,d) be a metric space, $\mathfrak{B}=\mathbb{R}$ with usual product and norm, $P=\mathbb{R}^+=[0,\infty)$ and let $\mathscr{A}$ be the family of all nonempty, closed bounded subsets (with respect to d) of X. Then the Pompeiu-Hausdorff metric on X, is a H-cone b-metric with respect to d.

Example 2.15. Let (X,d) be a cone metric space over $\mathscr{A}$. Then every H-cone metric on X (in the sense of Wardowski [20]) is a H-cone b-metric with respect to d.

Let $T\colon X\to 2^X$ and $g\colon X\to X$ be two mappings on a nonempty set X. The pair of mappings (T,g) is called a hybrid pair of mappings.

Let X be a nonempty set, $T\colon X\to 2^X$ and $g\colon X\to X$ and let $x\in X$. The point x is called a coincidence point of the hybrid pair pair (T,g) and gx is called the corresponding point of coincidence if (see also, [33, 34]) $gx\in Tx$. The set of all coincidence point of the pair (T,g) is denoted by $\mathrm{CO}(T,g)$.

The hybrid pair (T,g) is called weakly compatible if $g(T(x))\subseteq T(g(x))$ for all $x\in\mathrm{CO}(T,g)$.

A point $x\in X$ is called a common fixed point of the hybrid pair (T,g) if $x=gx\in Tx$. The set of all common fixed point of the pair (T,g) is denoted by $\mathrm{CF}(T,g)$.

Let (X,d) be a cone b-metric space over a Banach algebra $\mathfrak{B}$ and $A\subseteq X$ be nonempty. Then, the set A is called closed if for every sequence $\{x_n\}$ in A such that $x_n\to x$ as $n\to\infty$, we have $x\in A$.

Özavşar [23] introduced the Nadler type mappings in cone b-metric spaces and proved the following theorem:

Theorem 2.16. [23] *Let $T: X \to \mathscr{A}$ be a set-valued mapping. If there is $k \in P$ with $\rho(sk) \in [0,1)$ such that*

$$H(Tx,Ty) \preceq kd(x,y) \text{ for all } x,y \in X \tag{2.1}$$

then there is at least one $x \in X$ such that $x \in Tx$.

We generalize the above theorem by proving a coincidence point result for the hybrid pair (T,g). Moreover, we prove a common fixed point result for this hybrid pair.

3. Main Results

We first prove some auxiliary results.

Lemma 3.1. *Let (X,d) be a cone b-metric space over $\mathfrak{B}$, $\mathscr{A}$ be a nonempty collection of nonempty subsets of X and $H: \mathscr{A} \times \mathscr{A} \to \mathfrak{B}$ be an H-cone b-metric with respect to the cone b-metric d. If $A = \{x\}, B = \{y\} \in \mathscr{A}$, then $H(\{x\},\{y\}) = d(x,y)$.*

Proof. Suppose, (i) of (H4) is satisfied (similar proof is valid if (ii) of (H4) is satisfied). Then, for every given $c_n \in \mathfrak{B}$ with $\theta \ll c_n$ such that $c_n \to \theta$ $(n \to \infty)$ we have:

$$\begin{aligned} H(A,B) \preceq d(x,y) + c_n \quad &\implies H(\{x\},\{y\}) - d(x,y) \preceq c_n \\ d(x,y) \preceq H(A,B) + c_n \quad &\implies d(x,y) - H(\{x\},\{y\}) \preceq c_n. \end{aligned}$$

Since P is closed and $c_n \to \theta$ $(n \to \infty)$, the above inequalities implies that $H(\{x\},\{y\}) = d(x,y)$. □

Lemma 3.2. *Let (X,d) be a cone b-metric space over $\mathfrak{B}$, $\mathscr{A}$ be a nonempty collection of nonempty subsets of X and $H: \mathscr{A} \times \mathscr{A} \to \mathfrak{B}$ be an H-cone b-metric with respect to the cone b-metric d. Suppose that $A \in \mathscr{A}$ and for some $a \in X$ we have $\{a\} \in \mathscr{A}$. Then $a \in A$ if and only if $H(\{a\},A) = \theta$.*

Proof. Suppose, $a \in A$ and $B = \{a\}$. Suppose, (i) of (H4) is satisfied (similar proof is valid if (ii) of (H4) is satisfied), then for every given $c_n \in \mathfrak{B}$ with $\theta \ll c_n$ such that $c_n \to \theta$ $(n \to \infty)$ and for all $a' \in A$ there exists $b \in B$ such that:

$$H(B,A) \preceq d(b,a') + c_n.$$

Since B is singleton, we must have $b = a$. Also, since the above inequality is true for all $a' \in A$ hence we can replace a' by a (since $a \in A$). Hence, the above inequality yields

$$H(B,A) \preceq d(a,a) + c_n = c_n.$$

The above inequality implies that $c_n - H(B,A) \in P$. Since P is closed and $c_n \to \theta$ $(n \to \infty)$, we obtain $-H(B,A) \in P$. By definition we have $H(B,A) \in P$ and $P \cap (-P) = \{\theta\}$, hence

$$H(\{a\},A) = H(B,A) = \theta.$$

Converse follows directly from the definition of H. □

We next introduce the Nadler contraction and g-Nadler contractions on a cone b-metric spaces over Banach algebras.

Definition 3.3. Let (X,d) be a cone b-metric space over a Banach algebra $\mathfrak{B}$ and $\mathscr{A}$ be a nonempty collection of nonempty subsets of X. Let $T\colon X \to \mathscr{A}$ be a mapping. Then, the the mapping T is called a Nadler contraction with contractive vector k if there exists $k \in P^\circ$ such that $\rho(sk) \in [0,1)$ and the following condition is satisfied:

$$H(Tx,Ty) \preceq kd(x,y) \text{ for all } x,y \in X. \tag{3.1}$$

Definition 3.4. Let (X,d) be a cone b-metric space over a Banach algebra $\mathfrak{B}$ and $\mathscr{A}$ be a nonempty collection of nonempty closed subsets of X. Let $T\colon X \to \mathscr{A}$ and $g\colon X \to X$ be two mappings. Then, the mapping T is called a g-Nadler contraction with contractive vector k if there exists $k \in P^\circ$ such that $\rho(sk) \in [0,1)$ and the following condition is satisfied:

$$H(Tx,Ty) \preceq kd(gx,gy) \text{ for all } x,y \in X. \tag{3.2}$$

Note that, the class of Nadler contractions is a particular case of the class of g-Nadler contractions. In particular, every I_X-Nadler contraction is a Nadler contraction, where I_X is the identity mapping of X.

In the next theorem, we give a generalization of Theorem 2.16.

Theorem 3.5. *Let (X,d) be a cone b-metric space over a Banach algebra $\mathfrak{B}$. Let $\mathscr{A}$ be a nonempty collection of nonempty closed subsets of X. Let $g\colon X \to X$ be a mapping and let $T\colon X \to \mathscr{A}$ be a g-Nadler contraction with contractive vector k. If $Tx \subseteq g(X)$ for all $x \in X$ and $g(X)$ is complete, then $CO(T,g) \neq \emptyset$.*

Proof. Suppose, $x_0 \in X$. Since $Tx_0 \in \mathscr{A}$ the set Tx_0 is nonempty, hence let $z_1 \in Tx_0$. As, $Tx_0 \subseteq g(X)$ and $z_1 \in Tx_0$, there exists $x_1 \in X$ such that $z_1 = gx_1$. Let $z_0 = gx_0$. Since $z_1 \in Tx_0$, by definition of H-cone b-metric there exists $z_2 \in Tx_1$ such that

$$d(z_1,z_2) \preceq H(Tx_0,Tx_1) + c_1$$

where $c_1 \in P^\circ$ is chosen so that $\rho(c_1) < 1$. As, T is a g-Nadler contraction with contractive vector k, it follows from the above inequality that

$$d(z_1,z_2) \preceq kd(gx_0,gx_1) + c_1. \tag{3.3}$$

Again, since $z_2 \in Tx_1$ and $Tx_1 \subseteq g(X)$, there exists $x_2 \in X$ such that $z_2 = gx_2$. By definition of H-cone b-metric there exists $z_3 \in Tx_2$ such that

$$d(z_2,z_3) \preceq H(Tx_1,Tx_2) + c_2$$

where $c_2 \in P^\circ$ is chosen so that $\rho(c_2) < 1$. As, T is a g-Nadler contraction with contractive vector k, it follows from the above inequality that

$$d(z_2,z_3) \preceq kd(gx_1,gx_2) + c_2. \tag{3.4}$$

As, $z_3 \in Tx_2$ and $Tx_2 \subseteq g(X)$, so, there exists $x_3 \in X$ such that $z_3 = gx_3$.

Thus, we obtain a sequence $\{z_n\} = \{gx_n\}$ in $g(X)$ such that $z_n \in Tx_{n-1}$ for all $n \in \mathbb{N}$ and the following inequality holds:

$$d(z_n,z_{n+1}) \preceq kd(gx_{n-1},gx_n) + c_n = kd(z_{n-1},z_n) + c_n$$

where $c_n \in P^\circ$ is chosen so that $\rho(c_n) < 1$.

Successive use of the above inequality yields

$$\begin{aligned} d(z_n,z_{n+1}) &\preceq kd(z_{n-1},z_n)+c_n \\ &\preceq k[kd(z_{n-2},z_{n-1})+c_{n-1}]+c_n \\ &= k^2d(z_{n-2},z_{n-1})+kc_{n-1}+c_n \\ &= k^3d(z_{n-3},z_{n-2})+k^2c_{n-2}+kc_{n-1}+c_n \\ &\vdots \\ &\preceq k^nd(z_0,z_1)+k^{n-1}c_1+k^{n-2}c_2+\cdots+kc_{n-1}+c_n. \end{aligned}$$

Thus, for $n\in\mathbb{N}$ we have

$$d(z_n,z_{n+1})\preceq k^nd(z_0,z_1)+\sum_{i=0}^{n-1}k^ic_{n-i}.$$

As, $\rho(k)<1$, it follows from Lemma 2.6 that $\rho(k^n)<1$ for all $n\in\mathbb{N}$. Since $c_n\in P^\circ$ were chosen with the condition that $\rho(c_n)<1$ for all $n\in\mathbb{N}$, therefore, we can choose $c_n=k^{2n}$ for all $n\in\mathbb{N}$. As, $\rho(sk)<1$, so, $\rho(k)<1$ and the vector $e-k$ is invertible and we have $(e-k)^{-1}=\sum_{i=0}^{\infty}k^i$, and so, it follows from the above inequality that

$$\begin{aligned} d(z_n,z_{n+1}) &\preceq k^nd(z_0,z_1)+\sum_{i=0}^{n-1}k^{2n-i} \\ &\preceq k^nd(z_0,z_1)+k^n\sum_{i=0}^{\infty}k^i \\ &= k^nd(z_0,z_1)+k^n(e-k)^{-1}. \end{aligned}$$

Thus, for every $n\in\mathbb{N}$ we have

$$d(z_n,z_{n+1})\preceq k^nd(z_0,z_1)+k^n(e-k)^{-1}. \tag{3.5}$$

We next prove that $\{z_n\}$ is a Cauchy sequence in X. Suppose, $n,m\in\mathbb{N}$ and $m>n$, then we have

$$\begin{aligned} d(z_n,z_m) &\preceq s[d(z_n,z_{n+1})+d(z_{n+1},z_m)] \\ &\preceq sd(z_n,z_{n+1})+s^2[d(z_{n+1},z_{n+2})+d(z_{n+2},z_m)] \\ &\preceq sd(z_n,z_{n+1})+s^2d(z_{n+1},z_{n+2})+s^3[d(z_{n+2},z_{n+3})+d(z_{n+3},z_m)] \\ &\preceq sd(z_n,z_{n+1})+s^2d(z_{n+1},z_{n+2})+s^3d(z_{n+2},z_{n+3}) \\ &\quad +\cdots+s^{m-n-1}d(z_{m-2},z_{m-1})+s^{m-n-1}d(z_{m-1},z_m). \end{aligned}$$

Hence

$$d(z_n,z_m)\preceq s^{m-n-1}d(z_{m-1},z_m)+\sum_{j=n}^{m-2}s^{j-n+1}d(z_j,z_{j+1}). \tag{3.6}$$

As, $s\geq 1$, by the properties of P we have

$$s^{m-n-1}d(z_{m-1},z_m)\preceq s^{m-n}d(z_{m-1},z_m).$$

Using the above inequality in (3.6) we obtain

$$d(z_n,z_m)\preceq\sum_{j=n}^{m-1}s^{j-n+1}d(z_j,z_{j+1}). \tag{3.7}$$

Using (3.5) in (3.7) we obtain

$$\begin{aligned} d(z_n,z_m) &\preceq \sum_{j=n}^{m-1} s^{j-n+1}[k^j d(z_0,z_1)+k^j(e-k)^{-1}] \\ &= \sum_{j=n}^{m-1} s^{j-n+1}k^j[d(z_0,z_1)+(e-k)^{-1}] \\ &= s^{-n+1}\left[\sum_{j=n}^{j-n-1}(sk)^j\right][d(z_0,z_1)+(e-k)^{-1}] \\ &\preceq sk^n\left[\sum_{j=0}^{\infty}(sk)^j\right][d(z_0,z_1)+(e-k)^{-1}]. \end{aligned}$$

Since, $\rho(sk)<1$ the vector $e-sk$ is invertible and $(e-sk)^{-1}=\sum_{j=0}^{\infty}(sk)^j$. Therefore, it follows from the above inequality that

$$d(z_n,z_m)\preceq sk^n(e-sk)^{-1}\mu \tag{3.8}$$

where $\mu=d(z_0,z_1)+(e-k)^{-1}$. Since $\rho(k)<1$, by Remark 2.7 we have $\|k^n\|\to 0$, i.e., $k^n\to\theta$ as $n\to\infty$. Hence, $sk^n(e-sk)^{-1}\mu\to\theta$ as $n\to\infty$. Now by part (a) and (d) of Lemma 2.5 it follows that, for given $c\in P$, $\theta\ll c$ there exists $n_0\in\mathbb{N}$ such that

$$d(z_n,z_m)\ll c \text{ for all } n,m>n_0.$$

Thus, $\{z_n\}=\{gx_n\}$ is a Cauchy sequence. By completeness of $g(X)$, there exists $x^*\in X$ such that $z_n\to gx^*=y^*$ (say) as $n\to\infty$.

We shall show that x^* is a coincidence point of T and g.

Then, as $z_n=gx_n\in Tx_{n-1}$ for all $n\in\mathbb{N}$, by definition of H-cone b-metric, for each $n\in\mathbb{N}$ we have $y_n\in Tx^*$ such that

$$d(z_n,y_n)\preceq H(Tx_{n-1},Tx^*)+\varepsilon_n$$

where $\varepsilon_n\in P^\circ$ is chosen such that $\varepsilon_n\to\theta$ as $n\to\infty$. Since T is a g-Nadler contraction with contractive vector k, we obtain from the above inequality that

$$d(z_n,y_n)\preceq kd(gx_{n-1},gx^*)+\varepsilon_n.$$

The above inequality yields

$$\begin{aligned} d(y_n,gx^*) &\preceq s[d(y_n,z_n)+d(z_n,gx^*)] \\ &\preceq s[kd(gx_{n-1},gx^*)+\varepsilon_n]+sd(z_n,gx^*) \\ &= s[kd(z_{n-1},y^*)+\varepsilon_n]+sd(z_n,y^*). \end{aligned}$$

Since $z_n\to y^*$ and $\varepsilon_n\to\theta$ as $n\to\infty$, therefore given any $c\in P^\circ$, we can find $n_1\in\mathbb{N}$ such that $s[kd(z_{n-1},y^*)+\varepsilon_n]+sd(z_n,y^*)\ll c$ for all $n>n_1$. Now, by part (a) of Lemma 2.5 we have

$$d(y_n,gx^*)\ll c \quad \text{for all } n>n_1.$$

It shows that $y_n\to gx^*$ as $n\to\infty$. As, for all $n\in\mathbb{N}$ we have $y_n\in Tx^*$ and Tx^* is closed, we must have $y^*=gx^*\in Tx^*$.

Thus, x^* is a coincidence point of T and g. □

The following corollary is the main result of Özavşar [23].

Corollary 3.6. *Let (X,d) be a complete cone b-metric space over a Banach algebra $\mathfrak{B}$. Let $\mathscr{A}$ be a nonempty collection of nonempty closed subsets of X and let $T\colon X\to\mathscr{A}$ be a Nadler contraction with contractive vector k. Then there is at least one $x\in X$ such that $x\in Tx$.*

Proof. From $g=I_X$, the g-Nadler contraction becomes the Nadler contraction with same contractive vector. Also, if $g=I_X$, the the coincidence point of the hybrid pair (T,g) becomes the fixed point of T. Hence, the result follows by taking $g=I_X$ in Theorem 3.5. □

We now state some common fixed point results for the hybrid pair (T,g) which are set-valued version of the result of Abbas and Jungck [35] in cone b-metric spaces.

Theorem 3.7. *Let (X,d) be a cone b-metric space over a Banach algebra $\mathfrak{B}$. Let $\mathscr{A}$ be a nonempty collection of nonempty closed subsets of X. Let $g\colon X\to X$ be a mapping and let $T\colon X\to\mathscr{A}$ be a g-Nadler contraction with contractive vector k. If $Tx\subseteq g(X)$ for all $x\in X$ and $g(X)$ is complete, then $CO(T,g)\neq\emptyset$. In addition, if for every coincidence point x of the pair (T,g) we have $Tx=\{gx\}$, then the pair (T,g) has a unique point of coincidence. Further, if the hybrid pair (T,g) is weakly compatible, then the pair (T,g) has a unique common fixed point.*

Proof. Theorem 3.5 ensures the existence of coincidence point x^* and corresponding point of coincidence y^*. Now, by assumption we have $y^*=gx^*\in Tx^*=\{gx^*\}$. We first show that the point of coincidence y^* is unique. On contrary, suppose that z^* is a point of coincidence of the pair (T,g) and $y^*\neq z^*$. Then, we have $z^*=gw^*\in Tw^*=\{gw^*\}$ for some $w^*\in X$. Using Lemma 3.1 we have

$$\begin{aligned} d(y^*,z^*) &= d(gx^*,gw^*)\\ &= H(\{gx^*\},\{gw^*\})\\ &= H(Tx^*,Tw^*)\\ &\preceq kd(gx^*,gw^*)\\ &= kd(y^*,z^*). \end{aligned}$$

Therefore, on repeating the above process n times $(n\in\mathbb{N})$ we obtain:

$$d(y^*,z^*)\preceq k^n d(y^*,z^*) \text{ for all } n\in\mathbb{N}.$$

Now, it follows from the above inequality, Lemma 2.5 and Lemma 2.8 that: for every $c_m\in P^\circ, m\in\mathbb{N}$ there exists $n_0\in\mathbb{N}$ such that

$$d(y^*,z^*)\ll c_m d(y^*,z^*) \quad\text{for all}\quad n>n_0, m\in\mathbb{N}.$$

Choosing $c_m\to\theta$ as $m\to\infty$, and letting $m\to\infty$, we sees that the closedness of the cone P implies that

$$d(y^*,z^*)=\theta, \text{ i.e., } y^*=z^*.$$

This contradiction shows that the point of coincidence y^* of the pair (T,g) is unique.

Finally, we suppose that the hybrid pair (T,g) is weakly compatible. We shall prove that y^* is a common fixed point of the pair (T,g). For convenience, suppose, $u^*=gy^*$. By weak compatibility of the pair (T,g), we have

$$gTx^*\subseteq Tgx^*=Ty^*, \text{ i.e., } g(\{y^*\})\subseteq Ty^* \text{ or } u^*=gy^*\in Ty^*.$$

If $y^* \neq u^*$, then the above inclusion violates the uniqueness of the point of coincidence y^*. Hence, we must have

$$y^* = u^* = gy^* \in Ty^*.$$

Thus, y^* is a common fixed point of the pair (T,g).

The uniqueness of common fixed point follows from the uniqueness of the coincidence point. $\square$

Example 3.8. Let $\mathfrak{B} = C^1_{\mathbb{R}}[0,1]$ with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$ and usual multiplication of functions and with unit $e = 1$. Let $P = \{f \in \mathfrak{B} \colon f(t) \geq 0 \text{ for all } t \in [0,1]\}$. Let $X = [0,1]$. Then the mapping $d \colon X \times X \to \mathfrak{B}$ defined by

$$d(x,y) = \frac{1}{2}|x-y|^2 e^t \text{ for all } x,y \in X$$

is a cone b-metric on X with $s = 2$. Let $\mathscr{A} = \{[0,x] \colon x \in X\}$ and define a mapping $H \colon \mathscr{A} \times \mathscr{A} \to \mathfrak{B}$ by

$$H([0,x],[0,y]) = d(x,y) \text{ for all } x,y \in X.$$

Then, H is a H-cone b-metric on X. Define mappings $T \colon X \to \mathscr{A}$ and $g \colon X \to X$ by

$$Tx = \begin{cases} [0,2x], & \text{if } x \in [0,1/16]; \\ \{0\}, & \text{otherwise} \end{cases} \quad \text{and} \quad gx = \begin{cases} 8x, & \text{if } x \in [0,1/16]; \\ 1, & \text{otherwise.} \end{cases}$$

Then, $g(X) = \left[0, \frac{1}{2}\right] \cup \{1\}$. So, $Tx \subseteq g(X)$ for all $x \in X$. We show that the mapping T is a g-Nadler contraction with contractive vector $k(t) = e^t\lambda$, where $\lambda \in \left[\frac{1}{16}, \frac{1}{4e}\right)$. Then, we consider the following cases:

- If $x,y \in [0,1/16]$, then

$$\begin{aligned} H(Tx,Ty) &= d([0,2x],[0,2y]) = d(2x,2y) \\ &= \frac{1}{2}|2x-2y|^2 e^t = 2|x-y|^2 e^t \\ &\preceq k(t) \cdot \frac{1}{2}|8x-8y|^2 e^t \\ &= k(t) d(gx,gy). \end{aligned}$$

  Therefore, (3.2) is satisfied.
- If $x,y \in (1/16,1]$, then $H(Tx,Ty) = H(\{0\},\{0\}) = 0$. Hence (3.2) is satisfied trivially.
- If $x \in [0,1/16], y \in (1/16,1]$, then we have

$$\begin{aligned} H(Tx,Ty) &= d([0,2x],\{0\}) = d(2x,0) \\ &= \frac{1}{2}|2x-0|^2 e^t = 2x^2 e^t \\ &\preceq k(t) \cdot \frac{1}{2}|8x-1|^2 e^t \\ &= k(t) d(gx,gy). \end{aligned}$$

  Therefore, (3.2) is satisfied.

Similarly, (3.2) is satisfied in all other possible cases. Note that, $\rho(k) = \rho(e^t\lambda) = 2\lambda e$. So, $\rho(sk) = \rho(2k) < 1$. Thus, T is a g-Nadler contraction.

Hence all the conditions of Theorem 3.7 are satisfied and the hybrid pair (T,g) has a unique common fixed point. Namely, $CF(T,g) = \{0\}$.

Corollary 3.9. *Let (X,d) be a complete cone b-metric space over a Banach algebra $\mathfrak{B}$. Let $\mathscr{A}$ be a nonempty collection of nonempty closed subsets of X and let $T: X \to \mathscr{A}$ be a Nadler contraction with contractive vector k. Then there is at least one $x \in X$ such that $x \in Tx$. In addition, if for every x satisfying $x \in Tx$ we have $Tx = \{x\}$, then the mapping T has a unique fixed point.*

Proof. Since for $g = I_X$, the g-Nadler contraction becomes the Nadler contraction with same contractive vector. Also, if $g = I_X$, then the common fixed point of the hybrid pair (T,g) becomes the fixed point of T. Hence, the result follows by taking $g = I_X$ in Theorem 3.7. □

In the next theorem, we show that the completeness of space $g(X)$ in Theorem 3.5 can be replaced by another condition on T and g, and the coincidence point of the hybrid pair still exists.

Theorem 3.10. *Let (X,d) be a cone b-metric space over a Banach algebra $\mathfrak{B}$. Let $\mathscr{A}$ be a nonempty collection of nonempty closed subsets of X. Let $g: X \to X$ be a mapping and let $T: X \to \mathscr{A}$ be a g-Nadler contraction with contractive vector k. If $Tx \subseteq g(X)$, $\{x\} \in \mathscr{A}$ for all $x \in X$ and the following condition is satisfied: there exists $u \in X$ such that*

$$H(\{gu\},Tu) \preceq H(\{gx\},Tx) \text{ for all } x \in X. \tag{3.9}$$

Then $CO(T,g) \neq \emptyset$.

Proof. Let

$$D(x) = H(\{gx\},Tx) \text{ for all } x \in X.$$

Using the assumption, we have

$$D(u) \preceq D(x) \text{ for all } x \in X. \tag{3.10}$$

We claim that $gu \in Tu$, i.e., gu is a point of coincidence of the hybrid pair (T,g). On the contrary, suppose that $gu \notin Tu$. By Lemma 3.2, we have $\theta \prec D(u)$. Let $z \in Tu$ be arbitrary. Since $Tx \subseteq g(X)$, there exists $y \in X$ such that $z = gy \in Tu$. Since $gy \in Tu$, we find from Lemma 3.2 that $H(\{gy\},Tu) = \theta$. As, $(\mathscr{A},H)$ is a cone b-metric space (see Özavşar [23]), we have

$$\begin{aligned} H(\{gy\},Ty) &\preceq s[H(\{gy\},Tu)+H(Tu,Ty)] \\ &\preceq \theta + skd(gu,gy) \\ &= skd(gu,z). \end{aligned} \tag{3.11}$$

Using $gu \in \{gu\}$ and $(H3)$ for every given $c_n \in \mathfrak{B}$ with $\theta \ll c_n$ such that $c_n \to \theta$ $(n \to \infty)$, we have

$$d(z',gu) \preceq H(Tu,\{gu\}) + c_n \text{ for all } z' \in Tu \text{ and for all } n \in \mathbb{N}.$$

It shows that $H(Tu,\{gu\}) + c_n - d(z',gu) \in P$ for all $z' \in Tu$ and for all $n \in \mathbb{N}$. Letting $n \to \infty$ and using the fact that P is closed, we obtain $H(Tu,\{gu\}) + \theta - d(z',gu) \in P$, i.e.

$$d(z',gu) \preceq H(Tu,\{gu\}) \text{ for all } z' \in Tu.$$

As, $z \in Tu$ the above inequality with (3.11) yields $H(\{gy\},Ty) \preceq skH(Tu,\{gu\})$, i.e.

$$D(y) \preceq skD(u).$$

Using the above inequality in (3.10), we obtain

$$D(y) \preceq skD(y).$$

Repeating the above process $n \in \mathbb{N}$ times, we obtain

$$D(y) \preceq (sk)^n D(y) \text{ for all } n \in \mathbb{N}.$$

Since $\rho(sk) < 1$, we have from Remark 2.7 that $\|(sk)^n\| \to 0$, i.e., $(sk)^n \to \theta$ as $n \to \infty$. Hence, $k^n D(y) \to \theta$ as $n \to \infty$. Now, by part (a) and (d) of Lemma 2.5, it follows that, for given $c \in P$, $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that

$$D(y) \ll c \text{ for all } n > n_0.$$

Hence, by (c) of Lemma 2.5, we obtain $D(y) = \theta$, which together with (3.10) gives $D(u) = \theta$. This contradiction proves the result. □

Corollary 3.11. *Let (X,d) be a cone b-metric space over a Banach algebra $\mathfrak{B}$. Let $\mathscr{A}$ be a nonempty collection of nonempty closed subsets of X and let $T : X \to \mathscr{A}$ be a Nadler contraction with a contractive vector k. If $\{x\} \in \mathscr{A}$ for all $x \in X$ and the following condition is satisfied: there exists $u \in X$ such that*

$$H(\{u\}, Tu) \preceq H(\{x\}, Tx) \text{ for all } x \in X. \tag{3.12}$$

Then there is at least one $x \in X$ such that $x \in Tx$.

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