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STABILITY OF NONLINEAR NEUTRAL NABLA FRACTIONAL DIFFERENCE EQUATIONS

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Abstract. In this paper, we give sufficient conditions to guarantee the asymptotic stability of the solutions to a kind of nonlinear neutral nabla fractional difference equations of the Caputo type of order α $(0 < \alpha < 1)$. By using the Krasnoselskii's fixed point theorem and the discrete Arzela-Ascoli's theorem, we establish new results on the asymptotic stability of the solutions.

Keywords. Asymptotic stability; Caputo nabla fractional difference; Discrete Arzela-Ascoli's theorem; Krasnoselskii's fixed point theorem; Neutral nabla fractional difference equation.

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1. INTRODUCTION

Fractional differential and difference equations with and without delay arise from a variety of applications including in various fields of science and engineering such as applied sciences, practical problems concerning mechanics, economy, control systems, physics, chemistry, biology, medicine, atomic energy, information theory, harmonic oscillator, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, etc. In particular, problems concerning qualitative analysis of fractional differential and difference equations with and without delay have received the attention from many authors, see [1]-[23], [25]-[28] and the references therein.

Recently, Agarwal, Zhou and He [3] discussed the existence of solutions for the neutral fractional differential equation with bounded delay

$$\begin{cases} {}^{C}D^{\alpha}\left(x(t)-g(t,x_t)\right)=f\left(t,x_t\right),\ t\geq t_0, \\ x_{t_0}=\phi, \end{cases}$$

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where ${}^{C}D^{\alpha}$ is the standard Caputo's fractional derivative of order $0<\alpha<1$. By employing the Krasnoselskii's fixed point theorem, the authors obtained existence results.

The fractional difference equation

$$\begin{cases} \Delta^{\alpha}x(t)=f\left(t+\alpha,x(t+\alpha)\right),\ t\in\mathbb{N}_{1-\alpha}, \\ \Delta^{\alpha-1}x(t)\big|_{t=0}=x_0, \end{cases}$$

has been investigated in [12], where Δ^{α} denotes Riemann-Liouville like discrete fractional difference of order $0<\alpha<1$. By using the Krasnoselskii's fixed point theorem and the discrete Arzela-Ascoli's theorem, the asymptotic stability was established.

In [16], Jagan Mohan, Shobanadevi and Deekshitulu investigated the asymptotic stability of the solutions of the following nonlinear nabla fractional difference equation

$$\begin{cases} \nabla_{a*}^{\alpha}x(t)=f\left(t,x(t)\right),\ t\in\mathbb{N}_{a+1}, \\ x(a)=x_0, \end{cases}$$

where ∇_{a*}^{α} is a Caputo nabla fractional difference of order $0<\alpha<1$. By employing the Schauder's fixed point theorem and the discrete Arzela-Ascoli's theorem, the authors obtained asymptotic stability results.

Inspired and motivated by the works mentioned above and the corresponding results announced in [1]-[23], [25]-[28], we concentrate on the asymptotic stability of the solutions for the nonlinear neutral nabla fractional difference equation with variable delay

$$\begin{cases} \nabla_{a*}^{\alpha}\left[x(t)-g\left(t,x\left(t-\tau(t)\right)\right)\right]=f\left(t,x(t),x\left(t-\tau(t)\right)\right),\ t\in\mathbb{N}_{a+1}, \\ x(t)=\phi(t),\ t\in[m_0,a]\cap\mathbb{N}_{m_0}, \end{cases} \tag{1.1}$$

where $a\in\mathbb{R}^{+}$ is fixed, ∇_{a*}^{α} is a Caputo nabla fractional difference of order $0<\alpha<1$, $m_0=\inf_{t\in\mathbb{N}_a}\{t-\tau(t)\}$, $\mathbb{N}_t=\{t,t+1,t+2,\ldots\}$, $\tau:\mathbb{N}_a\to\mathbb{N}_a$ with $t-\tau(t)\to\infty$ as $t\to\infty$, $f:\mathbb{N}_a\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous in x and y, $g:\mathbb{N}_a\times\mathbb{R}\to\mathbb{R}$ is Lipschitz continuous in x, that is, there is a positive constant $K\in(0,1)$ such that

$$|g(t,x)-g(t,y)|\leq K|x-y|,\ g(t,0)=0. \tag{1.2}$$

The purpose of this paper is to use the Krasnoselskii's fixed point theorem and the discrete Arzela-Ascoli's theorem to show the asymptotic stability of solutions for (1.1). To apply the Krasnoselskii's fixed point theorem, we need to construct two mappings. One of the two is a contraction and the other is compact.

This paper is organized as follows. In section 2, we introduce some notations and lemmas, and state some preliminaries results needed in later sections. Also, we present the inversion of (1.1) and the Krasnoselskii's fixed point theorem. In Section 3, we give and prove our main results on the stability.

2. Preliminaries

In this section, we introduce preliminary facts which will be used throughout this paper.

Definition 2.1 ([15, 27]). Let $x:\mathbb{N}_a\to\mathbb{R}$ and $\alpha>0$ be given. Then the α^{th}-order nabla fractional sum of x is given by

$$\nabla_a^{-\alpha}x(t)=\frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}\left(t-\rho(s)\right)^{\overline{\alpha-1}}x(s)\ \text{ for } t\in\mathbb{N}_a, \tag{2.1}$$

where $\rho(s) = s-1$ and $t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}$ for $t \in \mathbb{R}\backslash\{...,-2,-1,0\}$ with $0^{\overline{\alpha}} = 0$.

Also, we define the trivial sum by $\nabla_a^{-0}x(t) = x(t)$ for $t \in \mathbb{N}_a$.

Definition 2.2 ([26])**.** Let $x : \mathbb{N}_a \to \mathbb{R}$, $\alpha > 0$ be given and let $N \in \mathbb{N}$ be chosen such that $N-1 < \alpha \leq N$. Then, the α^{th}-order Caputo nabla fractional difference of x is given by

$$\nabla_{a*}^{\alpha}x(t) = \nabla_a^{-(N-\alpha)}\left[\nabla^N x(t)\right] \text{ for } t \in \mathbb{N}_{a+N}. \tag{2.2}$$

For $\alpha = 0$, we set $\nabla_{a*}^{0}x(t) = x(t)$ for $t \in \mathbb{N}_a$.

Lemma 2.3 ([27])**.** *Let $h : \mathbb{N}_a \to \mathbb{R}$ and $0 < \alpha < 1$ be given. Then $x : \mathbb{N}_a \to \mathbb{R}$ is a solution of the initial value problem*

$$\begin{cases} \nabla_{a*}^{\alpha}x(t) = h(t),\ t \in \mathbb{N}_{a+1}, \\ x(a) = x_0, \end{cases} \tag{2.3}$$

if and only if x has the following representation

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\alpha-1}}h(s),\ t \in \mathbb{N}_a. \tag{2.4}$$

Lemma 2.4. *$x : \mathbb{N}_a \to \mathbb{R}$ is a solution of (1.1) if and only if x has the following representation*

$$\begin{aligned} x(t) &= \phi(a) - g(a,\phi(a-\tau(a))) + g(t,x(t-\tau(t))) \\ &+ \frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\alpha-1}}f(s,x(s),x(s-\tau(s))),\ t\ \in\ \mathbb{N}_a. \end{aligned} \tag{2.5}$$

Lemma 2.5 ([27])**.** *Let $\alpha > 0$ and $\mu > -1$. Then, for $t \in \mathbb{N}_a$, we have*

$$\nabla_a^{-\alpha}(t-a)^{\overline{\mu}} = \frac{\Gamma(1+\mu)}{\Gamma(1+\alpha+\mu)}(t-a)^{\overline{\alpha+\mu}}.$$

Definition 2.6. The solution $x = \varphi(t)$ of (1.1) is said to be

(i) stable, if for any $\varepsilon > 0$ and $a \in \mathbb{R}^+$, there exists a $\delta = \delta(a,\varepsilon) > 0$ such that $|\phi(t) - \varphi(t)| \leq \delta(a,\varepsilon)$ for $t \in [m_0,a] \cap \mathbb{N}_{m_0}$ implies

$$|x(t,\phi,a) - \varphi(t)| < \varepsilon,$$

for $t \geq a$,

(ii) attractive, if there exists $\eta(a) > 0$ such that $\|\phi\| \leq \eta$ implies

$$\lim_{t\to\infty} x(t,\phi,a) = 0,$$

(iii) asymptotically stable if it is stable and attractive.

The space $\ell_{m_0}^{\infty}$ is the set of real sequences defined on $\mathbb{N}_{m_0}$ where any individual sequence is bounded with respect to the usual supremum norm. It is well know that under the supremum norm $\ell_{m_0}^{\infty}$ is a Banach space.

Definition 2.7 ([14])**.** A set Ω of sequences in $\ell_{m_0}^{\infty}$ is uniformly Cauchy (or equi-Cauchy) if for every $\varepsilon > 0$, there exists an integer N such that $|x(i) - x(j)| < \varepsilon$ whenever $i, j > N$ for any $x = \{x(n)\}$ in Ω.

Theorem 2.8 ([14], Discrete Arzela-Ascoli's theorem)**.** *A bounded, uniformly Cauchy subset Ω of $\ell_{m_0}^{\infty}$ is relatively compact.*

Finally, we state Krasnoselskii's fixed point theorem which enables us to prove the stability of solutions to (1.1). For its proof we refer the reader to [24].

Theorem 2.9 ([24]). *Let Ω be a non-empty closed convex subset of a Banach space $(S, \|.\|)$. Suppose that T_1 and T_2 map Ω into S such that*

(i) $T_1x + T_2y \in \Omega$ for all $x, y \in \Omega$,

(ii) T_1 is continuous and $T_1\Omega$ is contained in a compact set of S,

(iii) T_2 is a contraction with constant $l < 1$.

Then there is a $x \in \Omega$ with $T_1x + T_2x = x$.

3. Main results

Let $\ell_{m_0}^{\infty}$ be the set of all real bounded sequences $x = \{x(t)\}_{t=m_0}^{\infty}$ with norm $\|x\| = \sup_{t\in\mathbb{N}_{m_0}} |x(t)|$. Then $\ell_{m_0}^{\infty}$ is a Banach space.

Define the operator $T : \ell_{m_0}^{\infty} \to \ell_{m_0}^{\infty}$ by $Tx(t) = \phi(t)$ for $t \in [m_0, a] \cap \mathbb{N}_{m_0}$ and

$$Tx(t) = T_1x(t) + T_2x(t) \text{ for } t \in \mathbb{N}_{a+1},$$

where

$$\begin{aligned} T_1x(t) &= \phi(a) - g(a, \phi(a - \tau(a))) + g(t, x(t - \tau(t))), \\ T_2x(t) &= \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{\alpha-1}} f(s, x(s), x(s - \tau(s))). \end{aligned} \tag{3.1}$$

Obviously, x is a solution of (1.1) if it is a fixed point of operator T.

Lemma 3.1. *Assume that (1.2) holds and the following condition is satisfied*

(H_1) there exist constants $\gamma_1, L_1 > 0$ such that

$$\begin{aligned} &|\phi(a)| + |g(a, \phi(a - \tau(a)))| + KL_1 (t - a)^{\overline{-\gamma_1}} \\ &+ \left| \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{\alpha-1}} f(s, x(s), x(s - \tau(s))) \right| \\ &\le L_1 (t - a)^{\overline{-\gamma_1}} \text{ for } t \in \mathbb{N}_{a+1}. \end{aligned}$$

Then there exists at least one solution x for (1.1).

Proof. Define the set

$$S_1 = \left\{ x \in \ell_{m_0}^{\infty} : x(t) = \phi(t) \text{ for } t \in [m_0, a] \cap \mathbb{N}_{m_0} \text{ and } |x(t)| \le L_1 (t - a)^{\overline{-\gamma_1}} \text{ for } t \in \mathbb{N}_{a+1} \right\}.$$

It is easy to know that S_1 is a closed, bounded and convex subset of $\ell_{m_0}^{\infty}$. In addition, for $t \in \mathbb{N}_{a+1}$, we have

$$(t - a)^{\overline{-\gamma_1}} = \frac{\Gamma(t - a - \gamma_1)}{\Gamma(t - a)} = (t - a)^{-\gamma_1} \left[1 + O\left(\frac{1}{t - a} \right) \right] \to 0 \text{ for } t \to \infty.$$

To prove that T has a fixed point, we first show that T maps S_1 to S_1. For $t \in \mathbb{N}_{a+1}$, condition (H_1) implies that

$$\begin{aligned}|Tx(t)| &\leq |\phi(a)| + |g(a, \phi(a - \tau(a)))| + KL_1(t-a)^{\overline{-\gamma_1}} \\ &+ \left| \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{\alpha-1}} f(s, x(s), x(s - \tau(s))) \right| \\ &\leq L_1(t-a)^{\overline{-\gamma_1}},\end{aligned}$$

which yields that $TS_1 \subset S_1$.

Next, we show that T_2 is continuous on S_1. Let $\varepsilon > 0$ be given, there exists a $N_1 \in \mathbb{N}_{a+1}$ such that $t > N_1$ implies that $L_1(t-a)^{\overline{-\gamma_1}} < \frac{\varepsilon}{2}$. Let $\{x_n\}$ be a sequence such that $x_n \to x$. For $t \in \{a+1, 2+a, ..., N_1\}$, applying the continuity of f and

$$\frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{\alpha-1}} = \frac{\Gamma(t - a + \alpha)}{\Gamma(\alpha + 1)\Gamma(t - a)},$$

we have

$$\begin{aligned}&|T_2x_n(t) - T_2x(t)| \\ &\leq \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{\alpha-1}} |f(s, x_n(s), x_n(s - \tau(s))) - f(s, x(s), x(s - \tau(s)))| \\ &\leq \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{\alpha-1}} \\ &\times \max_{s \in \{a+1, a+2, ..., N_1\}} |f(s, x_n(s), x_n(s - \tau(s))) - f(s, x(s), x(s - \tau(s)))| \\ &= \frac{\Gamma(t - a + \alpha)}{\Gamma(\alpha + 1)\Gamma(t - a)} \max_{s \in \{a+1, a+2, ..., N_1\}} |f(s, x_n(s), x_n(s - \tau(s))) - f(s, x(s), x(s - \tau(s)))| \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

For $t \in \{N_1 + 1, N_1 + 2, ...\}$, we have

$$\begin{aligned}&|T_2x_n(t) - T_2x(t)| \\ &= \left| \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{\alpha-1}} f(s, x_n(s), x_n(s - \tau(s))) \right. \\ &\left. - \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{\alpha-1}} f(s, x(s), x(s - \tau(s))) \right| \\ &\leq 2L_1(t-a)^{\overline{-\gamma_1}} < \varepsilon.\end{aligned}$$

Thus, for all $t \in \mathbb{N}_{a+1}$, we have

$$|T_2x_n(t) - T_2x(t)| \to 0 \text{ as } n \to \infty.$$

which means that T_2 is continuous.

Now, we show that T_2S_1 is relatively compact. Letting $t_1,t_2 \in \mathbb{N}_{a+1}$ and $t_2 > t_1 \geq N_1$, we have

$$\begin{aligned}
&\left|T_2x(t_2) - T_2x(t_1)\right| \\
&= \frac{1}{\Gamma(\alpha)}\left|\sum_{s=a+1}^{t_1}(t-\rho(s))^{\overline{\alpha-1}} f(s,x(s),x(s-\tau(s)))\right. \\
&\left. - \sum_{s=a+1}^{t_2}(t-\rho(s))^{\overline{\alpha-1}} f(s,x(s),x(s-\tau(s)))\right| \\
&\leq L_1(t-a)^{\overline{-\gamma_1}} + L_1(t-a)^{\overline{-\gamma_1}} < \varepsilon.
\end{aligned}$$

Then $\{T_2x : x \in S_1\}$ is a bounded and uniformly Cauchy subset. Hence, by Theorem 2.8, T_2S_1 is relatively compact.

Finally, we prove that T_1 is a contraction. Letting $x, y \in S_1$, we have

$$\begin{aligned}
\left|T_1x(t) - T_1y(t)\right| &= \left|g(t,x(t-\tau(t))) - g(t,y(t-\tau(t)))\right| \\
&\leq K\|x-y\|.
\end{aligned}$$

Then

$$\|T_1x - T_1y\| \leq K\|x-y\|,$$

which means that T_1 is a contraction by (1.2).

According to Theorem 2.9, we have that T has a fixed point in S_1 which is a solution of (1.1). This completes the proof. □

Theorem 3.2. *Assume that conditions (1.2) and (H_1) hold, then the solutions of (1.1) are attractive.*

Proof. By Lemma 3.1, the solutions of (1.1) exist and are in S_1. All functions x in S_1 tend to 0 as $t \to \infty$. Then the solutions of (1.1) tend to zero as $t \to \infty$. This completes the proof. □

Theorem 3.3. *Assume that (1.2) holds and the following condition is satisfied*

(H_2) *there exist constants $\gamma_2 \in (\alpha, 1)$ and $L_2 > 0$ such that*

$$\begin{aligned}
&\left|f(t,x(t),x(t-\tau(t))) - f(t,y(t),y(t-\tau(t)))\right| \\
&\leq L_2(t-a)^{\overline{-\gamma_2}}\|x-y\|,\ t \in \mathbb{N}_{a+1}.
\end{aligned}$$

Then the solutions of (1.1) are stable provided that

$$c = L_2\Gamma(1-\gamma_2) + K < 1. \tag{3.2}$$

Proof. Let x be a solution of (1.1), and let $\tilde{x}$ be a solution of (1.1) satisfying the initial condition $\tilde{x}(t) = \tilde{\phi}(t)$ for $t \in [m_0, a] \cap \mathbb{N}_{m_0}$. For $t \in \mathbb{N}_{a+1}$, applying (1.2) and condition (H_2) we have

$$\begin{aligned}
&|x(t) - \tilde{x}(t)| \\
&= \left|\phi(a) - \tilde{\phi}(a)\right| + \left|g(a, \phi(a - \tau(a))) - g\left(a, \tilde{\phi}(a - \tau(a))\right)\right| \\
&+ |g(t, x(t - \tau(t))) - g(t, \tilde{x}(t - \tau(t)))| \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{\alpha - 1}} |f(s, x(s), x(s - \tau(s))) - f(s, \tilde{x}(s), \tilde{x}(s - \tau(s)))| \\
&\leq \left|\phi(a) - \tilde{\phi}(a)\right| + K\left|\phi(a - \tau(a)) - \tilde{\phi}(a - \tau(a))\right| + K\|x - \tilde{x}\| \\
&+ \frac{L_2}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{\alpha - 1}} (t - a)^{\overline{-\gamma_2}} \|x - \tilde{x}\| \\
&= \left|\phi(a) - \tilde{\phi}(a)\right| + K\left|\phi(a - \tau(a)) - \tilde{\phi}(a - \tau(a))\right| + K\|x - \tilde{x}\| \\
&+ L_2 \nabla_a^{-\alpha} (t - a)^{\overline{-\gamma_2}} \|x - \tilde{x}\| \\
&= \left|\phi(a) - \tilde{\phi}(a)\right| + K\left|\phi(a - \tau(a)) - \tilde{\phi}(a - \tau(a))\right| + K\|x - \tilde{x}\| \\
&+ \frac{L_2 \Gamma(1 - \gamma_2)}{\Gamma(1 + \alpha - \gamma_2)} t^{\overline{\alpha - \gamma_2}} \|x - \tilde{x}\| \\
&\leq (1 + K)\left\|\phi - \tilde{\phi}\right\| + (K + L_2 \Gamma(1 - \gamma_2)) \|x - \tilde{x}\|,
\end{aligned}$$

which yields that

$$\|x - \tilde{x}\| \leq (1 + K)\left\|\phi - \tilde{\phi}\right\| + c\|x - \tilde{x}\|.$$

This further implies

$$\|x - \tilde{x}\| \leq \frac{1 + K}{1 - c}\left\|\phi - \tilde{\phi}\right\|.$$

Then, for any $\varepsilon > 0$, let $\delta = \frac{1-c}{1+K}\varepsilon$, $\left\|\phi - \tilde{\phi}\right\| < \delta$ implies that $\|x - \tilde{x}\| < \varepsilon$. Therefore, the solutions of (1.1) are stable. This completes the proof. □

Combining Theorem 2.8 and Theorem 2.9, we have

Theorem 3.4. *Assume that conditions (1.2), (H_1) and (H_2) hold, then the solutions of (1.1) are asymptotically stable provided that (3.2) holds.*

Lemma 3.5. *Assume that (1.2) holds and the following condition is satisfied*

(H_3) *there exist constants $\gamma_3 \in (\alpha, 1)$ and $L_3 > 0$ such that*

$$\begin{aligned}
&\left|\frac{\phi(a) - g(a, \phi(a - \tau(a))) + g(t, x(t - \tau(t)))}{\Gamma(1 - \alpha)}(s - a)^{\overline{\alpha}} + f(s, x(s), x(s - \tau(s)))\right| \\
&\leq L_3 (s - a)^{\overline{-\gamma_3}} \text{ for } s \in \mathbb{N}_{a+1},\ t \in \mathbb{N}_{a+1}.
\end{aligned}$$

Then exists at least one solution x for (1.1).

Proof. Define the set

$$\begin{aligned}
S_2 = \Big\{ x \in \ell_{m_0}^{\infty} : x(t) = \phi(t) \text{ for } t \in [m_0, a] \cap \mathbb{N}_{m_0} \\
\text{and } |x(t)| \leq \frac{L_3 \Gamma(1 - \gamma_3)}{\Gamma(1 + \alpha - \gamma_3)}(t - a)^{\overline{\alpha - \gamma_3}} \text{ for } t \in \mathbb{N}_{a+1} \Big\}.
\end{aligned}$$

From the above assumption of S_2, it is easy to know that S_2 is a closed, bounded and convex subset of $\ell_{m_0}^{\infty}$. First, we show that T maps S_2 to S_2. For $t \in \mathbb{N}_{a+1}$ from condition (H_3) and Lemma 2.5, we have

$$\begin{aligned}
&|Tx(t)| \\
&= \Bigg|\phi(a) - g(a, \phi(a - \tau(a))) + g(t, x(t - \tau(t))) \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{\alpha-1}} f(s, x(s), x(s - \tau(s)))\Bigg| \\
&= \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{\alpha-1}} \\
&\times \left|\frac{\phi(a) - g(a, \phi(a - \tau(a))) + g(t, x(t - \tau(t)))}{\Gamma(1-\alpha)} (s-a)^{\overline{\alpha}} + f(s, x(s), x(s - \tau(s)))\right| \\
&\leq \frac{L_3}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{\alpha-1}} (s-a)^{\overline{-\gamma_3}} \\
&= L_3 \nabla_a^{-\alpha} (t-a)^{\overline{-\gamma_3}} \\
&= \frac{L_3 \Gamma(1-\gamma_3)}{\Gamma(1+\alpha-\gamma_3)} (t-a)^{\overline{\alpha-\gamma_3}}.
\end{aligned}$$

Then $TS_2 \subset S_2$. The proofs that T_2 is continuous, T_2S_2 is relatively compact and T_1 is contractive are similar to that of Lemma 3.1. So, we omit it. By Theorem 2.9, we have that T has a fixed point in S_2 which is a solution of (1.1). This completes the proof. □

Theorem 3.6. *Assume that conditions (1.2) and (H_3) hold, then the solutions of (1.1) are attractive.*

Theorem 3.7. *Assume that conditions (1.2), (H_2) and (H_3) hold, then the solutions of (1.1) are asymptotically stable provided that (3.2) holds.*

Lemma 3.8. *Assume that (1.2) holds and the following condition is satisfied*

(H_4) *There exist constants $\beta > \frac{1}{1-\alpha}$ and $L_4 > 0$ such that*

$$\begin{aligned}
&\left|\frac{\phi(a) - g(a, \phi(a - \tau(a))) + g(t, x(t - \tau(t)))}{\Gamma(1-\alpha)} (s-a)^{\overline{\alpha}} + f(s, x(s), x(s - \tau(s)))\right| \\
&\leq L_4 |x(s+\gamma_4)|^{\beta} \text{ for } s \in \mathbb{N}_{a+1},\ t \in \mathbb{N}_{a+1}.
\end{aligned}$$

Then (1.1) exists at least one solution x on $\mathbb{N}_1$ provided that

$$\frac{L_4 \Gamma(1+\beta\gamma_4) \Gamma(1-\beta\gamma_4)}{\Gamma^{\beta}(1+\gamma_4) \Gamma(1+\alpha-\beta\gamma_4)} \leq 1, \tag{3.3}$$

where

$$\frac{\alpha}{\beta-1} < \gamma_4 < \frac{1}{\beta}. \tag{3.4}$$

Proof. From $\beta > \frac{1}{1-\alpha}$, we have that $\frac{\alpha}{\beta-1} < \frac{1}{\beta}$ which implies that γ_4 exists. In addition, $\gamma_4 < \frac{1}{\beta}$ means that $\Gamma(1-\beta\gamma_4) > 0$ and $\Gamma(1+\alpha-\beta\gamma_4) > 0$, $\frac{\alpha}{\beta-1} < \gamma_4$ implies that $\alpha - \beta\gamma_4 < -\gamma_4$. Define the set

$$S_3 = \left\{x \in \ell_{m_0}^{\infty} : x(t) = \phi(t) \text{ for } t \in [m_0, a] \cap \mathbb{N}_{m_0} \text{ and } |x(t)| \leq (t-a)^{\overline{-\gamma_4}} \text{ for } t \in \mathbb{N}_{a+1}\right\}.$$

We show that T maps S_3 to S_3. For $t \in \mathbb{N}_{a+1}$, applying condition (H_4) and (3.3), we have

$$\begin{aligned}
&|Tx(t)| \\
&= \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} \\
&\times \left| \frac{\phi(a) - g(a,\phi(a-\tau(a))) + g(t,x(t-\tau(t)))}{\Gamma(1-\alpha)} (s-a)^{\overline{\alpha}} + f(s,x(s),x(s-\tau(s))) \right| \\
&\le \frac{L_4}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} |x(s+\gamma_4)|^{\beta} \\
&\le \frac{L_4}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} \left[(t-a+\gamma_4)^{\overline{-\gamma_4}} \right]^{\beta} \\
&\le \frac{L_4 \Gamma(1+\beta\gamma_4)}{\Gamma(\alpha)\Gamma^{\beta}(1+\gamma_4)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} (t-a+\beta\gamma_4)^{\overline{-\beta\gamma_4}} \\
&= \frac{L_4 \Gamma(1+\beta\gamma_4)\Gamma(1-\beta\gamma_4)}{\Gamma^{\beta}(1+\gamma_4)\Gamma(1+\alpha-\beta\gamma_4)} (t-a)^{\overline{\alpha-\beta\gamma_4}} \\
&\le (t-a)^{\overline{\alpha-\beta\gamma_4}} \\
&\le (t-a)^{\overline{-\gamma_4}}.
\end{aligned}$$

Then $TS_3 \subset S_3$. Using Lemma 3.5, we find the desired conclusion immediately. □

Theorem 3.9. *Assume that conditions (1.2), (H_4) and (3.3) hold, then the solutions of (1.1) are attractive.*

Theorem 3.10. *Assume that conditions (1.2), (H_2) and (H_4) hold, then the solutions of (1.1) are asymptotically stable provided that (3.2) and (3.3) hold.*

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