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ON EXISTENCE RESULTS FOR A NEW CLASS OF VARIATIONAL-HEMIVARIATIONAL-LIKE INEQUALITIES IN BANACH SPACES

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Abstract. This paper is devoted to studying the existence of solutions for a new class of variational-hemivariational-like inequalities in reflexive Banach spaces. Using the notions of Φ-monotonicity and $\mathscr{H}$-hemicontinuity of a nonempty weakly compact-valued mapping, and the properties of Clarke's generalized directional derivative and Clarke's generalized gradient, we prove some existence results of solutions when the constrained set is nonempty, bounded (or unbounded), closed and convex.

Keywords. Variational-hemivariational-like inequality; Clarke's generalized directional derivative; $\mathscr{H}$-hemicontinuity; Φ-monotonicity; KKM mapping.

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1. INTRODUCTION

As a very powerful tool of the current mathematical technology, the variational inequalities, introduced and studied by Hartman and Stampacchia [1] in the early 1960s, have been extended and generalized to study a wide class of problems arising in mechanics, physics, optimization and control, nonlinear programming, economics, transportation equilibrium problems, engineering sciences and so on (see e.g., [2, 3, 4, 5, 6, 7]).

As an important and useful generalization of variational inequality, the hemivariational inequality was first introduced in order to formulate variational principles involving nonconvex and nonsmooth energy functions, and investigated by Panagiotopoulos [8] using the mathematical concepts of the Clarke's generalized directional derivative and the Clarke's generalized gradient. The hemivariational inequalities have been proved very efficient to describe a variety of mechanical and engineering problems, e.g.,

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non-monotone semipermeability problems, unilateral contact problems in nonlinear elasticity; see e.g., [9, 10] and the references therein.

On the other hand, the generalization of known theory and methods in the fields of optimization and nonlinear analysis from Euclidean spaces to Riemannian manifolds is in a certain sense natural, and advantageous in some cases. In 2003, Nemeth [11] introduced the notion of variational inequality on Hadamard manifolds. Since then, the theory of variational inequalities and equilibrium problems on Riemannian/Hadamard manifolds was developed by many researchers; see e.g., [12, 13, 14] and the references therein. Very recently, Tang, Zhou and Huang [15] introduced and studied the following hemivariational inequality problem on Hadamard manifold $\mathbf{M}$.

Let $\mathbf{M}$ be a Hadamard manifold and let K be a nonempty, closed and convex subset of $\mathbf{M}$. Let $A : K \to T\mathbf{M}$ be a vector field, that is, $A(x) \in T_x\mathbf{M}$ for each $x \in K$. Let $J : \mathbf{M} \to \mathbf{R}$ be a locally Lipschitz function. Then the hemivariational inequality problem consists in finding an element $x \in K$ such that

$$\langle A(x), \exp_x^{-1} y\rangle + J^\circ(x; \exp_x^{-1} y) \geq 0, \quad \forall y \in K,$$

where $J^\circ(x;z)$ denotes the generalized directional derivative in the sense of Clarke at the point $x \in K$ in the direction $z \in T_x\mathbf{M}$. Such a hemivariational inequality problem is denoted by HVIP(A,J,K).

Let X be a real reflexive Banach space with the norm denoted by $\|\cdot\|$ and let X^* be its dual space. We denote by $\langle\cdot,\cdot\rangle$ the duality pairing between X and its dual X^*. Let K be a nonempty, closed and convex subset of X and let $\eta : K \times K \to X$ be a mapping. Let $\Phi : X^* \times K \times K \to \mathbf{R}$ be a function and let $F : K \to 2^{X^*}$ be a set-valued mapping with nonempty values. Let $J^\circ(\cdot;\cdot)$ be Clarke's generalized directional derivative of a locally Lipschitz function $J : X \to \mathbf{R}$, and let $\phi : X \to \mathrm{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function such that

$$K_\phi = K \cap \mathrm{dom}\phi \neq \emptyset,$$

where $\mathrm{dom}\phi = \{x \in X : \phi(x) < +\infty\}$. Motivated and inspired by the research work going on this direction, in this paper, we investigate the existence of solutions for a new class of variational-hemivariational-like inequality problems in reflexive Banach spaces, that is, the following variational-hemivariational-like inequality problem:

(VHVLIP) Find $x \in K_\phi$ such that for some $u \in F(x)$

$$\Phi(u,x,y) + J^\circ(x; \eta(y,x)) + \phi(y) - \phi(x) \geq 0, \quad \forall y \in K. \tag{1.1}$$

Some special cases of the (VHVLIP) are stated as follows.

(I) If J is a constant, $\eta(x,y) = x - y$ for each $(x,y) \in K \times K$ and $\Phi(\nu,x,y) = \langle \nu, \eta(y,x)\rangle$ for each $(\nu,x,y) \in X^* \times K \times K$, then problem (1.1) reduces to the following generalized mixed variational inequality problem: find an element $x \in K_\phi$ such that for some $u \in F(x)$

$$\langle u, y - x\rangle + \phi(y) - \phi(x) \geq 0, \quad \forall y \in K, \tag{1.2}$$

which is just the problem considered in [16].

(II) If J is a constant, $F = A$ a single-valued mapping and $\Phi(\nu,x,y) = \langle \nu, \eta(y,x)\rangle$ for each $(\nu,x,y) \in X^* \times K \times K$, then problem (1.1) reduces to the following mixed variational-like inequality problem: find an element $x \in K_\phi$ such that

$$\langle A(x), \eta(y,x)\rangle + \phi(y) - \phi(x) \geq 0, \quad \forall y \in K, \tag{1.3}$$

which was studied by Schaible, Yao and Zeng [17] in the case of $X = H$ a Hilbert space. In addition, it was also considered in [18] as a special case of mixed quasivariational-like inequalities.

(III) If $\eta(x,y) = x - y$ for each $(x,y) \in K \times K$ and $\Phi(v,x,y) = \langle v, \eta(y,x)\rangle$ for each $(v,x,y) \in X^* \times K \times K$, then problem (1.1) reduces to the following variational-hemivariational inequality problem: find an element $x \in K_\phi$ such that for some $u \in F(x)$

$$\langle u, y - x\rangle + J^\circ(x; y - x) + \phi(y) - \phi(x) \geq 0, \quad \forall y \in K, \tag{1.4}$$

which is just the problem considered by Tang and Huang [19].

(IV) If $F = A$ a single-valued mapping and ϕ is a constant, then problem (1.4) becomes to finding an element $x \in K$ such that

$$\langle A(x), y - x\rangle + J^\circ(x; y - x) \geq 0, \quad \forall y \in K, \tag{1.5}$$

which is called the Hartman-Stampacchia hemivariational inequality considered extensively by several authors in two cases of space X (i.e., finite-dimensional and infinite-dimensional); see e.g., [20, 21, 22, 23] and the references therein.

Using the notions of Φ-monotonicity and $\mathscr{H}$-hemicontinuity of a nonempty weakly compact-valued mapping, and the properties of Clarke's generalized directional derivative and Clarke's generalized gradient, some existence results of solutions for the (VHVLIP) are proved when the constrained set is a nonempty, bounded (or unbounded), closed and convex set. In particular, a sufficient condition to the boundedness of the solution set and a necessary and sufficient condition to the existence of solutions are derived. The results presented in this paper generalize, improve and develop some known results in the earlier and recent literature.

The rest of the paper is organized in the following way. In the next section, we introduce and recall some definitions, notations and necessary materials. Section 3 is devoted to proving our main results. We show the existence of solutions in the case when the constraint set K is bounded and unbounded in Theorems 3.3 and 3.4, respectively. Theorem 3.5 provides a sufficient condition to the boundedness of the solution set. Theorem 3.4 gives a necessary and sufficient condition to the existence of solutions of the (VHVLIP).

2. Preliminaries

Throughout this paper, we assume that X is a real reflexive Banach space and the norms of X and its dual X^* are denoted by the same symbol $\|\cdot\|$. Let K be a nonempty, closed and convex subset of X. Let x_0 and $\{x_n\}$ be a point and a sequence in X, respectively. We use the notations $x_n \to x_0$ and $x_n \rightharpoonup x_0$ to indicate the strong convergence of $\{x_n\}$ to x_0 and the weak convergence of $\{x_n\}$ to x_0, respectively. Moreover, we denote by $\langle\cdot,\cdot\rangle$ the duality pairing between X and its dual X^*. Let $\eta : K \times K \to X$ be a mapping. η is said to be skew if $\eta(x,y) + \eta(y,x) = 0$ for each $(x,y) \in K \times K$.

Definition 2.1. Let $\Phi : X^* \times K \times K \to \mathbf{R}$ be a function and let $F : K \to 2^{X^*}$ be a set-valued mapping with nonempty values. F is said to be Φ-monotone if

$$\Phi(u,y,x) + \Phi(v,x,y) \geq 0, \quad \forall x,y \in K, \quad u \in F(x) \quad \text{and} \quad v \in F(y).$$

In particular, if $\Phi(v,x,y) = \langle v, \eta(y,x)\rangle$ for each $(v,x,y) \in X^* \times K \times K$, then Definition 2.1 is reduced to the following.

Definition 2.2. Let $\eta : K \times K \to X$ be a mapping, and let $F : K \to 2^{X^*}$ be a set-valued mapping with nonempty values. F is said to be η-monotone if

$$\langle u - v, \eta(x,y) \rangle \geq 0, \quad \forall x,y \in K, u \in F(x), v \in F(y).$$

If $F = A$ a single-valued mapping, then Definition 2.2 is reduced to the following.

Definition 2.3. Let $\eta : K \times K \to X$ be a mapping and let $A : K \to X^*$ be a single-valued mapping. A is said to be η-monotone, if

$$\langle A(x) - A(y), \eta(x,y) \rangle \geq 0, \quad \forall x,y \in K.$$

Assume that $J : X \to \mathbf{R}$ is a locally Lipschitz function, x is a given point and y is a vector in X. The Clarke's generalized directional derivative of J at x in the direction y, denoted by $J^\circ(x;y)$, is defined by

$$J^\circ(x;y) = \limsup_{z \to x \quad \lambda \downarrow 0} \frac{J(z + \lambda y) - J(z)}{\lambda},$$

by means of which the Clarke's generalized gradient of J at x, denoted by $\partial J(x)$, is the subset of the dual space X^* defined by

$$\partial J(x) = \{\xi \in X^* : J^\circ(x;y) \geq \langle \xi, y \rangle, \quad \forall y \in X\}.$$

The next lemma provides some basic properties for the Clarke's generalized directional derivative and the Clarke's generalized gradient; see e.g., [24] and the references therein.

Lemma 2.4. *Let X be a Banach space, $x,y \in X$ and $J : X \to \mathbf{R}$ be a locally Lipschitz function defined on X. Then*

(i) The function $y \mapsto J^\circ(x;y)$ is finite, positively homogeneous, subadditive and then convex on X;

(ii) $J^\circ(x;y)$ is upper semicontinuous on $X \times X$ as a function of (x,y), i.e., for all $x,y \in X$, $\{x_n\} \subseteq X$, $\{y_n\} \subseteq X$ such that $x_n \to x$ and $y_n \to y$ in X, we have that

$$\limsup_{n \to \infty} J^\circ(x_n;y_n) \leq J^\circ(x;y);$$

(iii) $J^\circ(x;-y) = (-J)^\circ(x;y)$;

(iv) For all $x \in X$, $\partial J(x)$ is a nonempty, convex, bounded and weak*-compact subset of X^*;

(v) For every $y \in X$, one has

$$J^\circ(x;y) = \max\{\langle \xi, y \rangle : \xi \in \partial J(x)\};$$

(vi) The graph of the Clarke's generalized gradient $\partial J(x)$ is closed in $X \times (w^*\text{-}X^*)$ topology, where $(w^*\text{-}X^*)$ denotes the space X^* equipped with weak* topology, i.e., if $\{x_n\} \subseteq X$ and $\{x_n^*\} \subseteq X^*$ are sequences such that $x_n^* \in \partial J(x_n)$, $x_n \to x$ in X and $x_n^* \to x^*$ weak*ly in X^*, then $x^* \in \partial J(x)$.

Let A_1, A_2 be nonempty subsets of a Banach space X. The Hausdorff metric $\mathscr{H}(\cdot,\cdot)$ between A_1 and A_2 is defined by

$$\mathscr{H}(A_1,A_2) = \max\{e(A_1,A_2), e(A_2,A_1)\},$$

where $e(A_1,A_2) = \sup_{a \in A_1} d(a,A_2)$ with $d(a,A_2) = \inf_{b \in A_2} \|a - b\|$. Note that [25] if A_1 and A_2 are nonempty, closed and bounded subsets in X, then for each $a \in A_1$ and each $\varepsilon > 0$ there exists $b \in A_2$ such that

$$\|a - b\| \leq (1 + \varepsilon)\mathscr{H}(A_1,A_2).$$

In particular, if A_1 and A_2 are compact subsets in X, then for each $a \in A_1$ there exists $b \in A_2$ such that

$$\|a-b\| \leq \mathscr{H}(A_1, A_2).$$

Definition 2.5. [26]. Let K be a nonempty, closed and convex subset of X. Let $\mathscr{H}(\cdot,\cdot)$ be the Hausdorff metric on the collection $CB(X^*)$ of all nonempty, closed and bounded subsets of X^*, which is defined by

$$\mathscr{H}(A,B) = \max\{e(A,B), e(B,A)\}$$

for A and B in $CB(X^*)$. A nonempty set-valued mapping $F : K \to CB(X^*)$ is said to be

(i) $\mathscr{H}$-hemicontinuous, if for any $x, y \in K$, the function $t \mapsto \mathscr{H}(F(x+t(y-x)), F(x))$ from $[0,1]$ into $\mathbf{R}^+ = [0,+\infty)$ is continuous at 0^+;

(ii) $\mathscr{H}$-continuous, if for any $\varepsilon > 0$ and any fixed $x_0 \in K$, there exists $\delta > 0$ such that for all $y \in K$ with $\|y - x_0\| < \delta$, one has $\mathscr{H}(F(y), F(x_0)) < \varepsilon$.

Remark 2.6. Clearly, the $\mathscr{H}$-continuity implies the $\mathscr{H}$-hemicontinuity, but the converse is not true in general.

Definition 2.7. A proper functional $\varphi : X \to \mathbf{R} \cup \{+\infty\}$ is said to be convex if, for all $x, y \in X$ and $t \in [0,1]$,

$$\varphi(x+t(y-x)) \leq (1-t)\varphi(x) + t\varphi(y).$$

In order to get an existence of solutions for the variational-hemivariational-like inequality problem (VHVLIP) in the case when the constraint set K is bounded, we now recall the following Fan-KKM lemma; see e.g., [27].

Lemma 2.8. *Let K be a nonempty subset of a Hausdorff topological vector space E and let $G : K \to 2^E$ be a set-valued mapping with nonempty-values such that the following properties hold:*

(i) G is a KKM mapping, i.e., for any $x_1, x_2, \ldots, x_n \in K$, there holds $\mathrm{co}(\{x_1, x_2, \ldots, x_n\}) \subseteq \bigcup_{i=1}^n G(x_i)$;

(ii) $G(x)$ is closed in E for every $x \in K$;

(iii) $G(x_0)$ is compact in E for some $x_0 \in K$.

Then $\bigcap_{x \in K} G(x) \neq \emptyset$.

Definition 2.9. Let X, Y be two real Banach spaces and $Q : X \times Y \to \mathbf{R}$ be a function. Q is said to be bi-sequentially weakly upper semicontinuous if, for any $\{x_n\} \subseteq X$ and $\{y_n\} \subseteq Y$ with $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$, one has $\limsup_{n\to\infty} Q(x_n, y_n) \leq Q(x,y)$.

3. Existence and boundedness results

We first establish the equivalence between the Hartman-Stampacchia variational-hemivariational-like inequality and the Minty variational-hemivariational-like inequality (i.e., problem (3.2)) under some suitable conditions, which is useful in proving the existence theorem in the case when the constraint set is bounded.

Proposition 3.1. *Let K be a nonempty, closed and convex subset of X, and let $\eta : K \times K \to X$ be a skew mapping that is affine in the first variable. Let the function $\Phi : X^* \times K \times K \to \mathbf{R}$ satisfy the conditions:*

(i) Φ is weakly upper semicontinuous in the first variable,

(ii) Φ is convex in the third variable, and

(iii) $\Phi(\nu,x,y)+\Phi(\nu,y,x)=0$ *for each* $(\nu,x,y)\in X^*\times K\times K$.

Let $F:K\to 2^{X^*}$ *be a nonempty weakly compact-valued mapping, which is* Φ*-monotone and* $\mathscr{H}$*-hemicontinuous. Let* $J:X\to\mathbf{R}$ *be a locally Lipschitz function and let* $\phi:X\to\mathrm{R}\cup\{+\infty\}$ *be a proper, convex and lower semicontinuous function such that* $K_\phi\neq\emptyset$*. Then* $x\in K_\phi$ *solves the (VHVLIP), i.e., for some* $u\in F(x)$*,*

$$\Phi(u,x,y)+J^\circ(x;\eta(y,x))+\phi(y)-\phi(x)\geq 0,\quad \forall y\in K, \tag{3.1}$$

if and only if

$$\Phi(v,x,y)+J^\circ(x;\eta(y,x))+\phi(y)-\phi(x)\geq 0,\quad \forall v\in F(y),\quad y\in K. \tag{3.2}$$

Proof. First, the implication (3.1)⇒(3.2). Indeed, since $\Phi(\nu,x,y)+\Phi(\nu,y,x)=0$ for each $(\nu,x,y)\in X^*\times K\times K$, from the Φ-monotonicity of F it follows that

$$\Phi(v,x,y)\geq-\Phi(u,y,x)=\Phi(u,x,y).$$

If $x\in K_\phi$ solves the (VHVLIP), then (3.2) becomes a direct consequence of the last inequality. Now we show that the inverse implication relation holds. Indeed, assume that $x\in K_\phi$ solves problem (3.2). Let $y\in K$ be arbitrarily fixed and set $y_t=x+t(y-x)$ for each $t\in[0,1]$. From (3.2), it follows that for any fixed $v_t\in F(y_t)$, $t\in(0,1]$,

$$\Phi(v_t,x,y_t)+J^\circ(x;\eta(y_t,x))+\phi(y_t)-\phi(x)\geq 0. \tag{3.3}$$

Since ϕ is a proper, convex and lower semicontinuous function, Φ is convex in the third variable, η is affine in the first variable and $J^\circ(x;\cdot)$ is positively homogeneous, we have

$$\begin{aligned}0&\leq\Phi(v_t,x,y_t)+J^\circ(x;\eta(y_t,x))+\phi(y_t)-\phi(x)\\&\leq(1-t)\Phi(v_t,x,x)+t\Phi(v_t,x,y)+J^\circ(x;(1-t)\eta(x,x)+t\eta(y,x))\\&\qquad+(1-t)\phi(x)+t\phi(y)-\phi(x)\\&=t[\Phi(v_t,x,y)+J^\circ(x;\eta(y,x))+\phi(y)-\phi(x)],\end{aligned}$$

which together with $t>0$, implies that

$$\Phi(v_t,x,y)+J^\circ(x;\eta(y,x))+\phi(y)-\phi(x)\geq 0. \tag{3.4}$$

Noting that F is a nonempty weakly compact-valued mapping, we know that $F(y_t)$ and $F(x)$ are nonempty weakly compact subsets in X. Hence, by Nadler's result [25] we know that for each $t\in(0,1]$ and each fixed $v_t\in F(y_t)$ there exists an $u_t\in F(x))$ such that

$$\|v_t-u_t\|\leq(1+t)\mathscr{H}(F(y_t),F(x)).$$

Since $F(x)$ is weakly compact, without loss of generality, we may assume that $u_t\rightharpoonup u\in F(x)$ as $t\to 0^+$. Since F is $\mathscr{H}$-hemicontinuous, by item (i) of Definition 2.5, we obtain that

$$\|v_t-u_t\|\leq(1+t)\mathscr{H}(F(y_t),F(x))=(1+t)\mathscr{H}(F(x+t\eta(y,x)),F(x))\to 0\quad\text{as}\quad t\to 0^+,$$

which immediately implies that for every $f\in X^*$

$$\begin{aligned}|f(v_t)-f(u)|&\leq|f(v_t)-f(u_t)|+|f(u_t)-f(u)|\\&\leq\|f\|\|v_t-u_t\|+|f(u_t)-f(u)|\to 0\quad\text{as}\quad t\to 0^+.\end{aligned}$$

This means that $v_t \rightharpoonup u$ as $t \to 0^+$. Since Φ is weakly upper semicontinuous in the first variable, taking the limit in (3.4) as $t \to 0^+$, we deduce that

$$\Phi(u,x,y) + J^\circ(x;\eta(y,x)) + \phi(y) - \phi(x) \geq 0. \tag{3.5}$$

Since $y \in K$ is arbitrary, it follows that x solves problem (3.1). This completes the proof. □

Remark 3.2. In terms of the notions of Φ-monotonicity and $\mathscr{H}$-hemicontinuity of a nonempty weakly compact-valued mapping, Proposition 3.1 generalizes, improves and develops Proposition 3.1 of Tang, Zhou and Huang [15] from the hemivariational inequality problem to the variational-hemivariational-like inequality problem (VHVLIP). It is worth pointing out that Proposition 3.1 of [15] includes Lemma 2 of Nemeth [11] as a special case. Indeed, whenever the function J is a constant, then Proposition 3.1 of [15] reduces to Lemma 2 of [11].

Next, utilizing the Fan-KKM lemma, we establish the first existence theorem for solutions of the (VHVLIP) in the case when the constraint set K is bounded.

Theorem 3.3. *Let K be a nonempty, bounded, closed and convex subset of X, and let $\eta : K \times K \to X$ be a skew mapping that is affine in the first variable and weakly continuous in the second variable. Let the function $\Phi : X^* \times K \times K \to \mathbf{R}$ satisfy the conditions:*

(i) $\Phi(\cdot,\cdot,y) : X^* \times K \to \mathbf{R}$ is bi-sequentially weakly upper semicontinuous for each $y \in K$,
(ii) Φ is convex in the third variable, and
(iii) $\Phi(v,x,y) + \Phi(v,y,x) = 0$ for each $(v,x,y) \in X^* \times K \times K$.

Let $F : K \to 2^{X^*}$ be a nonempty weakly compact-valued mapping, which is Φ-monotone and $\mathscr{H}$-hemicontinuous. Let $\phi : X \to \mathrm{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function such that $K_\phi \neq \emptyset$ and let $J : X \to \mathbf{R}$ be a locally Lipschitz function such that $J^\circ(\cdot;\cdot)$ is bi-sequentially weakly upper semicontinuous. Then the (VHVLIP) admits at least one solution.

Proof. For any $y \in K_\phi$, define two set-valued mappings $\Gamma, \tilde{\Gamma} : K_\phi \to 2^X$ as follows:

$$\Gamma(y) := \{x \in K_\phi : \Phi(u,x,y) + J^\circ(x;\eta(y,x)) + \phi(y) - \phi(x) \geq 0 \quad \text{for} \quad \text{some} \quad u \in F(x)\} \tag{3.6}$$

and

$$\tilde{\Gamma}(y) := \{x \in K_\phi : \Phi(v,x,y) + J^\circ(x;\eta(y,x)) + \phi(y) - \phi(x) \geq 0, \quad \forall v \in F(y)\}. \tag{3.7}$$

We divide the rest of the proof into several steps.

Step 1. Γ is a KKM mapping.

Suppose on the contrary that there exist $\{y_1, y_2, \ldots, y_n\} \subseteq K_\phi$ and a point $\hat{x}$ such that $\hat{x} \in \mathrm{co}(\{y_1, y_2, \ldots, y_n\})$ but $\hat{x} \notin \bigcup_{i=1}^n \Gamma(y_i)$, that is, for any $\hat{u} \in \Gamma(\hat{x})$

$$\Phi(\hat{u},\hat{x},y_i) + J^\circ(\hat{x};\eta(y_i,\hat{x})) + \phi(y_i) - \phi(\hat{x}) < 0, \quad \forall i \in \{1,2,\ldots,n\}. \tag{3.8}$$

Define a set

$$V := \{y \in K_\phi : \Phi(\hat{u},\hat{x},y) + J^\circ(\hat{x};\eta(y,\hat{x})) + \phi(y) - \phi(\hat{x}) < 0, \quad \forall \hat{u} \in F(\hat{x})\}.$$

Next, we claim that the set V is convex. Indeed, take y' and y'' in V such that $y' \neq y''$. We will show that the line segment $y_t = y' + t(y'' - y'), t \in [0,1]$ starting at y' and ending at y'' (i.e., $y_0 = y'$ and $y_1 = y''$) is contained in V, i.e., $y_t \in V$ for all $t \in [0,1]$. Since ϕ is a proper, convex and lower semicontinuous function, Φ is convex in the third variable, η is affine in the first variable and $J^\circ(\hat{x};\cdot)$ is positive homogeneous and subadditive, we deduce from $y',y''\in V$ that

$$\begin{aligned}
&\Phi(\hat{u},\hat{x},y_t)+J^\circ(\hat{x};\eta(y_t,\hat{x}))+\phi(y_t)-\phi(\hat{x})\\
&\le(1-t)\Phi(\hat{u},\hat{x},y')+t\Phi(\hat{u},\hat{x},y'')\\
&\qquad+J^\circ(\hat{x};(1-t)\eta(y',\hat{x})+t\eta(y'',\hat{x}))+\phi(y'+t(y''-y'))-\phi(\hat{x})\\
&\le(1-t)[\Phi(\hat{u},\hat{x},y')+J^\circ(\hat{x};\eta(y',\hat{x}))+\phi(y')-\phi(\hat{x})]\\
&\qquad+t[\Phi(\hat{u},\hat{x},y'')+J^\circ(\hat{x};\eta(y'',\hat{x}))+\phi(y'')-\phi(\hat{x})]\\
&<0,
\end{aligned}\tag{3.9}$$

and so $y_t\in V$. Therefore, the set V is convex. From (3.8), one knows that $y_i\in V$ for each $i\in\{1,2,...,n\}$. This, together with $\hat{x}\in\mathrm{co}(\{y_1,y_2,...,y_n\})$ and the convexity of V, implies that $\hat{x}\in V$, i.e.,

$$\Phi(\hat{u},\hat{x},\hat{x})+J^\circ(\hat{x};\eta(\hat{x},\hat{x}))+\phi(\hat{x})-\phi(\hat{x})<0.\tag{3.10}$$

On the other hand, it is easy to see that

$$\Phi(\hat{u},\hat{x},\hat{x})+J^\circ(\hat{x};\eta(\hat{x},\hat{x}))+\phi(\hat{x})-\phi(\hat{x})=0,$$

which contradicts (3.10). Therefore, mapping Γ is a KKM mapping.

Step 2. $\tilde{\Gamma}$ is a KKM mapping.

Since $\Phi(\nu,x,y)+\Phi(\nu,y,x)=0$ for each $(\nu,x,y)\in X^*\times K\times K$, we easily conclude from the Φ-monotonicity of F that $\Gamma(y)\subseteq\tilde{\Gamma}(y)$ for each $y\in K_\phi$. Combining the statement of Step 1, we obtain the conclusion.

Step 3. $\tilde{\Gamma}(y)$ is weakly closed for each $y\in K_\phi$.

To this end, for a sequence $\{x_n\}$ in $\tilde{\Gamma}(y)$ such that $x_n\rightharpoonup x$ as $n\to\infty$, we shall show that $x\in\tilde{\Gamma}(y)$. Since $x_n\in\tilde{\Gamma}(y)$ for each $n\ge 1$, we have

$$\Phi(v,x_n,y)+J^\circ(x_n;\eta(y,x_n))+\phi(y)-\phi(x_n)\ge 0,\quad\forall v\in F(y).\tag{3.11}$$

Since $\Phi(\cdot,\cdot,y):X^*\times K\to\mathbf{R}$ is bi-sequentially weakly upper semicontinuous, we have

$$\limsup_{n\to\infty}\Phi(v,x_n,y)\le\Phi(v,x,y).$$

Also, since η is weakly continuous in the second variable., we get that $\eta(y,x_n)\rightharpoonup\eta(y,x)$ as $n\to\infty$. Meantime, noting that ϕ is proper, convex and lower semicontinuous, we know that ϕ is weakly lower semicontinuous. Thus, $-\phi(\cdot)$ is weakly upper semicontinuous. Taking into account the assumption of bi-sequentially weakly upper semicontinuity of $J^\circ(\cdot;\cdot)$, we obtain

$$\limsup_{n\to\infty}J^\circ(x_n;\eta(y,x_n))\le J^\circ(x;\eta(y,x)).$$

So, taking the limsup in (3.11) as $n\to\infty$, we have

$$\Phi(v,x,y)+J^\circ(x;\eta(y,x))+\phi(y)-\phi(x)\ge 0,\quad\forall v\in F(y),$$

which immediately yields $x\in\tilde{\Gamma}(y)$.

Step 4. $\tilde{\Gamma}(y)$ is weakly compact for each $y\in K_\phi$.

Indeed, since K is bounded, so is $\tilde{\Gamma}(y)$ for each $y\in K_\phi$ from (3.7). Combining the statement of Step 3, we get the conclusion.

Therefore, applying Lemma 2.8 for each $G=\tilde{\Gamma}$, we obtain that $\bigcap_{y\in K_\phi}\tilde{\Gamma}(y)\neq\emptyset$. This means that there exists some $x\in K_\phi$ such that $x\in\tilde{\Gamma}(y)$ for all $y\in K_\phi$, i.e.,

$$\Phi(v,x,y)+J^\circ(x;\eta(y,x))+\phi(y)-\phi(x)\geq 0,\quad \forall v\in F(y),\quad y\in K.$$

From Proposition 3.1, it follows that $x\in K_\phi$ is a solution of the (VHVLIP). This completes the proof. □

Now we turn to the solvability of the (VHVLIP) in the case when the constrain set is unbounded. In this setting, we need some coercivity conditions (see conditions (a) and (b) below).

Theorem 3.4. *Assume that all the conditions in Theorem 3.3 are satisfied except for the boundedness one of K. Consider the following statements:*

(a) There exists a nonempty subset V_0 contained in a weakly compact subset V_1 of K_ϕ such that the set

$$D=\{x\in K_\phi:\Phi(v,x,y)+J^\circ(x;\eta(y,x))+\phi(y)-\phi(x)\geq 0,\quad \forall v\in F(y),\quad y\in V_0\} \tag{3.12}$$

is weakly compact or empty.

(b) There exist some $x_0\in K_\phi$ and $n_0\in\mathbf{N}$ such that for every $x\in K_\phi\setminus K_{n_0}$, there exists some $y_0\in K_\phi$ with $d(y_0,x_0)<d(x,x_0)$ such that

$$\Phi(u,x,y_0)+J^\circ(x;\eta(y_0,x))+\phi(y_0)-\phi(x)\leq 0,\quad \forall u\in F(x), \tag{3.13}$$

where $K_{n_0}=\{x\in K_\phi:d(x,x_0)\leq n_0\}$.

(c) The (VHVLIP) admits a solution.

Then, (a)⇒(b)⇒(c).

Proof. (a)⇒(b). First suppose that D is empty. Since V_0 is nonempty and contained in a weakly compact subset V_1 of K_ϕ, we conclude that there exist some $x_0\in K_\phi$ and $n_0\in\mathbf{N}$ such that $d(x,x_0)<n_0$ for all $x\in V_0$. From the emptiness of D, for each $x\in K_\phi\setminus K_{n_0}$, there exists $y_0\in V_0\neq\emptyset$ such that $d(y_0,x_0)<n_0$ and for some $v_0\in F(y_0)$,

$$\Phi(v_0,x,y_0)+J^\circ(x;\eta(y_0,x))+\phi(y_0)-\phi(x)<0. \tag{3.14}$$

Now if D is nonempty, then by (a), D is weakly compact. Since $D\cup V_0\subseteq D\cup V_1$, which is a weakly compact set, we conclude that there exist some $x_0\in K_\phi$ and $n_0\in\mathbf{N}$ such that $d(x,x_0)<n_0$ for all $x\in D\cup V_0$. Thus, for every $x\in K_\phi\setminus K_{n_0}$, there exists some $y_0\in V_0\neq\emptyset$ such that $d(y_0,x_0)<n_0$ and for some $v_0\in F(y_0)$,

$$\Phi(v_0,x,y_0)+J^\circ(x;\eta(y_0,x))+\phi(y_0)-\phi(x)<0$$

by $x\notin D$.

Hence, no matter the set D is empty or not, there exist some $x_0\in K_\phi$ and $n_0\in\mathbf{N}$ such that for any $x\in K_\phi\setminus K_{n_0}$, there exists some $y_0\in K_{n_0}$ such that $d(y_0,x_0)<n_0$ and (3.14) holds for some $v_0\in F(y_0)$. Combining $d(y_0,x_0)<n_0$ and $d(x,x_0)>n_0$ (by $x\in K_\phi\setminus K_{n_0}$), we conclude that this y_0 also satisfies $d(y_0,x_0)<d(x,x_0)$. Since $\Phi(\nu,x,y)+\Phi(\nu,y,x)=0$ for each $(\nu,x,y)\in X^*\times K\times K$, from the Φ-monotonicity of F, (3.13) follows.

(b)⇒(c). Let $m>n_0$ and denote $K_m=\{x\in K_\phi:d(x,x_0)\leq m\}$. Since K_m is a nonempty, bounded, closed and convex subset in X, we conclude that there exists $x_m\in K_m$ such that for some $u_m\in F(x_m)$

$$\Phi(u_m,x_m,y)+J^\circ(x_m;\eta(y,x_m))+\phi(y)-\phi(x_m)\geq 0,\quad \forall y\in K_m. \tag{3.15}$$

(i) First suppose that x_m satisfies $d(x_m,x_0)=m$. Then $x_m \in K_\phi \setminus K_{n_0}$. By condition (b), there is some $y_0 \in K_\phi$ with $d(y_0,x_0)<d(x_m,x_0)$ such that

$$\Phi(u_m,x_m,y_0)+J^\circ(x_m;\eta(y_0,x_m))+\phi(y_0)-\phi(x_m)\le 0, \quad \forall u_m \in F(x_m), \tag{3.16}$$

Let $y \in K$ be arbitrarily fixed. Since $d(y_0,x_0)<d(x_m,x_0)=m$, we know that $y_0 \in K_m$. Define $z_t = y_0+t(y-y_0)$ for each $t \in [0,1]$. Then it is easy to see that $z_t \in K_m$ for t small enough. Applying (3.15) for $y=z_t$, by the convexity of Φ in the third variable, the affinity of η in the first variable, the positive homogeneous and subadditive properties of $J^\circ(\cdot;\cdot)$ in the second variable, and the convexity of function ϕ, we deduce that for $t>0$ small enough

$$\begin{aligned}
0 &\le \Phi(u_m,x_m,z_t)+J^\circ(x_m;\eta(z_t,x_m))+\phi(z_t)-\phi(x_m)\\
&\le (1-t)\Phi(u_m,x_m,y_0)+t\Phi(u_m,x_m,y)+J^\circ(x_m;(1-t)\eta(y_0,x_m)+t\eta(y,x_m))\\
&\qquad +\phi(y_0+t(y-y_0))-\phi(x_m)\\
&\le (1-t)[\Phi(u_m,x_m,y_0)+J^\circ(x_m;\eta(y_0,x_m))+\phi(y_0)-\phi(x_m)]\\
&\qquad +t[\Phi(u_m,x_m,y)+J^\circ(x_m;\eta(y,x_m))+\phi(y)-\phi(x_m)]\\
&\le t[\Phi(u_m,x_m,y)+J^\circ(x_m;\eta(y,x_m))+\phi(y)-\phi(x_m)],
\end{aligned}$$

where the last inequality follows from (3.16) and $t>0$. Taking into account $t>0$, we obtain that

$$\Phi(u_m,x_m,y)+J^\circ(x_m;\eta(y,x_m))+\phi(y)-\phi(x_m)\ge 0, \quad \forall y \in K,$$

which shows that $x_m \in K_m \subset K_\phi$ is a solution of the (VHVLIP).

(ii) Now suppose that x_m satisfies $d(x_m,x_0)<m$. Take a fixed $y \in K$ arbitrarily and define $p_t = x_m+t(y-x_m)$ for each $t\in[0,1]$. Then it is easy to see that $p_t \in K_m$ for t small enough. Applying (3.15) for $y=p_t$, by the convexity of Φ in the third variable, the affinity of η in the first variable, the positive homogeneous and subadditive properties of $J^\circ(\cdot;\cdot)$ in the second variable, and the convexity of function ϕ, we deduce that for $t>0$ small enough

$$\begin{aligned}
0 &\le \Phi(u_m,x_m,p_t)+J^\circ(x_m;\eta(p_t,x_m))+\phi(p_t)-\phi(x_m)\\
&\le (1-t)\Phi(u_m,x_m,x_m)+t\Phi(u_m,x_m,y)+J^\circ(x_m;(1-t)\eta(x_m,x_m)+t\eta(y,x_m))\\
&\qquad +\phi(x_m+t(y-x_m))-\phi(x_m)\\
&\le (1-t)[\Phi(u_m,x_m,x_m)+J^\circ(x_m;\eta(x_m,x_m))+\phi(x_m)-\phi(x_m)]\\
&\qquad +t[\Phi(u_m,x_m,y)+J^\circ(x_m;\eta(y,x_m))+\phi(y)-\phi(x_m)]\\
&= t[\Phi(u_m,x_m,y)+J^\circ(x_m;\eta(y,x_m))+\phi(y)-\phi(x_m)].
\end{aligned}$$

Taking into account $t>0$, we obtain that

$$\Phi(u_m,x_m,y)+J^\circ(x_m;\eta(y,x_m))+\phi(y)-\phi(x_m)\ge 0, \quad \forall y \in K.$$

Therefore, $x_m \in K_\phi$ is a solution of the (VHVLIP). This completes the proof. □

When the constraint set is unbounded, one of interesting problems in the theory of hemivariational inequality is to explore some conditions to ensure the boundedness of the set of solutions for the underlying problem. To this end, we introduce condition (d) below in the sequel, which is slightly different from condition (b).

Theorem 3.5. *Assume that all the conditions in Theorem 3.3 are satisfied except for the boundedness one of K. If the following condition holds:*

(d) there exist some $x_0 \in K_\phi$ and $n_0 \in \mathbf{N}$ such that for every $x \in K_\phi \setminus K_{n_0}$, there exists some $y_0 \in K_\phi$ with $d(y_0,x_0) < d(x,x_0)$ such that

$$\Phi(u,x,y_0) + J^\circ(x;\eta(y_0,x)) + \phi(y_0) - \phi(x) < 0, \quad \forall u \in F(x), \tag{3.17}$$

where $K_{n_0} = \{x \in K_\phi : d(x,x_0) \le n_0\}$, then the set of solutions for the (VHVLIP) is nonempty and bounded.

Proof. It is easy to see that condition (d) implies (b). Thus, using Theorem 3.4, we deduce that the set of solutions of (VHVLIP) is nonempty. Next we shall show the boundness of the solution set by contradiction. To this end, suppose that the solution set is unbounded. Then for $x_0 \in K_\phi$ and $n_0 \in \mathbf{N}$ stated in condition (d), there exists $x \in K_\phi$ such that $d(x,x_0) > n_0$ and for some $u \in F(x)$

$$\Phi(u,x,y) + J^\circ(x;\eta(y,x)) + \phi(y) - \phi(x) \ge 0, \quad \forall y \in K. \tag{3.18}$$

Since $x \in K_\phi \setminus K_{n_0}$, by condition (d), there exists some $y_0 \in K_\phi$ such that

$$\Phi(u,x,y_0) + J^\circ(x;\eta(y_0,x)) + \phi(y_0) - \phi(x) < 0, \quad \forall u \in F(x),$$

which contradicts (3.18). This completes the proof. □

Before closing this section, we give a sufficient and necessary condition to characterize the solvability of (VHVLIP).

Theorem 3.6. *The (VHVLIP) admits a solution if and only if there exist $r > 0$ and $x_0 \in K_\phi$ such that the following problem, denoted by (VHVLIP)$_r$, consisting of finding $x_r \in K_r$ such that for some $u_r \in F(x_r)$*

$$\Phi(u_r,x_r,y) + J^\circ(x_r;\eta(y,x_r)) + \phi(y) - \phi(x_r) \ge 0, \quad \forall y \in K_r, \tag{3.19}$$

where $K_r = \{x \in K_\phi : d(x,x_0) \le r\}$, has a solution x_r which conforms to $d(x_r,x_0) < r$.

Proof. The necessity is evident. Conversely, suppose that there exists a solution x_r of the (VHVLIP)$_r$ with $d(x_r,x_0) < r$. We shall show that x_r is also a solution of the (VHVLIP). Take a fixed $y \in K$ arbitrarily and define $q_t = x_r + t(y - x_r)$ for each $t \in [0,1]$. Clearly, q_t satisfies $d(q_t,x_0) < r$ for t small enough. Applying (3.19) for $y = q_t$ and then using $\eta(q_t,x_r) = t\eta(y,x_r)$, the convexity of Φ in the third variable, the convexity of function ϕ, and the positive homogeneous property of $J^\circ(x_r;\cdot)$, we obtain

$$\begin{aligned}
0 &\le \Phi(u_r,x_r,q_t) + J^\circ(x_r;\eta(q_t,x_r)) + \phi(q_t) - \phi(x_r) \\
&\le (1-t)\Phi(u_r,x_r,x_r) + t\Phi(u_r,x_r,y) + J^\circ(x_r;t\eta(y,x_r)) \\
&\qquad + \phi(x_r + t(y - x_r)) - \phi(x_r) \\
&\le t[\Phi(u_r,x_r,y) + J^\circ(x_r;\eta(y,x_r)) + \phi(y) - \phi(x_r)].
\end{aligned}$$

Taking into account $t > 0$ and the arbitrariness of $y \in K$, we conclude that x_r is a solution of (VHVLIP). This completes the proof. □

Remark 3.7. In terms of the notions of Φ-monotonicity and $\mathscr{H}$-hemicontinuity of a nonempty weakly compact-valued mapping F, Theorems 3.3-3.5 generalize, improve and develop the corresponding Theorems 3.1-3.3 of Tang, Zhou and Huang [15] from the hemivariational inequality problem HVIP(A,J,K)

to the variational-hemivariational-like inequality problem (VHVLIP), respectively. In order to extend Theorems 3.1-3.3 of [15] from the finite-dimensional case to the infinite-dimensional case, we introduce the notion of bi-sequentially weakly upper semicontinuity of a bifunction.

Remark 3.8. Theorem 3.6 generalizes, improves and develops Theorem 3.4 of Tang, Zhou and Huang [15] from the HVIP(A,J,K) to the (VHVLIP) involved in three functionals $\Phi : X^* \times K \times K \to \mathbf{R}, J^\circ : X \times X \to \mathbf{R}, \phi : X \to \mathbf{R} \cup \{+\infty\}$ and two mappings $F : K \to 2^{X^*}, \eta : K \times K \to X$. It is worth pointing out that Theorem 3.4 of [15] includes Theorem III 1.7 of [28] as a special case and also extends a classical result (i.e., [29, Theorem 3]) for hemivariational inequality from Euclidean spaces to Hadamard manifolds.

4. Concluding remarks

In this paper, the existence of solutions for a new class of variational-hemivariational-like inequalities in reflexive Banach spaces is studied. Using the notions of Φ-monotonicity and $\mathscr{H}$-hemicontinuity of a nonempty weakly compact-valued mapping, and the properties of Clarke's generalized directional derivative and Clarke's generalized gradient, existence results of solutions when the constrained set is nonempty, bounded (or unbounded), closed and convex are obtained. In particular, a sufficient condition to the boundedness of the solution set and a necessary and sufficient condition to the existence of solutions are established. The results presented in this paper generalize and improve some known results in the earlier and recent literature. The purpose of this paper is to generalize and extend the main results (i.e., Theorems 3.1-3.4) of Tang, Zhou and Huang [15] from the hemivariational inequality problem (HVIP) to the variational-hemivariational-like inequality problem (VHVLIP). In order to accomplish this end, we first introduce the variational-hemivariational-like inequality problem (VHVLIP) involved in three functionals $\Phi : X^* \times K \times K \to \mathbf{R}, J^\circ : X \times X \to \mathbf{R}, \phi : X \to \mathbf{R} \cup \{+\infty\}$ and two mappings $F : K \to 2^{X^*}, \eta : K \times K \to X$, and then make some requirements which are very different from those in Theorems 3.1-3.4 of [15], for example, let $\eta : K \times K \to X$ be a skew mapping that is affine in the first variable and weakly continuous in the second variable; let the function $\Phi : X^* \times K \times K \to \mathbf{R}$ satisfy the conditions:

(i) $\Phi(\cdot,\cdot,y) : X^* \times K \to \mathbf{R}$ is bi-sequentially weakly upper semicontinuous for each $y \in K$,
(ii) Φ is convex in the third variable, and
(iii) $\Phi(v,x,y) + \Phi(v,y,x) = 0$ for each $(v,x,y) \in X^* \times K \times K$;

let $F : K \to 2^{X^*}$ be a nonempty weakly compact-valued mapping which is Φ-monotone and $\mathscr{H}$-hemicontinuous. In addition, by assuming the bi-sequentially weakly upper semicontinuity of $J^\circ(\cdot,\cdot)$, for Step 3 in the proof of Theorem 3.3, we make sure that $\tilde{\Gamma}(y)$ is weakly closed for each $y \in K_\phi$. So, it is known that $\tilde{\Gamma}(y)$ is weakly compact for each $y \in K_\phi$; see Step 4 in the proof of Theorem 3.3. Therefore, the notion of bi-sequentially weakly upper semicontinuity, plays a crucial role in the generalization of Theorems 3.1-3.4 of Tang, Zhou and Huang [15].

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