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SOME k-FRACTIONAL INTEGRAL INEQUALITIES OF HERMITE-HADAMARD TYPE CONCERNING TWICE DIFFERENTIABLE GENERALIZED RELATIVE SEMI-$(r;m,h_1,h_2)$-PREINVEX MAPPINGS

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Abstract. In this article, we first present some integral inequalities for Gauss-Jacobi type quadrature formula involving generalized relative semi-$(r;m,h_1,h_2)$-preinvex mappings. A new identity concerning twice differentiable mappings defined on m-invex set is derived. Based on the notion of generalized relative semi-$(r;m,h_1,h_2)$-preinvexity and an auxiliary result, some new estimates with respect to Hermite-Hadamard type inequalities via k-fractional integrals are established. It is pointed out that some new special cases can be deduced from the main results of this article.

Keywords. Hermite-Hadamard type inequality; Hölder's inequality; Power mean inequality; Minkowski inequality; Fractional integral.

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1. INTRODUCTION-PRELIMINARIES

The subsequent double inequality is known as the Hermite-Hadamard inequality.

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex mapping on an interval I of real numbers and $a,b \in I$ with $a < b$. Then the subsequent double inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

For recent results concerning the Hermite-Hadamard type inequalities through various classes of convex functions; one refers to [1], [2], [3], [4], [5], [6] and the references therein.

Now, let us evoke some definitions as follows.

Definition 1.2. [7] A set $M_\varphi \subseteq \mathbb{R}^n$ is said to be a relative convex (φ-convex) set if and only if there exists a function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that

$$t\varphi(x) + (1-t)\varphi(y) \in M_\varphi, \quad \forall\, x,y \in \mathbb{R}^n, \varphi(x), \varphi(y) \in M_\varphi, t \in [0,1].$$

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Definition 1.3. [7] A function f is said to be a relative convex (φ-convex) function on a relative convex (φ-convex) set M_φ if and only if there exists a function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that,

$$f(t\varphi(x)+(1-t)\varphi(y)) \leq tf(\varphi(x))+(1-t)f(\varphi(y)), \quad \forall x,y \in \mathbb{R}^n, \varphi(x),\varphi(y) \in M_\varphi, t \in [0,1].$$

Definition 1.4. [8] A nonnegative function $f : I \subseteq \mathbb{R} \longrightarrow [0,+\infty)$ is said to be P-function, if

$$f(tx+(1-t)y) \leq f(x)+f(y), \quad \forall x,y \in I,\ t \in [0,1].$$

Definition 1.5. [9] A set $K \subseteq \mathbb{R}^n$ is said to be invex respecting the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x+t\eta(y,x) \in K$ for every $x,y \in K$ and $t \in [0,1]$.

Definition 1.6. [10] Let $h : [0,1] \longrightarrow \mathbb{R}$ be a non-negative function and $h \neq 0$. The function f on the invex set K is said to be h-preinvex with respect to η if

$$f\big(x+t\eta(y,x)\big) \leq h(1-t)f(x)+h(t)f(y)$$

for each $x,y \in K$ and $t \in [0,1]$, where $f(\cdot) > 0$.

If we put $h(t) = t$ in Definition 1.6, then f becomes a preinvex function; see [11]. If mapping $\eta(y,x) = y-x$ in Definition 1.6, then the non-negative function f reduces to h-convex mappings; see [12].

Definition 1.7. [13] Let $f : K \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a non-negative function. A function $f : K \longrightarrow \mathbb{R}$ is said to be a tgs-convex function on K if the inequality

$$f\big((1-t)x+ty\big) \leq t(1-t)[f(x)+f(y)]$$

grips for all $x,y \in K$ and $t \in (0,1)$.

Definition 1.8. [14, 15] A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to MT-convex if it is non-negative and $\forall\, x,y \in I$ and $t \in (0,1)$ satisfies the subsequent inequality:

$$f(tx+(1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}f(x)+\frac{\sqrt{1-t}}{2\sqrt{t}}f(y).$$

Definition 1.9. [16] A function: $I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be m-MT-convex if f is positive and for $\forall x,y \in I$, and $t \in (0,1)$, among $m \in [0,1]$, satisfies the following inequality

$$f\big(tx+m(1-t)y\big) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}f(x)+\frac{m\sqrt{1-t}}{2\sqrt{t}}f(y).$$

Definition 1.10. [17] Let $K \subseteq \mathbb{R}$ be an open m-invex set respecting $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$. A function $f : K \longrightarrow \mathbb{R}$ and $h_1,h_2 : [0,1] \longrightarrow [0,+\infty)$ if

$$f\big(mx+t\eta(y,x,m)\big) \leq mh_1(t)f(x)+h_2(t)f(y)$$

is valid for all $x,y \in K$ and $t \in [0,1]$, together $m \in (0,1]$, then $f(x)$ is generalized (m,h_1,h_2)-preinvex with respect to η.

We need the subsequent Riemann-Liouville fractional calculus background.

Definition 1.11. [6] Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u} u^{\alpha-1} du$. Here $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.

If $\alpha = 1$, then the fractional integral reduces to the classical integral.

Due to the wide applications of the Riemann-Liouville fractional integrals, many authors have studied the Riemann-Liouville fractional inequalities via different classes of convex mappings: generalizations, variations and new inequalities; see [6], [17] and the references therein.

Definition 1.12. [18] If $k > 0$, then k-Gamma function Γ_k is defined as

$$\Gamma_k(\alpha) = \lim_{n \longrightarrow \infty} \frac{n! k^n (nk)^{\frac{\alpha}{k}} - 1}{(\alpha)_{n,k}}.$$

If $\mathrm{Re}(\alpha) > 0$, then k-Gamma function in integral form is defined as

$$\Gamma_k(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-\frac{t^k}{k}} dt,$$

with the property that

$$\Gamma_k(\alpha + k) = \alpha \Gamma_k(\alpha).$$

Definition 1.13. [19] Let $f \in L_1[a,b]$. Then k-fractional integrals of order $\alpha, k > 0$ with $a \geq 0$ are defined as

$$I^{\alpha,k}_{a+}f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x > a$$

and

$$I^{\alpha,k}_{b-}f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)dt, \quad b > x.$$

For $k = 1$, k-fractional integrals give Riemann-Liouville integrals. Now, let us recall the Gauss-Jacobi type quadrature formula as follows

$$\int_a^b (x-a)^p (b-x)^q f(x)dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R^{\star}_m |f|, \tag{1.2}$$

for certain $B_{m,k}, \gamma_k$ and rest $R^{\star}_m|f|$; see [20].

In [21], Liu obtained integral inequalities for P-function related to the left-hand side of (1.2). In [22], Özdemir Set and Alomari presented several integral inequalities concerning the left-hand side of (1.2) via some kinds of convexity. Motivated by the above literatures, the main objective of this article is to establish integral inequalities for the left side of Gauss-Jacobi type quadrature formula and some new estimates on Hermite-Hadamard type inequalities via k-fractional integrals associated with generalized relative semi-$(r;m,h_1,h_2)$-preinvex mappings. It is pointed out that some new special cases will be deduced from main results of the article.

2. Results involving Gauss-Jacobi type quadrature formula

Definition 2.1. [5] A set $K \subseteq \mathbb{R}^n$ is said to be m-invex with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\eta(y, mx) \in K$ grips for each $x, y \in K$ and any $t \in [0,1]$.

Remark 2.2. In Definition 2.1, under certain conditions, mapping $\eta(y, mx)$ could reduce to $\eta(y, x)$. For example, if $m = 1$, then the m-invex set degenerates an invex set on K.

We next introduce generalized relative semi-$(r; m, h_1, h_2)$-preinvex functions.

Definition 2.3. Let $K \subseteq \mathbb{R}$ be an open nonempty m-invex set with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ and let $\varphi : I \longrightarrow K$ be a continuous function. A function $f : K \longrightarrow (0, +\infty)$, $h_1, h_2 : [0,1] \longrightarrow [0, +\infty)$ is said to be generalized relative semi-$(r; m, h_1, h_2)$-preinvex if

$$f\big(m\varphi(x) + t\eta(\varphi(y), \varphi(x), m)\big) \leq M_r(h_1(t), h_2(t); f(x), f(y), m)$$

holds for all $x, y \in I$ and $t \in [0,1]$ for some fixed $m \in (0,1]$, where

$$M_r(h_1(t), h_2(t); f(x), f(y), m) = \begin{cases} \Big[mh_1(t) f^r(x) + h_2(t) f^r(y)\Big]^{\frac{1}{r}}, & \text{if } r \neq 0; \\ \big[f(x)\big]^{mh_1(t)} \big[f(y)\big]^{h_2(t)}, & \text{if } r = 0, \end{cases}$$

is the weighted power mean of order r for positive numbers $f(x)$ and $f(y)$.

Remark 2.4. In Definition 2.3, if $r = 1$ and $\varphi(x) = x$, then we get Definition 1.10.

Remark 2.5. For $r = 1$, let us discuss some special cases in Definition 2.3 as follows.

(I) If $h_1(t) = (1-t)^s$, $h_2(t) = t^s$ for $s \in (0,1]$, then we get generalized relative semi-$(m,s)-$ Breckner-preinvex functions.

(II) If $h_1(t) = h_2(t) = 1$, then we get generalized relative semi-(m, P)-preinvex functions.

(III) If $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ for $s \in (0,1]$, then we get generalized relative semi-(m,s)-Godunova-Levin-Dragomir-preinvex functions.

(IV) If $h_1(t) = h(1-t)$, $h_2(t) = h(t)$, then we get generalized relative semi-(m,h)-preinvex functions.

(V) If $h_1(t) = h_2(t) = t(1-t)$, then we get generalized relative semi-(m, tgs)-preinvex functions.

(VI) If $h_1(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$, then we get generalized relative semi-m-MT-preinvex functions.

It is worth mentioning here that to the best of our knowledge all the special cases discussed above are new in the literature. We claim the following integral identity.

Lemma 2.6. *Let $\varphi : I \longrightarrow K$ be a continuous function. Assume that*

$$f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow \mathbb{R}$$

is a continuous function on K° (the interior of K) with respect to

$$\eta : K \times K \times (0,1] \longrightarrow \mathbb{R},\ \eta(\varphi(b), \varphi(a), m) > 0.$$

Then for some fixed $m \in (0,1]$ and $p,q>0$, we have

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$
$$= \eta^{p+q+1}(\varphi(b),\varphi(a),m)\int_0^1 t^p(1-t)^q f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt.$$

Proof. It is easy to observe that

$$\begin{aligned}
&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx\\
&= \eta(\varphi(b),\varphi(a),m)\int_0^1 (m\varphi(a)+t\eta(\varphi(b),\varphi(a),m)-m\varphi(a))^p\\
&\quad\times (m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-m\varphi(a)-t\eta(\varphi(b),\varphi(a),m))^q\\
&\quad\times f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt\\
&= \eta^{p+q+1}(\varphi(b),\varphi(a),m)\int_0^1 t^p(1-t)^q f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt.
\end{aligned}$$

□

With the help of Lemma 2.6, we have the following results.

Theorem 2.7. *Suppose that $h_1,h_2:[0,1]\longrightarrow[0,+\infty)$ and $\varphi: I\longrightarrow K$ are continuous functions. Assume that*

$$f: K=[m\varphi(a),m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]\longrightarrow(0,+\infty)$$

is a continuous function on K° with respect to $\eta: K\times K\times(0,1]\longrightarrow\mathbb{R}$, for $\eta(\varphi(b),\varphi(a),m)>0$. Let $k>1$ and $0<r\leq 1$. If $f^{\frac{k}{k-1}}$ is generalized relative semi-$(r;m,h_1,h_2)$-preinvex on an open m-invex set K for some fixed $m\in(0,1]$, then for any fixed $p,q>0$, we have

$$\begin{aligned}
&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx\\
&\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1,kq+1)\\
&\quad\times\left[mf^{\frac{rk}{k-1}}(a)\Psi^r(h_1(t);r)+f^{\frac{rk}{k-1}}(b)\Psi^r(h_2(t);r)\right]^{\frac{k-1}{rk}},
\end{aligned}$$

where

$$\Psi(h_i(t);r):=\int_0^1 h_i^{\frac{1}{r}}(t)dt,\ \ \forall i=1,2.$$

Proof. Let $k>1$ and $0<r\leq 1$. Since $f^{\frac{k}{k-1}}$ is generalized relative semi-$(r;m,h_1,h_2)$-preinvex on K, combining with Lemma 2.6, Hölder inequality and Minkowski inequality for all $t\in[0,1]$ and for some

fixed $m \in (0,1]$, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p (m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\left[\int_0^1 t^{kp}(1-t)^{kq}dt\right]^{\frac{1}{k}}$$

$$\times \left[\int_0^1 f^{\frac{k}{k-1}}(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt\right]^{\frac{k-1}{k}}$$

$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1,kq+1)$$

$$\times \left[\int_0^1 \left[mh_1(t)f^{\frac{rk}{k-1}}(a)+h_2(t)f^{\frac{rk}{k-1}}(b)\right]^{\frac{1}{r}}dt\right]^{\frac{k-1}{k}}$$

$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1,kq+1)$$

$$\times \left\{\left(\int_0^1 m^{\frac{1}{r}}h_1^{\frac{1}{r}}(t)f^{\frac{k}{k-1}}(a)dt\right)^r+\left(\int_0^1 h_2^{\frac{1}{r}}(t)f^{\frac{k}{k-1}}(b)dt\right)^r\right\}^{\frac{k-1}{rk}}$$

$$=\eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1,kq+1)\left[mf^{\frac{rk}{k-1}}(a)\Psi^r(h_1(t);r)+f^{\frac{rk}{k-1}}(b)\Psi^r(h_2(t);r)\right]^{\frac{k-1}{rk}}.$$

□

Next, we give some special cases of Theorem 2.7.

Corollary 2.8. *Letting $r=1$ and $h_1(t)=h(1-t)$, $h_2(t)=h(t)$ in Theorem 2.7, we have the following inequality for generalized relative semi-(m,h)-preinvex functions*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p (m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1,kq+1)\Psi^{\frac{k-1}{k}}(h(t);1)\left[mf^{\frac{k}{k-1}}(a)+f^{\frac{k}{k-1}}(b)\right]^{\frac{k-1}{k}}.$$

Corollary 2.9. *Letting $r=1$ and $h_1(t)=(1-t)^s$, $h_2(t)=t^s$ in Theorem 2.7, we have the following inequality for generalized relative semi-(m,s)-Breckner-preinvex functions*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p (m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1,kq+1)\left[\frac{mf^{\frac{k}{k-1}}(a)+f^{\frac{k}{k-1}}(b)}{s+1}\right]^{\frac{k-1}{k}}.$$

Corollary 2.10. *Letting $r=1$ and $h_1(t)=(1-t)^{-s}$, $h_2(t)=t^{-s}$ in Theorem 2.7 , we get the following inequality for generalized relative semi-(m,s)-Godunova-Levin-Dragomir preinvex functions*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p (m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1,kq+1)\left[\frac{mf^{\frac{k}{k-1}}(a)+f^{\frac{k}{k-1}}(b)}{1-s}\right]^{\frac{k-1}{k}}.$$

Corollary 2.11. *Letting $r=1$ and $h_1(t)=h_2(t)=t(1-t)$ in Theorem 2.7, we obtain the following inequality for generalized relative semi-(m,tgs)-preinvex functions*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$
$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1,kq+1)\left[\frac{mf^{\frac{k}{k-1}}(a)+f^{\frac{k}{k-1}}(b)}{6}\right]^{\frac{k-1}{k}}.$$

Corollary 2.12. *Letting $r=1$ and $h_1(t)=\frac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t)=\frac{\sqrt{t}}{2\sqrt{1-t}}$ in Theorem 2.7, we deduce the following inequality for generalized relative semi-m-MT-preinvex functions*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$
$$\leq \left(\frac{\pi}{4}\right)^{\frac{k-1}{k}}\eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1,kq+1)\left[mf^{\frac{k}{k-1}}(a)+f^{\frac{k}{k-1}}(b)\right]^{\frac{k-1}{k}}.$$

Theorem 2.13. *Suppose that $h_1,h_2:[0,1]\longrightarrow[0,+\infty)$ and $\varphi:I\longrightarrow K$ are continuous functions. Assume that $f:K=[m\varphi(a),m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]\longrightarrow(0,+\infty)$ is a continuous function on K° with respect to $\eta:K\times K\times(0,1]\longrightarrow\mathbb{R}$, for $\eta(\varphi(b),\varphi(a),m)>0$. Let $l\geq 1$ and $0<r\leq 1$. If f^l is generalized relative semi-$(r;m,h_1,h_2)$-preinvex on an open m-invex set K for some fixed $m\in(0,1]$, then for any fixed $p,q>0$, we have*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$
$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)\left[mf^{rl}(a)I^r(h_1(t);r,p,q)+f^{rl}(b)I^r(h_2(t);r,p,q)\right]^{\frac{1}{rl}},$$

where

$$I(h_i(t);r,p,q)=\int_0^1 t^p(1-t)^q h_i^{\frac{1}{r}}(t)dt,\ \ \forall i=1,2.$$

Proof. Let $l\geq 1$ and $0<r\leq 1$. Since f^l is generalized relative semi-$(r;m,h_1,h_2)$-preinvex on K, combining with Lemma 2.6, the well-known power mean inequality and Minkowski inequality for all $t\in[0,1]$ and for some fixed $m\in(0,1]$, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$
$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\left[\int_0^1 t^p(1-t)^q dt\right]^{\frac{l-1}{l}}\left[\int_0^1 t^p(1-t)^q f^l(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt\right]^{\frac{1}{l}}$$
$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)\left[\int_0^1 t^p(1-t)^q\left[mh_1(t)f^{rl}(a)+h_2(t)f^{rl}(b)\right]^{\frac{1}{r}}dt\right]^{\frac{1}{l}}$$
$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)$$
$$\times\left\{\left(\int_0^1 m^{\frac{1}{r}}t^p(1-t)^q h_1^{\frac{1}{r}}(t)f^l(a)dt\right)^r+\left(\int_0^1 t^p(1-t)^q h_2^{\frac{1}{r}}(t)f^l(b)dt\right)^r\right\}^{\frac{1}{rl}}$$
$$=\eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)\left[mf^{rl}(a)I^r(h_1(t);r,p,q)+f^{rl}(b)I^r(h_2(t);r,p,q)\right]^{\frac{1}{rl}}.$$

This completes the proof. □

Let us discuss some special cases of Theorem 2.13.

Corollary 2.14. *Letting $r=1$ and $h_1(t)=h(1-t)$, $h_2(t)=h(t)$ in Theorem 2.13, we have the following inequality for generalized relative semi-(m,h)-preinvex functions*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$
$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)\Big[mf^l(a)I(h(t);1,p,q)+f^l(b)I(h(t);1,q,p)\Big]^{\frac{1}{l}}.$$

Corollary 2.15. *Letting $r=1$ and $h_1(t)=(1-t)^s$, $h_2(t)=t^s$ in Theorem 2.13, we have the following inequality for generalized relative semi-(m,s)-Breckner-preinvex functions*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$
$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)\Big[mf^l(a)\beta(p+1,q+s+1)+f^l(b)\beta(q+1,p+s+1)\Big]^{\frac{1}{l}}.$$

Corollary 2.16. *Letting $r=1$ and $h_1(t)=(1-t)^{-s}$, $h_2(t)=t^{-s}$ in Theorem 2.13, we get the following inequality for generalized relative semi-(m,s)-Godunova-Levin-Dragomir preinvex functions*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$
$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)\Big[mf^l(a)\beta(p+1,q-s+1)+f^l(b)\beta(q+1,p-s+1)\Big]^{\frac{1}{l}}.$$

Corollary 2.17. *Letting $r=1$ and $h_1(t)=h_2(t)=t(1-t)$ in Theorem 2.13, we obtain the following inequality for generalized relative semi-(m,tgs)-preinvex functions*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$
$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)\beta^{\frac{1}{l}}(p+2,q+2)\Big[mf^l(a)+f^l(b)\Big]^{\frac{1}{l}}.$$

Corollary 2.18. *Letting $r=1$ and $h_1(t)=\frac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t)=\frac{\sqrt{t}}{2\sqrt{1-t}}$ in Theorem 2.13, we deduce the following inequality for generalized relative semi-m-MT-preinvex functions*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$
$$\leq \left(\frac{1}{2}\right)^{\frac{1}{l}}\eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)$$
$$\times\left[mf^l(a)\beta\left(p+\frac{1}{2},q+\frac{3}{2}\right)+f^l(b)\beta\left(q+\frac{1}{2},p+\frac{3}{2}\right)\right]^{\frac{1}{l}}.$$

3. RESULTS INVOLVING k-FRACTIONAL INTEGRALS

To establish our main results regarding generalizations of Hermite-Hadamard type inequalities associated with generalized relative semi-$(r;m,h_1,h_2)$-preinvexity via k-fractional integrals, we need the following new interesting lemma.

Lemma 3.1. *Let $\varphi : I \longrightarrow K$ be a continuous function. Suppose that $K \subseteq \mathbb{R}$ is an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for some fixed $m \in (0,1]$ and $\eta(\varphi(b),\varphi(a),m) > 0$. Assume that $f : K \longrightarrow \mathbb{R}$ is a twice differentiable function on K° and $f'' \in L_1[m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]$. Then for any $\alpha, k > 0$, the following identity for k-fractional integrals holds:*

$$
\begin{aligned}
I_{\alpha,k}(x;f,\eta,\varphi,m,a,b) &= \frac{\eta^{\frac{\alpha}{k}+2}(\varphi(x),\varphi(a),m)}{\left(\frac{\alpha}{k}+1\right)\eta(\varphi(b),\varphi(a),m)} \\
&\times \Bigg[\frac{f'(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))}{\eta(\varphi(x),\varphi(a),m)} - \frac{\frac{\alpha}{k}+1}{\eta^2(\varphi(x),\varphi(a),m)} \\
&\times \Bigg\{ f(m\varphi(a)+\eta(\varphi(x),\varphi(a),m)) - \frac{\Gamma_k(\alpha+k)}{\eta^{\frac{\alpha}{k}+1}(\varphi(x),\varphi(a),m)} \times I^{\alpha,k}_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))-} f(m\varphi(a)) \Bigg\} \Bigg] \\
&- \frac{\eta^{\frac{\alpha}{k}+2}(\varphi(x),\varphi(b),m)}{\left(\frac{\alpha}{k}+1\right)\eta(\varphi(b),\varphi(a),m)} \\
&\times \Bigg[\frac{-f'(m\varphi(b))}{\eta(\varphi(x),\varphi(b),m)} + \frac{\frac{\alpha}{k}+1}{\eta^2(\varphi(x),\varphi(b),m)} \\
&\times \Bigg\{ -f(m\varphi(b)) + \frac{\Gamma_k(\alpha+k)}{\eta^{\frac{\alpha}{k}+1}(\varphi(x),\varphi(b),m)} \times I^{\alpha,k}_{(m\varphi(b))+} f(m\varphi(b)+\eta(\varphi(x),\varphi(b),m)) \Bigg\} \Bigg] \\
&= \frac{\eta^{\frac{\alpha}{k}+2}(\varphi(x),\varphi(a),m)}{\left(\frac{\alpha}{k}+1\right)\eta(\varphi(b),\varphi(a),m)} \int_0^1 t^{\frac{\alpha}{k}+1} f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))dt \\
&- \frac{\eta^{\frac{\alpha}{k}+2}(\varphi(x),\varphi(b),m)}{\left(\frac{\alpha}{k}+1\right)\eta(\varphi(b),\varphi(a),m)} \int_0^1 (1-t)^{\frac{\alpha}{k}+1} f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))dt. \qquad (3.1)
\end{aligned}
$$

Proof. A simple proof of inequality (3.1) can be done by performing two integrations by parts in the integrals above and changing the variable. So, we omit the proof here. □

Using Lemma 3.1, we now state the following theorems for the corresponding version for the power of second derivative.

Theorem 3.2. *Suppose $h_1, h_2 : [0,1] \longrightarrow [0,+\infty)$ and $\varphi : I \longrightarrow K$ are continuous functions. Let $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for some fixed $m \in (0,1]$ and let $\eta(\varphi(b),\varphi(a),m) > 0$. Assume that $f : K = [m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)] \longrightarrow (0,+\infty)$ is a twice differentiable function on K°. If $0 < r \le 1$ and $(f''(x))^q$ is generalized relative semi-$(r;m,h_1,h_2)$-preinvex on K, $q > 1$, $p^{-1}+q^{-1} = 1$, then for any $\alpha, k > 0$, the following inequality for k-fractional integrals holds:*

$$
\begin{aligned}
\left| I_{\alpha,k}(x;f,\eta,\varphi,m,a,b) \right| &\le \left(\frac{1}{p\left[\frac{\alpha}{k}+1\right]+1} \right)^{\frac{1}{p}} \frac{1}{\left(\frac{\alpha}{k}+1\right)\eta(\varphi(b),\varphi(a),m)} \\
&\times \Bigg\{ |\eta(\varphi(x),\varphi(a),m)|^{\frac{\alpha}{k}+2} \Big[m(f''(a))^{rq}\Psi^r(h_1(t);r) + (f''(x))^{rq}\Psi^r(h_2(t);r) \Big]^{\frac{1}{rq}} \qquad (3.2) \\
&+ |\eta(\varphi(x),\varphi(b),m)|^{\frac{\alpha}{k}+2} \Big[m(f''(b))^{rq}\Psi^r(h_1(t);r) + (f''(x))^{rq}\Psi^r(h_2(t);r) \Big]^{\frac{1}{rq}} \Bigg\},
\end{aligned}
$$

where $\Psi(h_i(t);r)$, $\forall i = 1,2$, are defined as in Theorem 2.7.

Proof. Suppose that $q>1$ and $0<r\leq 1$. Using relation (3.1), generalized relative semi-$(r;m,h_1,h_2)$-preinvexity of $(f''(x))^q$, Hölder inequality, Minkowski inequality and properties of the modulus, we have

$$
\begin{aligned}
&\left|I_{\alpha,k}(x;f,\eta,\varphi,m,a,b)\right| \\
&\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 t^{\frac{\alpha}{k}+1}|f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))|dt \\
&\quad+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 |1-t|^{\frac{\alpha}{k}+1}|f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))|dt \\
&\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 t^{p\left(\frac{\alpha}{k}+1\right)}dt\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_0^1 \left(f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))\right)^q dt\right)^{\frac{1}{q}}+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 (1-t)^{p\left(\frac{\alpha}{k}+1\right)}dt\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_0^1 \left(f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))\right)^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 t^{p\left(\frac{\alpha}{k}+1\right)}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left[mh_1(t)(f''(a))^{rq}+h_2(t)(f''(x))^{rq}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}} \\
&\quad+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 (1-t)^{p\left(\frac{\alpha}{k}+1\right)}dt\right)^{\frac{1}{p}}\times\left(\int_0^1 \left[mh_1(t)(f''(b))^{rq}+h_2(t)(f''(x))^{rq}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}} \\
&\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 t^{p\left(\frac{\alpha}{k}+1\right)}dt\right)^{\frac{1}{p}} \\
&\quad\times\left\{\left(\int_0^1 m^{\frac{1}{r}}h_1^{\frac{1}{r}}(t)(f''(a))^q dt\right)^r+\left(\int_0^1 h_2^{\frac{1}{r}}(t)(f''(x))^q dt\right)^r\right\}^{\frac{1}{rq}} \\
&\quad+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 (1-t)^{p\left(\frac{\alpha}{k}+1\right)}dt\right)^{\frac{1}{p}} \\
&\times\left\{\left(\int_0^1 m^{\frac{1}{r}}h_1^{\frac{1}{r}}(t)(f''(b))^q dt\right)^r+\left(\int_0^1 h_2^{\frac{1}{r}}(t)(f''(x))^q dt\right)^r\right\}^{\frac{1}{rq}} \\
&=\left(\frac{1}{p\left[\frac{\alpha}{k}+1\right]+1}\right)^{\frac{1}{p}}\frac{1}{\left(\frac{\alpha}{k}+1\right)\eta(\varphi(b),\varphi(a),m)} \\
&\quad\times\left\{|\eta(\varphi(x),\varphi(a),m)|^{\frac{\alpha}{k}+2}\left[m(f''(a))^{rq}\Psi^r(h_1(t);r)+(f''(x))^{rq}\Psi^r(h_2(t);r)\right]^{\frac{1}{rq}}\right. \\
&\quad\left.+|\eta(\varphi(x),\varphi(b),m)|^{\frac{\alpha}{k}+2}\left[m(f''(b))^{rq}\Psi^r(h_1(t);r)+(f''(x))^{rq}\Psi^r(h_2(t);r)\right]^{\frac{1}{rq}}\right\}.
\end{aligned}
$$

This completes the proof. □

Next, we give some special cases of Theorem 3.2.

Corollary 3.3. *Letting $h_1(t) = h(1-t)$, $h_2(t) = h(t)$ in Theorem 3.2, we have the following inequality for generalized relative semi-$(r;m,h)$-preinvex functions*

$$\begin{aligned}\left|I_{\alpha,k}(x;f,\eta,\varphi,m,a,b)\right| \leq & \left(\frac{1}{p\left[\frac{\alpha}{k}+1\right]+1}\right)^{\frac{1}{p}} \frac{\Psi^{\frac{1}{q}}(h(t);r)}{\left(\frac{\alpha}{k}+1\right)\eta(\varphi(b),\varphi(a),m)} \\ & \times\left\{|\eta(\varphi(x),\varphi(a),m)|^{\frac{\alpha}{k}+2}\left[m(f''(a))^{rq}+(f''(x))^{rq}\right]^{\frac{1}{rq}}\right. \\ & \left.+|\eta(\varphi(x),\varphi(b),m)|^{\frac{\alpha}{k}+2}\left[m(f''(b))^{rq}+(f''(x))^{rq}\right]^{\frac{1}{rq}}\right\}.\end{aligned} \tag{3.3}$$

Corollary 3.4. *Letting $h_1(t) = (1-t)^s$, $h_2(t) = t^s$ in Theorem 3.2, we have the following inequality for generalized relative semi-$(r;m,s)$-Breckner-preinvex functions*

$$\begin{aligned}\left|I_{\alpha,k}(x;f,\eta,\varphi,m,a,b)\right| \leq & \left(\frac{1}{p\left[\frac{\alpha}{k}+1\right]+1}\right)^{\frac{1}{p}} \left(\frac{r}{s+r}\right)^{\frac{1}{q}} \frac{1}{\left(\frac{\alpha}{k}+1\right)\eta(\varphi(b),\varphi(a),m)} \\ & \times\left\{|\eta(\varphi(x),\varphi(a),m)|^{\frac{\alpha}{k}+2}\left[m(f''(a))^{rq}+(f''(x))^{rq}\right]^{\frac{1}{rq}}\right. \\ & \left.+|\eta(\varphi(x),\varphi(b),m)|^{\frac{\alpha}{k}+2}\left[m(f''(b))^{rq}+(f''(x))^{rq}\right]^{\frac{1}{rq}}\right\}.\end{aligned} \tag{3.4}$$

Corollary 3.5. *Letting $0 < s < r$ and $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ in Theorem 3.2, we have the following inequality for generalized relative semi-$(r;m,s)$–Godunova-Levin-Dragomir-preinvex functions*

$$\begin{aligned}\left|I_{\alpha,k}(x;f,\eta,\varphi,m,a,b)\right| \leq & \left(\frac{1}{p\left[\frac{\alpha}{k}+1\right]+1}\right)^{\frac{1}{p}} \left(\frac{r}{r-s}\right)^{\frac{1}{q}} \frac{1}{\left(\frac{\alpha}{k}+1\right)\eta(\varphi(b),\varphi(a),m)} \\ & \times\left\{|\eta(\varphi(x),\varphi(a),m)|^{\frac{\alpha}{k}+2}\left[m(f''(a))^{rq}+(f''(x))^{rq}\right]^{\frac{1}{rq}}\right. \\ & \left.+|\eta(\varphi(x),\varphi(b),m)|^{\frac{\alpha}{k}+2}\left[m(f''(b))^{rq}+(f''(x))^{rq}\right]^{\frac{1}{rq}}\right\}.\end{aligned} \tag{3.5}$$

Corollary 3.6. *Letting $h_1(t) = h_2(t) = t(1-t)$ in Theorem 3.2, we have the following inequality for generalized relative semi-$(r;m,tgs)$-preinvex functions*

$$\begin{aligned}\left|I_{\alpha,k}(x;f,\eta,\varphi,m,a,b)\right| \leq & \left(\frac{1}{p\left[\frac{\alpha}{k}+1\right]+1}\right)^{\frac{1}{p}} \frac{\beta^{\frac{1}{q}}\left(1+\frac{1}{r},1+\frac{1}{r}\right)}{\left(\frac{\alpha}{k}+1\right)\eta(\varphi(b),\varphi(a),m)} \\ & \times\left\{|\eta(\varphi(x),\varphi(a),m)|^{\frac{\alpha}{k}+2}\left[m(f''(a))^{rq}+(f''(x))^{rq}\right]^{\frac{1}{rq}}\right. \\ & \left.+|\eta(\varphi(x),\varphi(b),m)|^{\frac{\alpha}{k}+2}\left[m(f''(b))^{rq}+(f''(x))^{rq}\right]^{\frac{1}{rq}}\right\}.\end{aligned} \tag{3.6}$$

Corollary 3.7. *Letting $h_1(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$ in Theorem 3.2, we have the following inequality for generalized relative semi-$(r;m)$-MT-preinvex functions*

$$\begin{aligned}\left|I_{\alpha,k}(x;f,\eta,\varphi,m,a,b)\right| \leq \left(\frac{1}{2}\right)^{\frac{1}{rq}} \left(\frac{1}{p\left[\frac{\alpha}{k}+1\right]+1}\right)^{\frac{1}{p}} \frac{\beta^{\frac{1}{q}}\left(1+\frac{1}{2r},1-\frac{1}{2r}\right)}{\left(\frac{\alpha}{k}+1\right)\eta(\varphi(b),\varphi(a),m)} \\ \times\left\{|\eta(\varphi(x),\varphi(a),m)|^{\frac{\alpha}{k}+2}\left[m(f''(a))^{rq}+(f''(x))^{rq}\right]^{\frac{1}{rq}}\right. \\ \left.+|\eta(\varphi(x),\varphi(b),m)|^{\frac{\alpha}{k}+2}\left[m(f''(b))^{rq}+(f''(x))^{rq}\right]^{\frac{1}{rq}}\right\}.\end{aligned} \tag{3.7}$$

Theorem 3.8. *Suppose that $h_1,h_2:[0,1]\longrightarrow[0,+\infty)$ and $\varphi:I\longrightarrow K$ are continuous functions. Let $K\subseteq\mathbb{R}$ be an open m-invex subset with respect to $\eta:K\times K\times(0,1]\longrightarrow\mathbb{R}$ for some fixed $m\in(0,1]$ and let $\eta(\varphi(b),\varphi(a),m)>0$. Assume that $f:K=[m\varphi(a),m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]\longrightarrow(0,+\infty)$ is a twice differentiable function on K°. If $0<r\leq 1$ and $(f''(x))^q$ is generalized relative semi-$(r;m,h_1,h_2)$-preinvex on K, $q\geq 1$, then, for any $\alpha,k>0$, the following inequality for k-fractional integrals holds:*

$$\begin{aligned}\left|I_{\alpha,k}(x;f,\eta,\varphi,m,a,b)\right| \leq \left(\frac{k}{\alpha+2k}\right)^{1-\frac{1}{q}} \frac{1}{\left(\frac{\alpha}{k}+1\right)\eta(\varphi(b),\varphi(a),m)} \\ \times\left\{|\eta(\varphi(x),\varphi(a),m)|^{\frac{\alpha}{k}+2}\left[m(f''(a))^{rq}\Psi^r(h_1(t);r,\alpha,k)+(f''(x))^{rq}\Psi^r(h_2(t);r,\alpha,k)\right]^{\frac{1}{rq}}\right. \\ \left.+|\eta(\varphi(x),\varphi(b),m)|^{\frac{\alpha}{k}+2}\left[m(f''(b))^{rq}\Omega^r(h_1(t);r,\alpha,k)+(f''(x))^{rq}\Omega^r(h_2(t);r,\alpha,k)\right]^{\frac{1}{rq}}\right\},\end{aligned} \tag{3.8}$$

where

$$\Psi(h_i(t);r,\alpha,k):=\int_0^1 t^{\frac{\alpha}{k}+1}h_i^{\frac{1}{r}}(t)dt,\ \ \forall i=1,2,$$

and

$$\Omega(h_i(t);r,\alpha,k):=\int_0^1 (1-t)^{\frac{\alpha}{k}+1}h_i^{\frac{1}{r}}(t)dt,\ \ \forall i=1,2.$$

Proof. Suppose that $q\geq 1$ and $0<r\leq 1$. Using relation (3.1), generalized relative semi-$(r;m,h_1,h_2)$-preinvexity of $(f''(x))^q$, the well-known power mean inequality, Minkowski inequality and properties of the modulus, we have

$$\begin{aligned}&\left|I_{\alpha,k}(x;f,\eta,\varphi,m,a,b)\right| \\ &\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 t^{\frac{\alpha}{k}+1}|f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))|dt \\ &\quad+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 |1-t|^{\frac{\alpha}{k}+1}|f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))|dt \\ &\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 t^{\frac{\alpha}{k}+1}dt\right)^{1-\frac{1}{q}}\times\left(\int_0^1 t^{\frac{\alpha}{k}+1}\left(f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))\right)^q dt\right)^{\frac{1}{q}} \\ &\quad+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 (1-t)^{\frac{\alpha}{k}+1}dt\right)^{1-\frac{1}{q}} \\ &\quad\times\left(\int_0^1 (1-t)^{\frac{\alpha}{k}+1}\left(f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))\right)^q dt\right)^{\frac{1}{q}}\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 t^{\frac{\alpha}{k}+1}dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 t^{\frac{\alpha}{k}+1}\left[mh_1(t)(f''(a))^{rq}+h_2(t)(f''(x))^{rq}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}} \\
&\quad+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 (1-t)^{\frac{\alpha}{k}+1}dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 (1-t)^{\frac{\alpha}{k}+1}\left[mh_1(t)(f''(b))^{rq}+h_2(t)(f''(x))^{rq}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}} \\
&\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 t^{\frac{\alpha}{k}+1}dt\right)^{1-\frac{1}{q}} \\
&\times\left\{\left(\int_0^1 m^{\frac{1}{r}}t^{\frac{\alpha}{k}+1}h_1^{\frac{1}{r}}(t)(f''(a))^q dt\right)^r+\left(\int_0^1 t^{\frac{\alpha}{k}+1}h_2^{\frac{1}{r}}(t)(f''(x))^q dt\right)^r\right\}^{\frac{1}{rq}} \\
&\quad+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\frac{\alpha}{k}+2}}{\left(\frac{\alpha}{k}+1\right)\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 (1-t)^{\frac{\alpha}{k}+1}dt\right)^{1-\frac{1}{q}} \\
&\times\left\{\left(\int_0^1 m^{\frac{1}{r}}(1-t)^{\frac{\alpha}{k}+1}h_1^{\frac{1}{r}}(t)(f''(b))^q dt\right)^r+\left(\int_0^1 (1-t)^{\frac{\alpha}{k}+1}h_2^{\frac{1}{r}}(t)(f''(x))^q dt\right)^r\right\}^{\frac{1}{rq}} \\
&=\left(\frac{k}{\alpha+2k}\right)^{1-\frac{1}{q}}\frac{1}{\left(\frac{\alpha}{k}+1\right)\eta(\varphi(b),\varphi(a),m)} \\
&\quad\times\left\{|\eta(\varphi(x),\varphi(a),m)|^{\frac{\alpha}{k}+2}\left[m(f''(a))^{rq}\Psi^r(h_1(t);r,\alpha,k)+(f''(x))^{rq}\Psi^r(h_2(t);r,\alpha,k)\right]^{\frac{1}{rq}}\right. \\
&\quad\left.+|\eta(\varphi(x),\varphi(b),m)|^{\frac{\alpha}{k}+2}\left[m(f''(b))^{rq}\Omega^r(h_1(t);r,\alpha,k)+(f''(x))^{rq}\Omega^r(h_2(t);r,\alpha,k)\right]^{\frac{1}{rq}}\right\}.
\end{aligned}
$$

This completes the proof. □

Next, we give some special cases of Theorem 3.8.

Corollary 3.9. *Letting $h_1(t)=h(1-t)$, $h_2(t)=h(t)$ in Theorem 3.8, we have the following inequality for generalized relative semi-$(r;m,h)$-preinvex functions*

$$
\begin{aligned}
&\left|I_{\alpha,k}(x;f,\eta,\varphi,m,a,b)\right| \leq \left(\frac{k}{\alpha+2k}\right)^{1-\frac{1}{q}}\frac{1}{\left(\frac{\alpha}{k}+1\right)\eta(\varphi(b),\varphi(a),m)} \\
&\quad\times\left\{|\eta(\varphi(x),\varphi(a),m)|^{\frac{\alpha}{k}+2}\left[m(f''(a))^{rq}\Psi^r(h(1-t);r,\alpha,k)+(f''(x))^{rq}\Psi^r(h(t);r,\alpha,k)\right]^{\frac{1}{rq}}\right. \\
&\quad\left.+|\eta(\varphi(x),\varphi(b),m)|^{\frac{\alpha}{k}+2}\left[m(f''(b))^{rq}\Psi^r(h(t);r,\alpha,k)+(f''(x))^{rq}\Psi^r(h(1-t);r,\alpha,k)\right]^{\frac{1}{rq}}\right\}.
\end{aligned}
\tag{3.9}
$$

Corollary 3.10. *Letting $h_1(t)=(1-t)^s$, $h_2(t)=t^s$ in Theorem 3.8, we have the following inequality for generalized relative semi-$(r;m,s)$-Breckner-preinvex functions*

$$\begin{aligned}\left|I_{\alpha,k}(x;f,\eta,\varphi,m,a,b)\right| &\leq \left(\frac{k}{\alpha+2k}\right)^{1-\frac{1}{q}} \frac{1}{\left(\frac{\alpha}{k}+1\right)\eta(\varphi(b),\varphi(a),m)} \\ &\times\left\{|\eta(\varphi(x),\varphi(a),m)|^{\frac{\alpha}{k}+2}\left[m(f''(a))^{rq}\beta^r\left(\frac{\alpha}{k}+2,\frac{s}{r}+1\right)+(f''(x))^{rq}\left(\frac{1}{\frac{\alpha}{k}+\frac{s}{r}+2}\right)^r\right]^{\frac{1}{rq}}\right. \\ &\left.+|\eta(\varphi(x),\varphi(b),m)|^{\frac{\alpha}{k}+2}\left[m(f''(b))^{rq}\left(\frac{1}{\frac{\alpha}{k}+\frac{s}{r}+2}\right)^r+(f''(x))^{rq}\beta^r\left(\frac{\alpha}{k}+2,\frac{s}{r}+1\right)\right]^{\frac{1}{rq}}\right\}.\end{aligned} \tag{3.10}$$

Corollary 3.11. *Letting $h_1(t)=(1-t)^{-s}$, $h_2(t)=t^{-s}$ in Theorem 3.8, we have the following inequality for generalized relative semi-$(r;m,s)$–Godunova-Levin-Dragomir-preinvex functions*

$$\begin{aligned}\left|I_{\alpha,k}(x;f,\eta,\varphi,m,a,b)\right| &\leq \left(\frac{k}{\alpha+2k}\right)^{1-\frac{1}{q}} \frac{1}{\left(\frac{\alpha}{k}+1\right)\eta(\varphi(b),\varphi(a),m)} \\ &\times\left\{|\eta(\varphi(x),\varphi(a),m)|^{\frac{\alpha}{k}+2}\left[m(f''(a))^{rq}\beta^r\left(\frac{\alpha}{k}+2,1-\frac{s}{r}\right)+(f''(x))^{rq}\left(\frac{1}{\frac{\alpha}{k}-\frac{s}{r}+2}\right)^r\right]^{\frac{1}{rq}}\right. \\ &\left.+|\eta(\varphi(x),\varphi(b),m)|^{\frac{\alpha}{k}+2}\left[m(f''(b))^{rq}\left(\frac{1}{\frac{\alpha}{k}-\frac{s}{r}+2}\right)^r+(f''(x))^{rq}\beta^r\left(\frac{\alpha}{k}+2,1-\frac{s}{r}\right)\right]^{\frac{1}{rq}}\right\}.\end{aligned} \tag{3.11}$$

Corollary 3.12. *Letting $h_1(t)=h_2(t)=t(1-t)$ in Theorem 3.8, we have the following inequality for generalized relative semi-$(r;m,tgs)$-preinvex functions*

$$\begin{aligned}\left|I_{\alpha,k}(x;f,\eta,\varphi,m,a,b)\right| &\leq \left(\frac{k}{\alpha+2k}\right)^{1-\frac{1}{q}} \frac{\beta^{\frac{1}{q}}\left(\frac{\alpha}{k}+\frac{1}{r}+2,\frac{1}{r}+1\right)}{\left(\frac{\alpha}{k}+1\right)\eta(\varphi(b),\varphi(a),m)} \\ &\times\left\{|\eta(\varphi(x),\varphi(a),m)|^{\frac{\alpha}{k}+2}\left[m(f''(a))^{rq}+(f''(x))^{rq}\right]^{\frac{1}{rq}}\right. \\ &\left.+|\eta(\varphi(x),\varphi(b),m)|^{\frac{\alpha}{k}+2}\left[m(f''(b))^{rq}+(f''(x))^{rq}\right]^{\frac{1}{rq}}\right\}.\end{aligned} \tag{3.12}$$

Corollary 3.13. *Letting $h_1(t)=\frac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t)=\frac{\sqrt{t}}{2\sqrt{1-t}}$ in Theorem 3.8, we have the following inequality for generalized relative semi-$(r;m)$-MT-preinvex functions*

$$\begin{aligned}
\left|I_{\alpha,k}(x;f,\eta,\varphi,m,a,b)\right| &\leq \left(\frac{1}{2}\right)^{\frac{1}{rq}}\left(\frac{k}{\alpha+2k}\right)^{1-\frac{1}{q}}\frac{1}{\left(\frac{\alpha}{k}+1\right)\eta(\varphi(b),\varphi(a),m)} \\
&\times\left\{|\eta(\varphi(x),\varphi(a),m)|^{\frac{\alpha}{k}+2}\left[m(f''(a))^{rq}\beta^{r}\left(\frac{\alpha}{k}-\frac{1}{2r}+2,\frac{1}{2r}+1\right)\right.\right. \\
&\left.+(f''(x))^{rq}\beta^{r}\left(\frac{\alpha}{k}+\frac{1}{2r}+2,1-\frac{1}{2r}\right)\right]^{\frac{1}{rq}} \\
&+|\eta(\varphi(x),\varphi(b),m)|^{\frac{\alpha}{k}+2}\left[m(f''(b))^{rq}\beta^{r}\left(\frac{\alpha}{k}+\frac{1}{2r}+2,1-\frac{1}{2r}\right)\right. \\
&\left.\left.+(f''(x))^{rq}\beta^{r}\left(\frac{\alpha}{k}-\frac{1}{2r}+2,\frac{1}{2r}+1\right)\right]^{\frac{1}{rq}}\right\}.
\end{aligned} \tag{3.13}$$

Remark 3.14. For $k=1$, by Theorems 3.2 and 3.8, we can get some new special Hermite-Hadamard type inequalities associated with generalized relative semi-$(r;m,h_1,h_2)$-preinvex functions via fractional integrals. Applying Theorems 3.2 and 3.8, we can also deduce some new inequalities using special means associated with generalized relative semi-$(r;m,h_1,h_2)$-preinvex functions.

4. Conclusions

In this article, we first presented some integral inequalities for Gauss-Jacobi type quadrature formula involving generalized relative semi-$(r;m,h_1,h_2)$-preinvex mappings. A new identity concerning twice differentiable mappings defined on m-invex set is derived. By using the notion of generalized relative semi-$(r;m,h_1,h_2)$-preinvexity and the obtained identity as an auxiliary result, some new estimates with respect to Hermite-Hadamard type inequalities via k-fractional integrals are established. It is pointed out that some new special cases are deduced from main results of the article. We conclude that our methods considered here may be a stimulant for further investigations concerning Hermite-Hadamard, Ostrowski and Simpson type integral inequalities for various kinds of preinvex functions involving local fractional integrals, fractional integral operators, Caputo k-fractional derivatives, q-calculus, (p,q)-calculus, time scale calculus and conformable fractional integrals.

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