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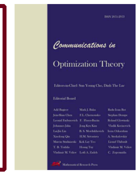

INVARIANT APPROXIMATIONS IN BANACH SPACES

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Abstract. In this paper, we first introduce a new approach to the classical fixed point theorems for H^+-type nonexpansive multivalued mappings in Banach spaces and obtain a generalization of the classical Nadler's fixed point theorem. Based on this generalization of Nadler's fixed point theorem, we study the invariant approximation and proved several new results by replacing multivalued nonexpansive mappings with H^+-type multivalued nonexpansive mappings. Several examples are constructed to illustrate the main results.

Keywords. Invariant approximation; H^+-type multi-valued nonexpansive mapping; Demiclosedness; Opial's condition; Proximinal set.

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1. Introduction

The study of the invariant approximation is fascinating and active both in approximation theory and the geometry of Banach spaces. Recent interest and a large amount of mathematical activity in this area have led to many significant new results and applications in different areas. Recently, many new results on the invariant approximation were obtained and applied to real world problems. The study of fixed points for multivalued contractions and nonexpansive mappings using the Hausdoff metric was first initiated independently by Markin [1] and Nadler [2]. Later on an interesting and rich fixed point theory for such mappings developed; see [3] and the references therein. Inspired from the Nadler fixed point theorems [2], the fixed point theory of multivalued contractions was further developed in different directions by many authors, see, Beg and Azam [4], Feng and Liu [5], Kaneko [6], Klim and Wardowski [7], Lami Dozo [8], Lim [9], Mizoguchi and Takahashi [10], Pathak and Shahzad [11], Radenovic and Vetro [12], Reich [13, 14], Suzuki [15].

In this paper, we first introduce a new approach to the classical fixed point theorems for H^+-type nonexpansive multivalued mappings in Banach spaces by reformulating the arguments in an ultrapower

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context which helps to illuminate many underlying ideas and obtain a generalization of the classical Nadler fixed point theorem [2]. More precisely, our method may provide an efficient way to recover all of the classical Banach space results. Based on this generalization of Nadler's fixed point theorem, we also study the invariant approximation. Using the fact that H^+-type multivalued nonexpansiveness is much weaker than the multivalued nonexpansiveness with respect to Hausdorff metric, we obtain several results on invariant approximation by replacing multivalued nonexpansive mappings with H^+-type multivalued nonexpansive mappings. Several examples are constructed to illustrate our main results.

2. Preliminaries

Let X be a normed space and let K be a nonempty subset of X. Let 2^K, $\mathscr{CB}(K)$ and $\mathscr{K}(K)$ denote the collection of all nonempty subsets of K, the collection of all nonempty closed bounded subsets of K and the collection of all compact subsets of K, respectively.

For $A,B \in \mathscr{CB}(X)$, let

$$H(A,B) = \max\Big\{\rho(A,B), \rho(B,A)\Big\},$$

$$H^+(A,B) = \frac{1}{2}\Big\{\rho(A,B) + \rho(B,A)\Big\},$$

where $\rho(A,B) = \sup_{x\in A} dist(x,B)$ and $dist(x,B) = \inf_{y\in B}\|x-y\|$. It is known that H is a metric on $\mathscr{CB}(X)$ and such a map H is called the *Hausdorff metric* induced by the norm of X.

A set-valued mapping $T : X \to \mathscr{CB}(X)$ is said to be a

(i) *multi-valued k-contraction mapping* if there exists a fixed real number k $(0 < k < 1)$ such that

$$H(Tx,Ty) \le k\|x-y\|, \quad \forall x,y \in X.$$

(ii) *multi-valued nonexpansive mapping* if

$$H(Tx,Ty) \le \|x-y\|, \quad x,y \in X.$$

Proposition 2.1. [16] *H^+ is a metric on $CB(X)$.*

Notice that the two metrics H and H^+ are equivalent [17] since

$$\frac{1}{2}H(A,B) \le H^+(A,B) \le H(A,B).$$

In the light of this equivalence and referring to Kuratowski [17], we conclude that $(\mathscr{CB}(X), H^+)$ is complete whenever (X,d) is complete. Indeed, it is a simple consequence of the completeness of the Hausdorff metric H. Notice also that $H^+ : \mathscr{CB}(X) \times \mathscr{CB}(X) \to \mathbf{R}$ is a continuous function. To see this, we observe that the inequality

$$H^+(A,B) \le H^+(A,C) + H^+(C,B), \quad \forall A,B,C \in \mathscr{CB}(X)$$

holds. Pick $(A_0,B_0) \in \mathscr{CB}(X) \times \mathscr{CB}(X)$, for a given $\varepsilon > 0$, we can choose a positive number $\delta = \frac{\varepsilon}{2}$ such that

$$|H^+(A,B) - H^+(A_0,B_0)| \le H^+(A,A_0) + H^+(B_0,B) < \delta + \delta = 2\delta = \varepsilon$$

whenever $H^+(A,A_0) < \delta, H^+(B_0,B) < \delta$. This shows that H^+ is continous at (A_0,B_0).

It is routine to prove the following.

Proposition 2.2. *Let $(X, \|\cdot\|)$ be a normed linear space. For any λ (real or complex), $A, B \in \mathscr{CB}(X)$, we have*
(i) $H^+(\lambda A, \lambda B) = |\lambda| H^+(A, B)$,
(ii) $H^+(A + a, B + a) = H^+(A, B)$.

In classical approach, one can easily prove Theorem 2.3, Proposition 2.5 and Proposition 2.6 stated below.

Theorem 2.3. *If $a, b \in X$ and $A, B \in \mathscr{CB}(X)$, then the relations:*
(1) $\|a - b\| = H^+(\{a\}, \{b\})$;
(2) $A \subset \overline{S}(B; r_1), B \subset \overline{S}(A; r_2) \Rightarrow H^+(A, B) \leq r$, *where* $r = \frac{r_1 + r_2}{2}$;
and
(3) $H^+(A, B) < r \Rightarrow \exists r_1, r_2 > 0$ *such that* $\frac{r_1 + r_2}{2} = r$ *and* $A \subset S(B; r_1), B \subset S(A; r_2)$
hold.

Remark 2.4. The relations (2) and (3) implies that the following relations
$(2')$ $A \subset S(B; r_1), B \subset S(A; r_2) \Rightarrow H^+(A, B) \leq r$ where $r = \frac{r_1 + r_2}{2}$;
and
$(3'^+(A, B) < r \Rightarrow \exists r_1, r_2 > 0$ such that $\frac{r_1 + r_2}{2} = r$ and $A \subset \overline{S}(B; r_1), B \subset \overline{S}(A; r_2)$
hold.

Proposition 2.5. *If $A, B \in \mathscr{CB}(X)$, then the equalities*
(4) $H^+(A, B) = \inf\{r > 0 : A \subset S(B; r_1), B \subset S(A; r_2), r = \frac{r_1 + r_2}{2}\}$,
$(4'^+(A, B) = \inf\{r > 0 : A \subset \overline{S}(B; r_1), A \subset \overline{S}(B; r_2), r = \frac{r_1 + r_2}{2}\}$
hold.

Proposition 2.6. *If $b \in X$ and $A \in \mathscr{CB}(X)$, then there exists $a \in A$ such that the following equality $\|a - b\| \leq H^+(A, \{b\})$ holds.*

To observe this fact, let us consider $X = \mathbb{R}$ endowed with usual norm $\|\cdot\|$. Let $A = [0, 1]$ and $B = \{2\}$. Then $\rho(A, \{2\}) = 2$, $\rho(\{2\}, A) = 1$. It is clear that, for all $a \in [\frac{1}{2}, 1]$, $\|a - 2\| \leq \frac{3}{2} = H^+(A, \{2\})$.

Theorem 2.7. [16] *If the metric space (X, d) is complete, then $(\mathscr{CB}(X), H^+)$ is also complete and $\mathscr{K}(X)$ is a closed subspace of $(\mathscr{CB}(X), H^+)$.*

An element $x \in X$ is called a fixed point of a multivalued map $T : K \subset X \to 2^X$ if $x \in T(x)$. We denote by $F(T)$ the set of fixed points of T. In what follows, let throughout the paper $\to$ and $\overset{w}{\to}$ denote strong and weak convergence, respectively.

A multivalued map T of $K \subseteq X$ into 2^X is said to be demiclosed if for every sequence $\{x_n\} \subset K$ and any $y_n \in T(x_n), n = 1, 2, \cdots$ such that $x_n \overset{w}{\to} x$ and $y_n \to y$, we have $x \in X$ and $y \in T(x)$.

We recall the following definitions.

Definition 2.8. [16] (i) Let (X, d) be a metric space. A multi-valued map $T : X \to \mathscr{CB}(X)$ is said to be H^+-contractive if
(1) there exists k in $(0, 1)$ such that

$$H^+(Tx, Ty) \leq k d(x, y), \quad \forall x, y \in X,$$

(2) for every x in X, y in $T(x)$ and $\varepsilon > 0$, there exists z in $T(y)$ such that

$$d(y,z) \leq H^+(T(y),T(x)) + \varepsilon.$$

(ii) Let $(X, \|\cdot\|)$ be a Banach space. A multi-valued map $T : X \to \mathscr{CB}(X)$ is said to be H^+-nonexpansive if

$$(H^+(T(x),T(y))) \leq \|x-y\| \quad \forall x,y \in X,$$

(2′) for every x in X, y in $T(x)$ and $\varepsilon > 0$, there exists z in $T(y)$ such that

$$\|y-z\| \leq H^+(T(y),T(x)) + \varepsilon.$$

Nadler [2] proved the following result.

Theorem 2.9. [2] *Let (X,d) be a complete metric space and $T : X \to \mathscr{CB}(X)$ be a multi-valued contraction mapping. Then T has a fixed point.*

Recently, Pathak and Shahzad [16] generalized the above Nadler'result of by weakening the multi-valued contraction. Indeed, they proved the following result.

Theorem 2.10. *Every H^+-type multi-valued contraction mapping $T : X \to \mathscr{CB}(X)$ with Lipschitz constant $k < 1$ has a fixed point.*

Remark 2.11. It is interesting to observe that the condition (2) in Definition 2.8 strengthen the condition (1) in the definition of the H^+-type multi-valued contraction. Notice that the notion of the H^+-type multi-valued contraction is weaker than the notion of the multi-valued contraction. It seems that the condition (2) is necessary for the existence of fixed points. To ensure this, we furnish Examples 2.12 and 2.13 below.

To show the significance of condition (2) in the definition of the H^+-contraction, we furnish two examples showing that the condition (2) is necessary for the existence of fixed points of an H^+-contraction mapping.

Example 2.12. Let $X = \mathbf{N}$, where $\mathbf{N}$ denotes the natural numbers and define the metric d on X by setting $d(n,n) = 0$ and $d(n,m) = 1 + \frac{1}{n} \Leftrightarrow n < m$. Then (X,d) is complete because there are no nontrivial Cauchy sequences. For $n \in X$, set $T(n) = \{n+1, n+2, \cdots\}$. If $m,n \in X$ and $m > n$,

$$\begin{aligned}
H^+(T(m),T(n)) &= \frac{1}{2}\rho(T(m),T(n)) \\
&= \frac{1}{2}\left(1 + \frac{1}{n+1}\right) \\
&= \frac{1}{2}H(T(m),T(n)) \\
&< \frac{1}{2}\left(1 + \frac{1}{n}\right) = \frac{1}{2}d(m,n).
\end{aligned}$$

Further, we observe that for any $n \in X$, $m \in T(n) = \{n+1, n+2, \cdots\}$ and $p \in T(m) = \{m+1, m+2, \cdots\}$,

$$d(m,p) = \left(1 + \frac{1}{m}\right) > \frac{1}{2}\left(1 + \frac{1}{n+1}\right) = H^+(T(m),T(n)).$$

Note that $\max_{n\in\mathbf{N}} \frac{1}{n+1} = \frac{1}{2}$ and $\min_{m>n} \frac{1}{m} \to 0$. Further, we observe that for any $n \in X$, $m \in T(n)$, $\varepsilon > 0$ and $p \in T(m)$,

$$d(m,p) \leq H^+(T(m),T(n)) + \varepsilon \Leftrightarrow \varepsilon \geq \frac{1}{2} + \Big[\frac{1}{m} - \frac{1}{2}\Big(\frac{1}{n+1}\Big)\Big].$$

Thus, we conclude from the above observation that for any $n \in X$, $m \in T(n)$ and $0 < \varepsilon < \frac{1}{4}$ there exists no $p \in T(m)$ such that

$$d(m,p) \leq H^+(T(m),T(n)) + \varepsilon.$$

It follows that T has Lipschitz constant $\frac{1}{2}$, but it fails to satisfy condition (2) and clearly T is fixed point free.

Example 2.13. Let X be the real line interval $[0,\infty)$ with the metric: $d(x,y) = \frac{|x-y|}{|x-y|+1}$ $\forall x,y \in [0,\infty)$. Then (X,d) is a complete metric space with diameter 1. Define $T : X \to CB(X)$ by setting $T(x) = [x+1,\infty)$. If $x > y, T(x) \subseteq T(y)$ and

$$\begin{aligned} H^+(T(x),T(y)) &= \frac{1}{2}\rho(T(x),T(y)) \\ &= \frac{1}{2}\frac{|(x+1)-(y+1)|}{|(x+1)-(y+1)|+1} \\ &= \frac{1}{2}\frac{|x-y|}{|x-y|+1} \\ &= \frac{1}{2}d(x,y). \end{aligned}$$

Further, we observe that for any $x \in X$, $y \in T(x) = [x+1,\infty)$ and $z \in T(y) = [y+1,\infty)$,

$$d(y,z) = \frac{|y-z|}{|y-z|+1} \geq \frac{1}{2}\frac{|x-y|}{|x-y|+1} = H^+(T(x),T(y)).$$

Note that $\max \frac{|x-y|}{|x-y|+1} \to 1$ and $\min \frac{|y-z|}{|y-z|+1} = \frac{1}{2}$. Further, we observe that for any $x \in X$, $y \in T(x)$, $\varepsilon > 0$ and $z \in T(y)$,

$$d(y,z) \leq H^+(T(x),T(y)) + \varepsilon \Leftrightarrow \varepsilon \geq \frac{|y-z|}{|y-z|+1} - \frac{1}{2}\frac{|x-y|}{|x-y|+1}.$$

Observation that for any $x \in X$, $y = x+1 \in T(x) = [x+1,\infty)$ and $0 < \varepsilon < \frac{1}{4}$,

$$\min_{z\in T(y)} d(y,z) = \frac{1}{2} > \frac{1}{4} + \varepsilon = H^+(T(x),T(y)) + \varepsilon.$$

Thus, for any $x \in X$, $y = x+1 \in T(x) = [x+1,\infty)$ and $0 < \varepsilon < \frac{1}{4}$, there exists no $z \in T(y) = [x+2,\infty)$ such that

$$d(y,z) \leq H^+(T(x),T(y)) + \varepsilon.$$

Therefore T has Lipschitz constant $\frac{1}{2}$, but it fails to satisfy condition (2) and clearly T is fixed point free.

Let K be a nonempty convex weakly compact subset of a Banach space X. X is said to satisfies the Opial's condition if for each x_0 in X and each sequence $\{x_n\}$ weakly converging to x_0, the inequality

$$\liminf_{n\to\infty} \|x_n - x_0\| < \liminf_{n\to\infty} \|x_n - x\|$$

holds for all $x \neq x_0$.

A Banach space which satisfies the Opial's property is called an Opial space. Notice that all Hilbert spaces and $l^p(1 < p < \infty)$ are Opial spaces while L^p spaces $(p \neq 2)$ are not Opial spaces [8, 18].

Proposition 2.14. [16, Proposition 4.3] *Let $T : K \to \mathscr{K}(X)$ be an H^+-type multi-valued nonexpansive mapping and let X satisfy Opial's condition. Then $I - T$ is demiclosed, where $I : X \to X$ is an identity map.*

Let (x_n) be a bounded sequence in a Banach space X. For $x \in X$, set $r(x,(x_n)) = \limsup_{n\to\infty} d(x,x_n)$. The asymptotic radius $r((x_n))$ of (x_n) is given by

$$r((x_n)) = \inf\{r(x,(x_n)) : x \in X\}.$$

The asymptotic center $A((x_n))$ of (x_n) is the set

$$A((x_n)) = \{x \in X : r(x,(x_n)) = r((x_n))\}.$$

Recall that a bounded sequence (x_n) is regular if $r(x_n) = r(u_n)$ for every subsequence (u_n) of (x_n). It is easy to see that every bounded sequence in X has a regular subsequence (see, e.g., [19, p.166]).

We will also need the following important fact about asymptotic centers.

Proposition 2.15. [20] *If K is a closed convex subset of X and (x_n) is a bounded sequence in K, then the asymptotic center of (x_n) is in K.*

Let C be a subset of a Banach space X. Let us consider X as a metric space with respect to the metric d defined by $d(x,y) = ||x-y||$ for all $x,y \in X$. For convenience and brevity we work in an ultrapower setting. Let $\mathscr{U}$ be a nontrivial ultrafilter on the natural numbers $\mathbf{N}$. Fix $p \in X$, and let $\tilde{X}$ denote the metric space ultrapower of X over $\mathscr{U}$ relative to p. Thus the elements of $\tilde{X}$ consist of equivalence classes $[(x_i)]_{i\in\mathbf{N}}$ for which

$$\lim_{\mathscr{U}} d(x_i,p) < \infty,$$

with $(u_i) \in [(x_i)]$ if and only if $\lim_{\mathscr{U}} d(x_i,u_i) = 0$.

For $C \subseteq X$, let

$$\tilde{C} = \{x = [(x_n)] : x_n \in C \text{ for each } n\},$$

and for $x \in X$, let $\dot{x} = [(x_n)]$, where $x_n = x$ for each $n \in \mathbf{N}$. Finally, let $\dot{X}$ and $\dot{C}$ denote the respective canonical isometric copy of X and C in $\tilde{X}$.

Now let $\mathscr{U}$ be a nontrivial ultrafilter over the natural numbers $\mathbf{N}$ and let $\tilde{X}$ denote the Banach space ultrapower of X over $\mathscr{U}$. We will use the standard notation for this setting, see for example [21]. Notice that an H^+-type nonexpansive set-valued mapping $T : C \to \mathscr{CB}(X)$ induces an H^+-type nonexpansive set-valued mapping $\tilde{T}$ defined on $\tilde{C}$ as follows:

$$\tilde{T}(\tilde{x}) = \{\tilde{u} \in \tilde{X} : \exists \text{ a representative } (u_n) \text{ of } \tilde{u} \text{ with } u_n \in T(x_n) \text{ for each } n\}.$$

To see that $\tilde{T}$ is H^+-type nonexpansive (and hence well-defined), let $\tilde{x},\tilde{y} \in \tilde{C}$, with $\tilde{x} = [(x_n)]$ and $\tilde{y} = [(y_n)]$. Then

$$\begin{aligned} H^+(\tilde{T}(\tilde{x}),\tilde{T}(\tilde{y})) &\le \lim_{\mathscr{U}} H^+(T(x_n);T(y_n)) \\ &\le \lim_{\mathscr{U}} d(x_n,y_n) \\ &= d_{\mathscr{U}}(\tilde{x},\tilde{y}). \end{aligned}$$

Since for each $n \in \mathbf{N}$, $x_n \in C$, $y_n \in T(x_n)$ and $\varepsilon > 0$, there exists $z_n \in T(y_n)$ such that $d(y_n, z_n) \leq H^+(T(x_n), T(y_n)) + \varepsilon$. It follows that

$$\lim_{\mathscr{U}} d(y_n, z_n) \leq \lim_{\mathscr{U}} H^+(T(x_n), T(y_n)) + \varepsilon.$$

Thus, for any $\tilde{x} = [(x_n)] \in \tilde{C}$, $\tilde{y} = [(y_n)] \in \tilde{T}(\tilde{x})$ and $\varepsilon > 0$, there exists $\tilde{z} = [(z_n)] \in \tilde{T}(\tilde{y})$ such that

$$d_{\mathscr{U}}(\tilde{y}, \tilde{z}) \leq H^+(\tilde{T}(\tilde{x}), \tilde{T}(\tilde{y}) + \varepsilon.$$

For our further discussion, the following fact will be needed.

$$\text{If } S \subseteq C \text{ is compact, then } \dot{S} = \tilde{S}. \tag{2.3}$$

Proposition 2.16. [20] *Element x is the asymptotic center of a regular sequence $(x_n) \subset X$ if and only if $\dot{x}$ is the unique point of $\dot{X}$ which is nearest to $\tilde{x} := [(x_n)]$ in the ultrapower $\tilde{X}$.*

3. Invariant approximations in Banach spaces

For each $x \in X$ and $K \in 2^X$, we denoted by $P_K(x)$ the set $\{y \in K : \|x - y\| = dist(x, K)\}$, which is usually known as the set of *best K-approximants* to x. The set $P_K(x)$ is closed and bounded, and is convex if K is convex. We also recall that K is *Chebyshev* if for every $x \in X$, there is a unique element $u \in K$ such that $\|x - y\| = dist(x, K)$. A subset K is said to be *starshaped* with respect to a point $p \in K$ if, for all $x \in K$, $\{(1-t)p + tx : 0 \leq t \leq 1\} \subset K$. The point p is called a *starcenter* for K. Clearly, each convex set is starshaped with repsect to each of its points. It is well known that, in the strictly convex Banach space, if $P_K(x)$ is nonempty and starshaped, then it contains a unique element. If T is a singlevalued or multivalued map defined on X with $T(K) \subset K$, then K is called *T-invariant* subset of X.

Recall that a subset C of X is said to be (uniquely) *proximinal* if each point $x \in X$ has a (unique) nearest point in C.

The following interesting result was proved in [20].

Lemma 3.1. [20] *Let K be a subset of a Banach space X. Suppose that $T : K \to 2^X$, is a nonexpansive mapping and there exists $x_0 \in K$ such that $x_0 \in T(x_0)$. Suppose that C is a subset of K for which $T : C \to \mathscr{K}(C)$, and C is uniquely proximinal in K. Then T has a fixed point in C. Indeed, the point of C which is nearest to x_0 is a fixed point of T.*

Notice that the above result can be easily extended to H^+-type nonexpansive multi-valued mappings. Indeed, we prove the following result which is the basis for all of our Banach space results.

Lemma 3.2. *Let K be a subset of a Banach space X. Suppose that $T : K \to 2^X$, is H^+-type nonexpansive and there exists $x_0 \in K$ such that $T(x_0) = \{x_0\}$. Suppose that C is a subset of K for which $T : C \to \mathscr{K}(C)$, and C is uniquely proximinal in K. Then T has a fixed point in C. Indeed, the point of C which is nearest to x_0 is a fixed point of T.*

Proof. If $x_0 \in C$, then $T(x_0) = \{x_0\}$ yields that T has a fixed point in C. Otherwise, let x be the unique point of C nearest to x_0. We now assert that $x \in T(x)$. Since $T(x_0) = \{x_0\}$, by Proposition 2.5, x_0 must lie in a ρ_1-neighborhood of $T(x)$ for $\rho_1 \leq \rho_0 = H^+(T(x_0), T(x))$. Therefore, since $T(x)$ is compact, $\text{dist}(x_0, T(x)) = \|x_0 - u\| \leq \rho_1$ for some $u \in T(x)$. But since T is H^+-type nonexpansive, $T(x) \subset C$ and if $u \neq x$, we have

$$\|x_0 - u\| > \|x_0 - x\| \geq H^+(T(x_0), T(x)) = \rho_0 \geq \rho_1,$$

a contradiction. This in turn implies that $u = x \in T(x)$. □

Remark 3.3. **(i)** It is worth noting that boundedness of K is not needed in the proof of Lemma 3.2 as was the case in Lemma 3.1.
(ii) It might be worth noting that Lemma 3.2 holds for mappings taking only closed values if it is assumed that the space is uniformly convex.

Theorem 3.4. *Let X be a uniformly convex Banach space, and let C be a closed convex subset of X. If $T : C \to \mathscr{K}(C)$ is H^+-type nonexpansive such that*

$$dist(x_n, T(x_n)) \to 0 \text{ as } n \to \infty \tag{3.1}$$

for a unique bounded sequence (x_n) in C, then T has a fixed point.

Proof. Let $\tilde{x} = [(x_n)] \in \tilde{C}$. As we have observed, $\tilde{T} : \tilde{C} \to 2^{\tilde{C}} \backslash \emptyset$ is H^+-type nonexpansive. Also (3.1) implies $\tilde{T}(\tilde{x}) = \{\tilde{x}\}$. Since the uniform convexity is a super property, $\tilde{X}$ is uniformly convex and then $\tilde{x}$ has a unique nearest point $\dot{x} \in \dot{C}$. Since $\dot{T} : \dot{C} \to \mathscr{K}(\dot{C})$, Lemma 3.2 implies there exists $\dot{x} \in \dot{C}$ such that $\dot{x} \in \tilde{T}(\dot{x})$. However by (3.1) $\tilde{T}(\dot{x}) = T(\dot{x})$. This in turn implies that $x \in T(x)$. □

If X has the Opial property, then the assumption that $T : C \to \mathscr{K}(C)$ can be weakened to $T : C \to \mathscr{K}(X)$. For this we will make use of the following fact.

Proposition 3.5. [22] *Let X be a Banach space with the Opial property. Then $x \in X$ is the weak limit of a regular sequence $(x_n) \subset X$ if and only if $\dot{x}$ is the unique point of $\dot{X}$ which is nearest to $\tilde{x} := [(x_n)]$ in the ultrapower $\tilde{X}$.*

Theorem 3.6. *Let X be a Banach space with the Opial property, and let C be a weakly compact subset of X. If $T : C \to \mathscr{K}(X)$ is H^+-type nonexpansive such that*

$$dist(x_n, T(x_n)) \to 0 \text{ as } n \to \infty \tag{3.2}$$

for a unique bounded sequence (x_n) in C, then T has a fixed point.

Proof. By passing to a subsequence if necessary we may suppose that (x_n) is regular and converges weakly, say to $x \in C$. By Proposition 3.5, $\dot{x}$ is the unique point of $\dot{X}$ which is nearest to $\tilde{x} := [(x_n)]$. Note also that $x \in C$ and $\dot{x} \in \dot{C}$. Since (3.2) implies $\tilde{T}(\tilde{x}) = \{\tilde{x}\}$, $\tilde{x}$ must lie in a ρ_1-neighborhood of $\tilde{T}(\dot{x})$ for $\rho_1 \leq \rho_0 = H^+(\{\tilde{x}\}, \tilde{T}(\dot{x}))$. Since $\tilde{T}(\dot{x})$ is compact, $dist_{\mathscr{U}}(\tilde{x}, \tilde{T}(\dot{x})) = \|\tilde{x} - \dot{u}\|_{\mathscr{U}} \leq \rho_1$ for some $\dot{u} \in \tilde{T}(\dot{x})$. But $\tilde{T}(\dot{x}) \subset \dot{X}$, if $\dot{u} \neq \dot{x}$ and we assume that $\dot{x} \notin \tilde{T}(\dot{x})$, then we have the contradiction

$$\|\tilde{x} - \dot{u}\|_{\mathscr{U}} > \|\tilde{x} - \dot{x}\|_{\mathscr{U}} \geq H^+(\{\tilde{x}\}, \tilde{T}(\dot{x})) = \rho_0 \geq \rho_1.$$

Therefore $\dot{x} = \dot{u} \in \tilde{T}(\dot{x})$. Since $\tilde{T}(\dot{x}) = \widetilde{T(x)}$, we find from (3.1) that $x \in T(x)$. □

Theorem 3.7. *Let K be a subset of a uniformly convex Banach space X. Suppose that $T : K \to 2^X$ is H^+-type nonexpansive, and there exists $x_0 \in K$ such that $T(x_0) = \{x_0\}$. Suppose C is a closed convex subset of K for which $T : C \to \mathscr{CB}(C)$. Then the point of C which is nearest to x_0 is a fixed point of T.*

Proof. If $x_0 \in C$, then $T(x_0) = \{x_0\}$ yields that T has a fixed point in C. Otherwise, let x be the unique point of C nearest to x_0. We now assert that $x \in T(x)$. If not, since $T(x_0) = \{x_0\}$, by Proposition 2.5, x_0 must lie in a ρ_1-neighborhood of $T(x)$ for $\rho_1 \leq \rho_0 = H^+(T(x_0), T(x))$. If dist $(x_0, T(x)) > \|x_0 - x\|$ we have a contradiction as in the proof of Lemma 3.2. On the other hand, if dist $(x_0, T(x)) = \|x_0 - x\|$, then there exists a sequence $(u_n) \subset T(x)$ such that $\|x_0 - u_n\| > \|x_0 - x\|$ as $n \to \infty$. Since $\|x_0 - \frac{x+u_n}{2}\| > \|x_0 - x\|$, the uniform convexity of X yields $\|x - u_n\| \to 0$ as $n \to \infty$. Since $T(x)$ is closed, then $x \in T(x)$. □

Singh [23] has extended the well known result of Brosowski [24], relaxing the condition of linearity of the function and the convexity of the set. More precisely, he proved the following.

Theorem 3.8. [23] *Let X be a Banach space and let $T : X \to X$ be a single valued nonexpansive map. Let K be a T-invariant subset of X and let $x_0 \in F(T)$. If $P_K(x_0)$ is nonempty, compact, and starshaped, then $P_K(x_0) \cap F(T) \neq \emptyset$.*

Latif and Bano [25] further extended Theorem 3.8 to multivalued nonexpansive maps.

Theorem 3.9. [25] *Let X be a Banach space and let $T : X \to \mathscr{CB}(X)$ be a nonexpansive mapping such that $T(x_0) = \{x_0\}$ for some $x_0 \in X$. Let K be a nonempty T-invariant subset of X. Assume that $P_K(x_0)$ is nonempty, weakly compact and starshaped. If $I - T$ is demiclosed on $P_K(x_0)$, then $P_K(x_0) \cap F(T) \neq \emptyset$.*

Let M be a nonempty starshaped subset of a Banach space X. Let $T : M \to \mathscr{CB}(M)$ be a multivalued mapping. For a fixed $q \in M$ and any $x \in M$, we define the segment $[q,x]$ by $[q,x] = \{y \in M : y = (1-\lambda)q + \lambda x, 0 \le \lambda \le 1\}$. We call T is q-redundant if $Ty = Tx$ for all $y \in [q,x]$.

Note the fact that H^+-type multivalued nonexpansiveness is much weaker than multivalued nonexpansiveness with respect to Hausdorff metric. We obtain a result on invariant approximation by replacing multivalued nonexpansive mapping in Theorem 3.9 with the H^+-type multivalued nonexpansive mapping. We will apply weakly compact and starshaped conditions on certain nonempty subset of $P_K(x_0)$. Indeed, we prove the following result.

Theorem 3.10. *Let X be a Banach space and let $T : X \to \mathscr{CB}(X)$ be a H^+-type multivalued nonexpansive map such that $T(x_0) = \{x_0\}$ for some $x_0 \in X$. Let K be a nonempty T-invariant subset of X. Assume that $(\cup_{u \in P_K(x_0)} T(u)) \cap P_K(x_0)$ is nonempty, weakly compact and starshaped. If q be the starcenter of M such that T is q-redundant and $I - T$ is demiclosed on $(\cup_{u \in P_K(x_0)} T(u)) \cap P_K(x_0)$, then $P_K(x_0) \cap F(T) \neq \emptyset$.*

Proof. Let $u \in P_K(x_0)$. Then $u \in K$ and $\|x_0 - u\| = dist(x_0, K)$. By Proposition 2.6 there exists $v \in T(u) \subset K$ such that

$$\|v - x_0\| \le H^+(T(u), T(x_0)) \le \|u - x_0\| = \|u - x_0\| = dist(x_0, K).$$

On the other hand, $dist(x_0, K) \le \|v - x_0\|$. It follows that $\|v - x_0\| = dist(x_0, K)$. This implies that $v \in P_K(x_0)$. Let M_u be the set of all such $v \in T(u)$. It follows that $M_u = T(u) \cap P_K(x_0)$. Let $M = \cup_{u \in P_K(x_0)} M_u = (\cup_{u \in P_K(x_0)} T(u)) \cap P_K(x_0)$. Now let q be the starcenter of M. Then, for $x \in M$ and $\mu \in [0,1]$, $(1-\mu)q + \mu x \in M$. Take a sequence $\{\mu_n\}$ of real numbers such that $0 < \mu_n < 1$ and $\mu_n \to 1$ as $n \to \infty$.

Now, for each n define a multivalued mapping T_n by setting

$$T_n(x) = (1-\mu_n)q + \mu_n T(x), \forall x \in M.$$

Clearly, each T_n is a mapping from M into $\mathscr{CB}(M)$. Furthermore, for any $x, y \in M$, we have

$$H^+(T_n(x)), T_n(y)) = \mu_n H^+(T(x), T(y)) \le \mu_n \|x - y\|.$$

Now let $\varepsilon > 0$ be given. By $(2')$, corresponding to any y in $T(x)$, i.e., in turn, for any $y' = (1-\mu_n)q + \mu_n y$ in $T_n(x)$, there exists $z \in T(y)$ and, in turn, there exists $z' = (1-\mu_n)q + \mu_n z$ in $T_n(y)$. Hence, in

$$T_n(y') = (1-\mu_n)q + \mu_n T(y') = (1-\mu_n)q + \mu_n T((1-\mu_n)q + \mu_n y)) = (1-\mu_n)q + \mu_n T(y) = T_n(y)$$

such that $\|y-z\| \leq H^+(T(y),T(x))+\varepsilon$. Thus, for all $n \in \mathbf{N}$, we have

$$\begin{aligned}\|y'-z'\| &= \mu_n\|y-z\| \\ &\leq \mu_n(H^+(T(y),T(x))+\varepsilon) \\ &= H^+(T_n(y),T_n(x))+\mu_n\varepsilon \\ &< H^+(T_n(y'),T_n(x))+\varepsilon.\end{aligned}$$

Hence T_n is a μ_n-contraction mapping for all $n \in \mathbf{N}$. Also, since M is a complete metric space, therefore it follows from Theorem 2.10, that for each $n \in \mathbf{N}$, there exists $x_n \in M$ such that $x_n \in T_n(x_n)$. This implies that $\exists y_n \in T(x_n)$ such that

$$x_n = (1-\mu_n)q + \mu_n y_n \text{ for all } n \in \mathbf{N}.$$

Since M is weakly compact, for a convenient subsequence still denoted by $\{x_n\}$, we have $x_n \overset{w}{\to} z \in M$. Now

$$\|x_n - y_n\| = (1-\mu_n)\|q-y_n\| \text{ for all } n \in \mathbf{N}.$$

Setting $z_n = x_n - y_n \in (I-T)x_n$ for all $n \in \mathbf{N}$, we find that $z_n \to 0$ as $n \to \infty$. Since $I-T$ is demiclosed on M, it follows that $0 \in (I-T)z$, that is, $z \in F(T)$. Thus we obtain $M \cap F(T) \neq \emptyset$. This implies that $P_K(x_0) \cap F(T) \neq \emptyset$. □

Example 3.11. Let $X = \mathbf{R}^+ = [0,\infty)$ be endowed with the standard norm $\|\cdot\|$ defined by $\|x\| = |x|$ for all $x \in X$. Let $T : X \to \mathscr{CB}(X)$ be defined by

$$T(x) = \begin{cases} [0,x], & x \neq 1, \\ \{0,1\}, & x = 1. \end{cases}$$

It is routine to check that T is a H^+-type multivalued nonexpansive map and $F(T) = \mathbf{R}^+$. Indeed, to check the condition $(1')$ we have to consider the following cases:

(i) If $x = 0, y = 1$, then we have

$$H^+(T0,T1) = H^+(\{0\},\{0,1\}) = [0+1]/2 = 1/2 < \|0-1\|.$$

(ii) If $0 < y \leq x < \infty, x,y \neq 1$, then we have

$$H^+(Tx,Ty) = H^+([0,x],[0,y]) = [\|x-y\|+0]/2 < \|x-y\|/2 = \|x-y\|,$$

(iii) If $0 < x \leq \frac{1}{2}, y = 1$, then we have

$$H^+(Tx,T1) = H^+([0,x],\{0,1\}) = [x+(1-x)]/2 = 1/2 \leq 1-x = \|x-1\|.$$

(iv) If $\frac{1}{2} < x < 1, y = 0$, then we have

$$H^+(Tx,T1) = H^+([0,x],\{0,1\}) = [(1-x)+(1-x)]/2 = 1-x = \|x-1\|.$$

(v) If $x > 1, y = 1$, then we have

$$H^+(Tx,T1) = H^+([0,x],\{0,1\}) = [(x-1)+(x-1)]/2 = x-1 = \|x-1\|.$$

Rest of the cases follows by the symmetry of H^+. To check the condition $(2')$, we consider the following cases:

Case 1 (a). For any $x \in X, x < 1, y \in Tx = [0,x]$ and $\varepsilon > 0$, there exists $z \in Ty = [0,y]$ with $|y-z| = \frac{1}{2}|x-y|$ satisfying the inequality $(2')$ i.e., $\|y-z\| \leq H^+(Ty,Tx)+\varepsilon$.

(b_1) If $x = 1$, then any $y \in Tx = \{0,1\}$, say $y = 0$ and $\varepsilon > 0$, there exists $z(=0) \in Ty = \{0\}$ satisfying

the inequality $(2')$.
(b_2) If $x = 1$, then any $y \in Tx = \{0, 1\}$, say $y = 1$ and $\varepsilon > 0$, there exists $z(= 1) \in Ty = \{0, 1\}$ satisfying the inequality $(2')$.
Case 2 (a). For any $x \in X, x > 1$, $y(\neq 1) \in Tx = [0, x]$ and $\varepsilon > 0$, there exists $z \in Ty = [0, y]$ with $|y - z| = \frac{1}{2}|x - y|$ satisfying the inequality $(2')$.
(b). For any $x \in X, x > 1$, $y(= 1) \in Tx = [0, x]$ and $\varepsilon > 0$, there exists $z(= 1) \in Ty = \{0, 1\}$ satisfying the inequality $(2')$.

Now take $x_0 = 0$ and $K = [0, 1]$. Then K is a nonempty T-invariant subset of X and $P_K(x_0) = \{0\}$. Clearly, $(\cup_{u \in P_K(x_0)} T(u)) \cap P_K(x_0) = \{0\}$ is nonempty, weakly compact and starshaped with star-center $q = 0$. Notice that $I - T$ is demiclosed on $(\cup_{u \in P_K(x_0)} T(u)) \cap P_K(x_0) = \{0\}$. Clearly, T is q-redundant. And we have $P_K(x_0) \cap F(T) \neq \emptyset$.

In Example 3.11, the mapping $T : X \to \mathscr{CB}(X)$ is nonexpansive in the usual sense and hence it is H^+-type nonexpansive. We now present an example validating all conditions of Theorem 3.10 where T is H^+-type nonexpansive but not nonexpansive in the usual sense.

Example 3.12. Let $X = [0, 1]$ be endowed with the standard norm $\|\cdot\|$ defined by $\|x\| = |x|$ for all $x \in X$. Let $T : X \to \mathscr{CB}(X)$ be defined by

$$T(x) = \begin{cases} [0, 2x], & x \in [0, \frac{1}{2}), \\ [0, 1], & x \in [\frac{1}{2}, 1] \end{cases}$$

It is routine to check that T is a H^+-type multivalued nonexpansive map and $F(T) = [0, 1]$. Indeed, one need to consider the following cases:
(i) If $0 \leq y \leq x < \frac{1}{2}$, then we have

$$H^+(Tx, Ty) = H^+([0, 2x], [0, 2y]) = [2\|x - y\| + 0]/2 = \|x - y\|.$$

(ii) If $0 \leq x < \frac{1}{2}, \frac{1}{2} \leq y \leq 1$, then we have

$$H^+(Tx, T1) = H^+([0, 2x], [0, 1]) = [0 + (1 - 2x)]/2 = \frac{1}{2} - x < y - x = \|x - y\|.$$

(iii) If $\frac{1}{2} \leq x \leq y \leq 1$, then we have

$$H^+(Tx, T1) = H^+([0, 1], [0, 1]) = [0 + 0]/2 = 0 \leq \|x - y\|.$$

Rest of the cases follows by the symmetry of H^+. To check the condition $(2')$, we consider the following cases:
Case 1 (a_1). For any $x \in X, 0 \leq x < \frac{1}{2}$, $y(< \frac{1}{2}) \in Tx = [0, 2x]$ and $\varepsilon > 0$, there exists $z(= y) \in Ty = [0, 2y]$ satisfying the inequality $(2')$ i.e., $0 = \|y - z\| \leq \|x - y\| \leq H^+(Ty, Tx) + \varepsilon$.
(a_2). For any $x \in X, 0 \leq x < \frac{1}{2}$, $y(\frac{1}{2} \leq y < 1) \in Tx = [0, 2x]$ and $\varepsilon > 0$, there exists $z(= y) \in Ty = [0, 1]$ satisfying the inequality $(2')$ i.e., $0 = \|y - z\| \leq \|\frac{1}{2} - x\| \leq H^+(Ty, Tx) + \varepsilon$.
Case 2 (b_1). For any $x \in X, \frac{1}{2} \leq x \leq 1$, $y(< \frac{1}{2}) \in Tx = [0, 1]$ and $\varepsilon > 0$, there exists $z(= y) \in Ty = [0, 2y]$ satisfying the inequality $(2')$ i.e., $0 = \|y - z\| \leq \|\frac{1}{2} - y\| \leq H^+(Ty, Tx) + \varepsilon$.
(b_2) If $\frac{1}{2} \leq x \leq 1$, then for any $y(\frac{1}{2} \leq y \leq 1) \in Tx = [0, 1]$ and $\varepsilon > 0$, there exists $z(= y) \in Ty = [0, 1]$ satisfying the inequality $(2')$ i.e., $\|y - z\| = 0 \leq 0 \leq H^+(Ty, Tx) + \varepsilon$.

Hence T is a H^+-type multivalued nonexpansive mapping. However, one may notice that T is not a multivalued nonexpansive mapping in the usual sense. If $0 \le y < x < 1$, then

$$H(Tx,Ty) = H([0,2x],[0,2y]) = \max\{2\|x-y\|,0\} = 2\|x-y\| > \|x-y\|.$$

Now take $x_0 = 0$ and $K = [0,1]$. Then K is a nonempty T-invariant subset of X and $P_K(x_0) = \{0\}$. Clearly, $(\cup_{u\in P_K(x_0)} T(u)) \cap P_K(x_0) = \{0\}$ is nonempty, weakly compact and starshaped with star-center $p = 0$. Notice that $I-T$ is demiclosed on $(\cup_{u\in P_K(x_0)} T(u)) \cap P_K(x_0) = \{0\}$. Clearly, T is q-redundant. And we have $P_K(x_0) \cap F(T) \neq \emptyset$.

Indeed, we also have the following result on invariant approximation for Opial spaces.

Corollary 3.13. *Let X be an Opial space, and let $T : X \to \mathcal{K}(X)$ be H^+-type nonexpansive map such that $T(x_0) = \{x_0\}$ for some $x_0 \in X$. Let K be a nonempty T-invariant subset of X. If $(\cup_{u\in P_K(x_0)} T(u)) \cap P_K(x_0)$ is nonempty, weakly compact and starshaped, then $P_K(x_0) \cap F(T) \neq \emptyset$.*

Proof. Following the proof of Theorem 3.10, we observe that T maps $M = (\cup_{u\in P_K(x_0)} T(u)) \cap P_K(x_0)$ into $\mathcal{K}(M)$. Note that $I-T$ is demiclosed on M. Now the result follows from Theorem 3.10. □

Remark 3.14. We remarked that
(i) In [26, Theorems 3.1 and 4.1] the domain of the mappings is assumed to be weakly compact and convex. However weak compactness suffices – the convexity assumption may be dropped. To see this one could either use Theorem 3.6 in lieu of the demiclosedness principal in the proofs of those theorems, or observe that convexity is not needed in the proof of the demiclosedness principal itself ([26, Lemma 2.1])
(ii) The classical result of Hicks and Humphries (see, e.g. [3]) is a particular case of Corollary 3.13 .
(iii) If X is a strictly convex Banach space and $P_K(x_0)$ is nonempty and starshaped in Theorem 3.10, then we find that $P_K(x_0) \subset F(T)$.

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