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ON THE SOLVABILITY AND OPTIMAL CONTROL OF SOME PARTIAL FUNCTIONAL INTEGRODIFFERENTIAL EQUATIONS WITH INFINITE DELAY IN BANACH SPACES

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Abstract. This paper concerns the study of the solvability and optimal control of some partial functional integrodifferential equations with infinite delay in Banach spaces. We assume that the undelayed part admits a resolvent operator in the sense of Grimmer. We investigate the existence of mild solutions and prove the existence of optimal controls for the integrodifferential equation. An example is also provided to support our results.
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1. Introduction

In this paper, we study the existence of mild solutions and the optimal controls of some systems that arise in the analysis of the heat conduction in materials with memory [1], and viscoelasticity and take the form of the following partial functional integrodifferential equation with infinite delay in a Banach space $(X, \|\cdot\|)$:

$$\begin{cases} x'(t) = Ax(t) + \int_0^t \gamma(t-s)x(s)ds + f(t,x_t) + C(t)u(t) \ \text{ for } t \in I = [0,b], \\ x_0 = \varphi \in \mathscr{B}, \end{cases} \tag{1.1}$$

where $A : \mathscr{D}(A) \to X$ is the infinitesimal generator of a C_0-semigroup $\left(T(t)\right)_{t\geq 0}$ on a separable reflexive Banach space X; for $t \geq 0$, $\gamma(t)$ is a closed linear operator with domain $\mathscr{D}(\gamma(t)) \supset \mathscr{D}(A)$. The control u takes values from another separable reflexive Banach space U. The operator $C(t)$ belongs to $\mathscr{L}(U,X)$, which is the Banach space of bounded linear operators from U into X, and the phase space $\mathscr{B}$ is a linear space of functions mapping $]-\infty,0]$ into X satisfying axioms, which will be described later, for every

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$t \geq 0$, x_t denotes the history function of $\mathscr{B}$ defined by

$$x_t(\theta) = x(t+\theta) \text{ for } -\infty \leq \theta \leq 0,$$

$f : I \times \mathscr{B} \to X$ is a continuous function satisfying some conditions. Recently, some results were devoted to equations with finite delay, the phase space is the space of continuous functions on $[-r, 0]$, for some $r > 0$, endowed with the uniform norm topology. But when the delay is unbounded, the selection of the phase space $\mathscr{B}$ plays an important role in both qualitative and quantitative theories. A usual choice is a normed space satisfying some suitable axioms, which was introduced by Hale and Kato [2]. Problems of controllability and existence of optimal controls for nonlinear differential equations have been studied extensively by many authors under various hypotheses; see, e.g., [3, 4, 5, 6, 7, 8] and the references therein, but little is known and done on the existence of optimal controls for integrodifferential equations with delay using the resolvent operator theory. In [9], Wang, Zhou and Medved considered the following fractional integrodifferential equation with infinite delay in Banach spaces

$$\begin{cases} {}^{C}D_t^q x(t) = Ax(t) + f\left(t, x_t, \int_0^t g(t,s,x_s)ds\right) + C(t)u(t) \text{ for } t \in I = [0,b], \\ x_0 = \varphi \in \mathscr{B}, \end{cases}$$

where ${}^{C}D_t^q$ denotes the Caputo fractional derivative of order $q \in (0,1)$. Using the techniques of *a priori* estimation, they studied the existence and continuous dependence of mild solutions and the optimal controls of the associated Lagrange problem. In [10], Wang and Zhou discussed the optimal controls of a Lagrange problem for the following fractional evolution equations:

$$\begin{cases} D^q x(t) = -Ax(t) + f(t, x(t)) + C(t)u(t) \text{ for } t \in [0,b], \\ x(0) = x_0 \in X, \end{cases}$$

where D^q denotes the Caputo fractional derivative of order $q \in (0,1)$ and $-A : \mathscr{D}(A) \to X$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators. In [11], Li and Liu studied the existence of mild solutions and the optimal controls of a Lagrange problem for the following impulsive fractional semilinear differential equations

$$\begin{cases} {}^{C}D_t^\alpha x(t) = Ax(t) + f\left(t, x(t), \int_0^t g(t,s,x(s))ds\right) + C(t)u(t) \text{ for } t \in [0,b],\ t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k^-)),\ k = 1,2,\cdots m, \\ x(0) = x_0 \in X, \end{cases}$$

where ${}^{C}D_t^\alpha$ denotes the Caputo fractional derivative of order $\alpha \in (0,1]$ with lower limit zero and $A : \mathscr{D}(A) \to X$ is the infinitesimal generator of a C_0-semigroup. They used the techniques of *a priori* estimation. In [12], Pan, Li and Zhao considered the following semilinear control systems with Riemann-Liouville fractional derivatives:

$$\begin{cases} {}^{L}D_t^\alpha x(t) = Ax(t) + f(t, x(t)) + Cu(t) \text{ for } t \in (0,b], \\ I_{0^+}^{1-\alpha} x(t)|_{t=0} = x_0 \in X, \end{cases}$$

where $0 < \alpha < 1$, ${}^{L}D_t^\alpha$ denotes the Riemann-Liouville fractional derivative of order α with the lower limit zero, and $A : \mathscr{D}(A) \to X$ is the infinitesimal generator of a C_0-semigroup. In [13], Ezzinbi and N-dambomve considered the following partial functional integrodifferential equation with classical Cauchy condition in Banach spaces

$$\begin{cases} x'(t) = Ax(t) + \int_0^t \gamma(t-s)x(s)ds + f(t,x(t)) + C(t)u(t) \text{ for } t \in I = [0,b], \\ x(0) = x_0 \in X, \end{cases} \tag{1.2}$$

where $A : \mathscr{D}(A) \to X$ is the infinitesimal generator of a C_0-semigroup $\left(T(t)\right)_{t\geq 0}$. Using the using the techniques of *a priori* estimation, they studied the existence and continuous dependence of mild solutions and the optimal controls of the associated Lagrange problem. In [14], Zhou considered a controlled stochastic delay partial differential equation with Neumann boundary conditions and studied the optimal control problem by means of the associated backward stochastic differential equations. In [15], Motta and Rampazzo discussed the asymptotic controllability and the optimal control of some control system where the state approaches asymptotically a target, while paying an integral cost with a nonnegative Lagrangian.

In [10], Wand and Zhou discussed the optimal controls of a Lagrange problem for fractional evolution equations. In [16], Wei, Xiang and Peng studied the optimal controls for nonlinear impulsive integrodifferential equations of mixed type on Banach spaces. In [11], Li and Liu studied the existence of mild solutions and the optimal controls of a Lagrange problem for some impulsive fractional semilinear differential equations, using the techniques of *a priori* estimation. Motivated by these works, we investigate the solvability and the existence of optimal controls of a Lagrange problem for equation (1.1) based on the techniques of *a priori* estimation of mild solutions. The existence and uniqueness of mild solutions is obtained using the theory of resolvent operators for integral equations.

As a motivation for the problem studied in this paper, we consider a heat flow in a rigid body Ω of a material with memory. Let $w(t,\xi)$, $e(t,\xi)$, $q(t,\xi)$ and $s(t,\xi)$ denote respectively the temperature, the internal energy, the heat flux, and the external heat supply at time t and position ξ. The balance law for the heat transfer is given by:

$$e_t(t,\xi) + \operatorname{div} q(t,\xi) = s(t,\xi) \tag{1.3}$$

and the physical properties of the body suggest the dependence of e and q on w and ∇w, respectively. For instance assuming the Fourier Law, i.e.,

$$e(t,\xi) = c_1 w(t,\xi), \tag{1.4}$$

$$q(t,\xi) = -c_2 \nabla w(t,\xi), \tag{1.5}$$

where c_1, c_2 are positive constants, one deduces from (1.3) the classical heat equation

$$w_t(t,\xi) = c\Delta w(t,\xi) + f(t,\xi) \tag{1.6}$$

with $c = c_1^{-1}c_2$ and $g(t,\xi) = c_1^{-1}s(t,\xi)$. In many results, the assumptions (1.4), (1.5) are not justified because they do not take the memory effects into account: several models have been proposed to overcome this difficulty, see, e.g. [17, 18, 19]: one of them consists of substituting (1.5) with

$$q(t,\xi) = -c_2\nabla w(t,\xi) - \int_{-\infty}^t h(t-s)\nabla w(s,\xi)ds. \tag{1.7}$$

If $c_1 = c_2 = 1$, then we get from (1.3), (1.4) and (1.7) that

$$w_t(t,\xi) = \Delta w(t,\xi) + \int_{-\infty}^t h(t-s)\Delta w(s,\xi)ds + s(t,\xi). \tag{1.8}$$

If we assume that the thermal history w of the body Ω is known up to $t=0$ and the temperature of the boundary $\partial\Omega$ of Ω is constant (=0) for all t, we are led to the following system:

$$\begin{cases} w_t(t,\xi) = \Delta w(t,\xi) + \int_0^t h(t-s)\Delta w(s,\xi)ds + g(t,\xi), \ \ (t,\xi)\in[0,b]\times\Omega, \\ w(t,\xi)=0, \ \ (t,\xi)\in[0,b]\times\partial\Omega, \end{cases} \tag{1.9}$$

where $b>0$ is arbitrarily fixed. If we prescribe h (in addition to g), then (1.9) is a Cauchy-Dirichlet problem for an integrodifferential equation in the unknown w, which has been studied by several authors in the last decades, see, e.g., [20, 21, 22] and references therein. Now, if we consider that the thermal history of the body Ω is known at all time less than or equal up to the present time t, the temperature of the boundary $\partial\Omega$ of Ω is constant (= 0) for all t, and the external heat flux depends on the this thermal history of the body, we have that system (1.9) becomes the following integrodifferential equation with finite delay:

$$\begin{cases} w_t(t,\xi) = \Delta w(t,\xi) + \int_0^t h(t-s)\Delta w(s,\xi)ds + \int_{-\infty}^0 g(\theta, w(t+\theta,\xi))\,d\theta, \ \ (t,\xi)\in[0,b]\times\Omega, \\ w(t,\xi)=\psi(t,\xi) \ \text{ for } t\in]-\infty,0] \text{ and } x\in\overline{\Omega}, \end{cases} \tag{1.10}$$

where ψ is a given initial function and the integral $\int_{-\infty}^0 g(\theta,w(t+\theta,\xi))\,d\theta$ is assumed to converge under some sufficient conditions

Now we define

$$\begin{aligned} x(t)(\xi) &= w(t,\xi), \\ Ax &= \Delta x, \\ \varphi(\theta)(\xi) = \psi(\theta,\xi), & \ \ \theta\in]-\infty,0], \ \ \xi\in\Omega, \\ f(t,\varphi)(\xi) &= \int_{-\infty}^0 g(t,\varphi(\theta)(\xi))\,d\theta \quad \text{for } t\in[0,b] \text{ and } \xi\in\Omega, \\ (\gamma(t)x)(\xi) &= h(t)\Delta x(t)(\xi) \ \text{ for } \ t\in[0,b], \text{ and } \ \xi\in\Omega. \end{aligned}$$

Then, equation (1.10) can be transformed into the following abstract form:

$$\begin{cases} x'(t) = Ax(t) \ + \ \int_0^t \gamma(t-s)x(s)ds \ + \ f(t,x_t) \ \text{ for } t\in I=[0,b], \\ x_0=\varphi\in\mathscr{B}, \end{cases} \tag{1.11}$$

where X is a Banach space and $\mathscr{B}$ is a phase space defined axiomatically. An example of a material with memory is *Shape-memory polymers* (SMPs), which are polymeric smart materials that have the ability to return from a deformed state (temporary shape) to their original (permanent) shape induced by an external stimulus (trigger), such as, temperature change. That is, they act adaptively to their environment and they can easily be shaped into different forms at a low temperature. In order to evaluate the performance of a system quantatively, the designer selects a performance measure or a cost function. In certain cases, the statement of the problem may clearly indicate what to select for a cost function, whereas in other cases, the selection of a cost function is a subjective matter. Suppose that the objective in this example is to make the material take a particular form, with minimum heating. The optimal control problem: Find an admissible control u^* which causes the system

$$x'(t) = Ax(t) \ + \ \int_0^t \gamma(t-s)x(s)ds \ + \ f(t,x_t) + C(t)u(t) \ \text{ for } t\in I$$

to follow an admissible state x^* that minimize the cost function

$$\mathscr{J}(u) := \int_0^b \mathscr{L}\Big(t, x_t^u, x^u(t), u(t)\Big)\, dt,$$

where x^u denotes the mild solution of (1.11) corresponding to the control $u \in \mathscr{U}_{ad}$, the space of admissible controls. u^* is called an optimal control and x^* is called an optimal state.

Recently, equation (1.11) has been studied by many authors (see e.g., [23] and the references therein). To the best of our knowledge, this equation has never been considered for optimal control. The rest of the paper is organized as follows. In Section 2, we present the problem and give a motivation from the analysis of heat flow in rigid body. In Section 3, we present some basic definitions and preliminaries results, which will be used in the subsequent sections. In Section 4, sufficient conditions are established for the existence and uniqueness of mild solutions of equation (1.1) based on a well known fixed point theorem. In Section 5, we obtain an *a priori* estimation of mild solutions of equation (1.1) and establish the continuous dependence of solutions on the initial data and the controls. We also investigate the existence of optimal controls of a Lagrange optimal control problem for equation (1.1). Finally, in Section 6, an example is given to illustrate the main results of this paper.

2. Resolvent Operators and Balder's Theorem

In this section we introduce some definitions and Lemmas that will be used next.

Let $I = [0,b], \quad b > 0$ and let X be a Banach space. A measurable function $x : I \to X$ is Bochner integrable if and only if $\|x\|$ is Lebesgue integrable. We denote by $L^1(I,X)$ the Banach space of Bochner integrable functions $x : I \to X$ normed by

$$\|x\|_{L^1} = \int_0^b \|x(t)\|dt.$$

Consider the following linear homogeneous equation:

$$\begin{cases} x'(t) = Ax(t) + \displaystyle\int_0^t \gamma(t-s)x(s)ds \ \text{ for } \quad t \geq 0, \\ x(0) = x_0 \in X, \end{cases} \tag{2.1}$$

where A and $\gamma(t)$ are closed linear operators on a Banach space X.

In the sequel, we assume A and $\big(\gamma(t)\big)_{t\geq 0}$ satisfy the following conditions:

$(\mathbf{H_1})$ A is a densely defined closed linear operator in X. Hence $\mathscr{D}(A)$ is a Banach space equipped with the graph norm defined by, $|y| = \|Ay\| + \|y\|$ which will be denoted by $(X_1, |\cdot|)$.

$(\mathbf{H_2})$ $\big(\gamma(t)\big)_{t\geq 0}$ is a family of linear operators on X such that $\gamma(t)$ is continuous when regarded as a linear map from $(X_1, |\cdot|)$ into $(X, \|\cdot\|)$ for almost all $t \geq 0$ and the map $t \mapsto \gamma(t)y$ is measurable for all $y \in X_1$ and $t \geq 0$, and belongs to $W^{1,1}(\mathbb{R}^+, X)$. Moreover there is a locally integrable function $b : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|\gamma(t)y\| \ \leq \ b(t)|y| \quad \text{ and } \quad \left\|\frac{d}{dt}\gamma(t)y\right\| \ \leq \ b(t)|y| \,.$$

Remark 2.1. Note that $(\mathbf{H_2})$ is satisfied in the modelling of heat conduction in materials with memory and viscosity. More details can be found in [24].

Let $\mathscr{L}(X)$ be the Banach space of bounded linear operators on X.

Definition 2.2. [23] A resolvent operator $\big(R(t)\big)_{t\geq 0}$ for equation (2.1) is a bounded operator valued function

$$R:\ [0,+\infty)\ \longrightarrow\ \mathscr{L}(X)$$

such that

- **(i)** $R(0)=Id_X$ and $\|R(t)\|\leq Ne^{\beta t}$ for some constants N and β.
- **(ii)** For all $x\in X$, the map $t\mapsto R(t)x$ is continuous for $t\geq 0$.
- **(iii)** Moreover for $x\in X_1$, $R(\cdot)x\in\mathscr{C}^1(\mathbb{R}^+;X)\cap\mathscr{C}(\mathbb{R}^+;X_1)$ and

$$\begin{aligned} R'(t)x &= AR(t)x+\int_0^t\gamma(t-s)R(s)xds \\ &= R(t)Ax+\int_0^t R(t-s)\gamma(s)xds. \end{aligned}$$

Observe that the map defined on $\mathbb{R}^+$ by $t\mapsto R(t)x_0$ solves equation (2.1) for $x_0\in\mathscr{D}(A)$.

Theorem 2.3. [1] *Assume that* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold. Then, linear equation (2.1) has a unique resolvent operator* $\big(R(t)\big)_{t\geq 0}$.

Remark 2.4. In general, the resolvent operator $\big(R(t)\big)_{t\geq 0}$ for equation (2.1) does not satisfy the semi-group law, namely,

$$R(t+s)\ \neq\ R(t)R(s)\qquad \text{for some }\ t,s>0\,.$$

The following Theorem is needed for the proof of the existence of optimal controls.

Theorem 2.5. *(Balder's Theorem, [25]) Let* $(\Sigma,\mathscr{F},\mu)$ *be a finite nonatomic measure space and let* $(X,\|\cdot\|)$ *be a separable Banach space. Let* $(V,|\cdot|)$ *be a separable reflexive Banach space, and let* V' *be its dual. Let* $\theta:\Sigma\times X\times V\to(-\infty,+\infty]$ *be a given* $\mathscr{F}\times\mathscr{L}(X\times V)$*-measurable function. The associated integral functional* $I_\theta: L^1_X\times L^1_V\to[-\infty,+\infty]$ *is defined by:*

$$I_\theta(x,v)=\int_\Sigma\theta(t,x(t),v(t))\mu(dt),$$

where L^1_X *denotes the set of all absolutely summable functions from* Σ *to* X*. The following three conditions*

- **(i)** $\theta(t,\cdot,\cdot)$ *is sequencially lower semicontinuous on* $X\times V$, μ*-a.e.,*
- **(ii)** $\theta(t,x,\cdot)$ *is convex on* V *for* $x\in X$, μ*-a.e.,*
- **(iii)** *there exist* $\sigma>0$ *and* $\varphi\in L^1_{\mathbb{R}}$ *such that*

$$\theta(t,x,v)\geq\varphi(t)-\sigma(\|x\|+|v|),\ \textit{for all}\ \ x\in X,\ v\in V,\ \mu\textit{-a.e.},$$

are sufficient for sequential strong-weak lower semicontinuity I_θ *on* $L^1_X\times L^1_V$*. Moreover, they are also necessary, provided that* $I_\theta(\bar{x},\bar{v})<+\infty$ *for some* $\bar{x}\in L^1_X,\bar{v}\in L^1_V$*.*

Theorem 2.6. *(Mazur's Lemma, [26]) Let Z be a Banach space and let G be a convex and closed set in Z. Then G is weakly closed in Z.*

3. EXISTENCE OF MILD SOLUTIONS FOR EQUATION (1.1)

In this section, we will employ an axiomatic definition of the phase space $\mathscr{B}$ introduced by Hale and Kato [2]. Thus, $(\mathscr{B}, \|\cdot\|_{\mathscr{B}})$ will be a normed linear space of functions mapping $]-\infty, 0]$ into X and satisfying the following axioms:

$(\mathbf{A_1})$ There exist positive constant H and functions $K : \mathbb{R}^+ \to \mathbb{R}^+$ continuous and $M : \mathbb{R}^+ \to \mathbb{R}^+$ locally bounded, such that for $a > 0$, if $x :]-\infty, a] \to X$ is continuous on $[0,a]$ and $x_0 \in \mathscr{B}$, then for every $t \in [0,a]$, the following conditions hold:

(i) $x_t \in \mathscr{B}$,

(ii) $\|x(t)\| \leq H\|x_t\|_{\mathscr{B}}$, which is equivalent to $\|\varphi(0)\| \leq H\|\varphi\|_{\mathscr{B}}$ for every $\varphi \in \mathscr{B}$,

(iii) $\|x_t\|_{\mathscr{B}} \leq K(t) \sup\limits_{0 \leq s \leq t} \|x(s)\| + M(t)\|x_0\|_{\mathscr{B}}$.

$(\mathbf{A_2})$ For the function x in $(\mathbf{A_1})$, $t \to x_t$ is a $\mathscr{B}$-valued continuous function for $t \in [0,a]$.

$(\mathbf{A_3})$ The space $\mathscr{B}$ is complete.

We assume the following hypotheses.

$(\mathbf{H_3})$ The function $f : I \times \mathscr{B} \to X$ satisfies the following conditions:

(i) $f(\cdot, \psi)$ is measurable for $\psi \in \mathscr{B}$,

(ii) for any $\rho > 0$, there exists $L_f(\rho) > 0$ such that

$$\|f(t,\psi_1) - f(t,\psi_2)\| \leq L_f(\rho)\|\psi_1 - \psi_2\|_{\mathscr{B}} \quad \text{for } \|\psi_1\|_{\mathscr{B}} \leq \rho,\ \|\psi_2\|_{\mathscr{B}} \leq \rho \text{ and } t \in [0,b],$$

(iii) there exists $a_f > 0$ such that

$$\|f(t,\psi)\| \leq a_f(1 + \|\psi\|_{\mathscr{B}}) \quad \text{for all} \quad \psi \in \mathscr{B} \quad \text{and} \quad t \in [0,b].$$

$(\mathbf{H_4})$ Let U be the separable reflexive Banach space from which the control u takes values and assume $C \in L^\infty(I; \mathscr{L}(U,X))$.

$(\mathbf{H_5})$ The multivalued map $\Gamma : I \to 2^U \setminus \{\emptyset\}$ has closed, convex, and bounded values, Γ is graph measurable, and $\Gamma(\cdot) \subseteq \Omega$ where Ω is a bounded set in U.

We denote by $\mathscr{U}_{ad}$ the set of admissible controls defined by:

$$\mathscr{U}_{ad} = \Big\{ u : I \to U\, ;\, u \text{ is measurable and } u(t) \in \Gamma(t),\, a.e. \Big\}.$$

Then, we have the following.

Theorem 3.1. [9] *$\mathscr{U}_{ad} \neq \emptyset$ and $\mathscr{U}_{ad} \subset L^2(I,U)$ is bounded, closed and convex. Also, $Cu \in L^2(I,U)$ for all $u \in \mathscr{U}_{ad}$.*

Definition 3.2. Let $u \in \mathscr{U}_{ad}$ and $\varphi \in \mathscr{B}$. A function $x :]-\infty, b] \to X$ is called a mild solution of equation (1.1) if $x \in \mathscr{C}([0,b]; X)$ and

$$x(t) = \begin{cases} R(t)\varphi(0) + \displaystyle\int_0^t R(t-s)\,[f(s,x_s) + C(s)u(s)]\, ds \ \text{ for } t \in I, \\ \varphi(t) \ \text{ for } \ -\infty \leq t \leq 0. \end{cases} \tag{3.1}$$

Let

$$R_b = \sup_{t\in[0,b]} \|R(t)\|,\ \ K_b = \sup_{t\in[0,b]} \|K(t)\|,\ \ M_b = \sup_{t\in[0,b]} \|M(t)\|,\ \ \|C\| = \sup_{t\in[0,b]} \|C(t)\|_{\mathscr{L}(U,X)}.$$

We have the following theorem on existence of mild solutions of equation (1.1) with respect to a given control $u \in \mathscr{U}_{ad}$.

Theorem 3.3. *Assume that* $(\mathbf{H_1})-(\mathbf{H_5})$ *hold. Then, for each* $u \in \mathscr{U}_{ad}$*, equation* (1.1) *has a unique mild solution on* $]-\infty, b]$.

Proof Let $b_1 \leq b$, $\rho > 0$, and $\psi \in \mathscr{B}$ such that $\|\psi\|_{\mathscr{B}} \leq \rho$. For $t \in [0, b_1]$, we have by the local Lipschitz condition on f that

$$\|f(t,\psi)\| \leq L_f(\rho)\|\psi\|_{\mathscr{B}} + \|f(t,0)\| \leq L_f(\rho)\rho + \sup_{s\in[0,b_1]} \|f(s,0)\|.$$

b_1 will be chosen sufficiently small enough to get the local existence of mild solutions. Let $\varphi \in \mathscr{B}$, $\rho = \|\varphi\|_{\mathscr{B}} + 1$ and $\rho^* = L_f(\rho)\rho + \sup_{s\in[0,b_1]} \|f(s,0)\|$. Define the function

$$y(t) = \begin{cases} R(t)\varphi(0) & \text{for } t \in [0,b], \\ \varphi(t) & \text{for } t \in]-\infty, 0]. \end{cases}$$

By axioms $(\mathbf{A_1})-(\mathbf{i})$ and $(\mathbf{A_2})$, we deduce that $y_t \in \mathscr{B}$ and $t \mapsto y_t$ is continuous for $t \in [0,b]$. Then, for $\tau_1 \in]0,1[$, there exists $\tau_2 \in]0,1[$ such that

$$\|y_t - \varphi\|_{\mathscr{B}} \leq \tau_1 \quad \text{for } t \in [0, \tau_2].$$

Let $b_1 \in [0, \tau_2]$ be such that

$$K_b R_b \left(\rho^* b_1 + \|C\| \, \|u\|_{L^2} \sqrt{b_1}\right) < 1 - \tau_1. \tag{3.2}$$

For $x \in \mathscr{C}([0,b], X)$ such that $x(0) = \varphi(0)$, we define its extension $\widetilde{x}$ on $]-\infty, b]$ by

$$\widetilde{x}(t) = \begin{cases} x(t) & \text{for } t \in [0,b], \\ \varphi(t) & \text{for } t \in]-\infty, 0]. \end{cases}$$

We introduce the following space

$$\mathscr{E}_\varphi = \left\{ x \in \mathscr{C}([0,b_1];X) \text{ such that } x(0) = \varphi(0) \text{ and } \sup_{s\in[0,b_1]} \|\widetilde{x}_s - \varphi\|_{\mathscr{B}} \leq 1 \right\},$$

provided with the uniform norm topology. Define the operator $P : \mathscr{E}_\varphi \to \mathscr{C}([0,b_1];X)$ by

$$(Px)(t) = R(t)\varphi(0) + \int_0^t R(t-s)\left[f(s,\widetilde{x}_s) + C(s)u(s)\right] ds \quad \text{for } t \in [0,b_1].$$

We claim that $P\left(\mathscr{E}_\varphi\right) \subset \mathscr{E}_\varphi$. In fact, let $x \in \mathscr{E}_\varphi$. By $(\mathbf{H_3})$ and axiom $(\mathbf{A_1})$, the function $s \mapsto f(s,\widetilde{x}_s)$ is continuous on $[0,b_1]$. Using Definition 2.2, we imply that

$$s \mapsto \int_0^t R(t-s)\left[f(s,\widetilde{x}_s) + C(s)u(s)\right] ds$$

is continuous on $[0,b_1]$, consequently, $v = Px$ is continuous on $[0,b_1]$. We claim that $v \in \mathscr{E}_\varphi$. In fact, for any $t \in [0,b_1]$, we have

$$\|\widetilde{v}_t - \varphi\|_{\mathscr{B}} \leq \|\widetilde{v}_t - y_t\|_{\mathscr{B}} + \|y_t - \varphi\|_{\mathscr{B}} \leq \|\widetilde{v}_t - y_t\|_{\mathscr{B}} + \tau_1.$$

Since $\|\widetilde{x}_s - \varphi\|_{\mathscr{B}} \leq 1$ for $s \in [0,b_1]$, we have $\|\widetilde{x}_s\|_{\mathscr{B}} \leq \rho$ for $s \in [0,b_1]$. Then,

$$\|f(s,\widetilde{x}_s\| \leq L_f(\rho)\|\widetilde{x}_s\|_{\mathscr{B}} + \|f(s,0)\| \leq \rho^*.$$

By axiom $(\mathbf{A_3})-(\mathbf{iii})$, for any $t \in [0,b_1]$, we have

$$\|\widetilde{v}_t - y_t\|_{\mathscr{B}} \leq K_b \sup_{s\in[0,t]} \|v(s) - y(s)\|$$

and

$$\begin{aligned}\|v(t)-y(t)\| &\leq \int_0^t \left\|R(t-s)\left[f(s,\widetilde{x}_s)+C(s)u(s)\right]\right\| ds\\ &\leq R_b\,\rho^* b_1 + R_b\,\|C\|\,\|u\|_{L^2}\sqrt{b_1}\\ &< \frac{1}{K_b}(1-\tau_1).\end{aligned}$$

Consequently, one has $\|\widetilde{v}_t - y_t\|_{\mathscr{B}} < 1-\tau_1$. Thus, we deduce that $\|\widetilde{v}_t - \varphi\|_{\mathscr{B}} < 1$ for any $t\in[0,b_1]$, which implies that $v\in\mathscr{E}_\varphi$. Therefore $P\Big(\mathscr{E}_\varphi\Big)\subset\mathscr{E}_\varphi$.

We now show that P is a strict contraction on $[0,b_1]$. In fact, let $x^1, x^2\in\mathscr{E}_\varphi$ and $t\in[0,b_1]$. Then, $\|\widetilde{x}^i_s\|\leq\rho$ for $i=1,2$. We have that

$$\begin{aligned}\|(Px^1)(t)-(Px^2)(t)\| &\leq R_b\int_0^t \|f(s,\widetilde{x}^1_s)-f(s,\widetilde{x}^2_s)\|\,ds\\ &\leq R_b\,L_f(\rho)\int_0^t \|\widetilde{x}^1_s-\widetilde{x}^2_s\|\,ds\\ &\leq R_b\,L_f(\rho)K_b\int_0^t \sup_{\sigma\in[0,s]}\|x^1(\sigma)-x^2(\sigma)\|\,d\sigma\\ &\leq K_bR_b\,L_f(\rho)\,b_1\,\|x^1-x^2\|.\end{aligned}$$

It follows that

$$\|(Px^1)(t)-(Px^2)(t)\|\leq K_bR_b\,L_f(\rho)\,b_1\,\|x^1-x^2\|.$$

Note that

$$K_bR_b\,L_f(\rho)\,b_1\leq K_bR_b\,\rho^*\,b_1 < K_bR_b\,\rho^* b_1 + K_bR_b\,\|C\|\,\|u\|_{L^2}\sqrt{b_1}.$$

Condition (3.2) implies that

$$K_bR_b\,L_f(\rho)\,b_1<1.$$

Thus, P is a strict contraction on $\mathscr{E}_\varphi$. It follows from the contraction mapping principle that P has a unique fixed point $x\in\mathscr{E}_\varphi$, which is the unique mild solution of equation (1.1) with respect to u on $]-\infty,b_1]$. Using the same arguments, we can show that x can be extended to a maximal interval of existence $[0,t_{\max}[$.

Lemma 3.4. [23] *If $t_{\max}<b$, then* $\limsup_{t\to t_{\max}}\|x(t)\|=\infty$.

We show that $t_{\max}=b$. Assume on the contrary that $t_{\max}<b$. For $t\in[0,t_{\max}]$, we have that

$$x(t)=R(t)\varphi(0)+\int_0^t R(t-s)\left[f(s,x_s)+C(s)u(s)\right]ds.$$

It follows that

$$\begin{aligned}
\|x(t)\| &\leq R_b\|\varphi(0)\|+R_b\int_0^t\|f(s,x_s)\|\,ds+R_b\int_0^t\|C(s)u(s)\|\,ds\\
&\leq R_b\|\varphi(0)\|+R_b\,t_{\max}\,a_f+R_b a_f\int_0^t\|x_s\|\,ds+R_b\|C\|\int_0^t\|u(s)\|\,ds\\
&\leq R_b\|\varphi(0)\|+R_b\,t_{\max}\,a_f+R_b\sqrt{t_{\max}}\,\|C\|\|u\|_{L^2}+R_b\,a_f\int_0^t\|x_s\|\,ds\\
&\leq R_b\|\varphi(0)\|+R_b\,t_{\max}\,a_f+R_b\sqrt{t_{\max}}\,\|C\|\|u\|_{L^2}+R_b\,M_{t_{\max}}\,a_f\,t_{\max}\|\varphi(0)\|\\
&\quad +R_b\,K_{t_{\max}}\,a_f\int_0^t\sup_{\tau\in[0,s]}\|x(\tau)\|\,d\tau.
\end{aligned}$$

This implies that

$$\begin{aligned}
\sup_{s\in[0,t]}\|x(s)\| &\leq R_b\|\varphi(0)\|+R_b\,t_{\max}\,a_f+R_b\sqrt{t_{\max}}\,\|C\|\|u\|_{L^2}+R_b\,M_{t_{\max}}\,a_f\,t_{\max}\|\varphi(0)\|\\
&\quad +R_b\,K_{t_{\max}}\,a_f\int_0^t\sup_{\tau\in[0,s]}\|x(\tau)\|\,d\tau,
\end{aligned}$$

where

$$K_{t_{\max}}=\sup_{s\in[0,t_{\max}]}K(s)\ \text{ and }\ M_{t_{\max}}=\sup_{s\in[0,t_{\max}]}M(s).$$

It follows by Gronwall's inequality that

$$\|x(t)\|\leq\beta^*e^{(R_b\,K_{t_{\max}}\,a_f\,t)}\ \text{ for }\ t\in[0,t_{\max}],$$

where $\beta^*=R_b\|\varphi(0)\|+R_b\,t_{\max}\,a_f+R_b\sqrt{t_{\max}}\,\|C\|\|u\|_{L^2}+R_b\,M_{t_{\max}}\,a_f\,t_{\max}\|\varphi(0)\|$. Thus

$$\lim_{t\to t_{\max}}\|x(t)\|\leq\beta^*e^{(R_b\,K_{t_{\max}}\,a_f\,t_{\max})}<\infty.$$

This contradicts Lemma 3.4. Therefore, $t_{\max}=b$ and hence, equation (1.1) has a unique mild solution on $(-\infty,b]$.

4. Continuous Dependence and Existence of the Optimal Control

In this section, we discuss the continuous dependence of the mild solutions of equation (1.1) on the controls and initial states, and the existence of solutions of the Lagrange problem associated to equation (1.1).

We have the following a priori estimation.

Lemma 4.1. *Suppose that* $(\mathbf{H_3})$ *holds and equation (1.1) has a mild solution* x_u *on* $]-\infty,b]$ *with respect to* $u\in\mathscr{U}_{ad}$*. Then, there exists a constant* $\rho>0$ *independent of* u *such that* $\|x_u(t)\|\leq\rho$ *for* $t\in[0,b]$*, (*ρ *depends only on* $\mathscr{U}_{ad}$ *and* φ*).*

Proof. Let $\varphi\in\mathscr{B}$. For $x\in\mathscr{C}([0,b],X)$ such that $x(0)=\varphi(0)$, we define its extension $\widetilde{x}$ on $]-\infty,b]$ by

$$\widetilde{x}(t)=\begin{cases}x(t) & \text{for } t\in[0,b],\\ \varphi(t) & \text{for } t\in]-\infty,0].\end{cases}$$

Also, we define the function $y\in\mathscr{C}([0,b],X)$ by $y(t)=R(t)\varphi(0)$ and its extension $\widetilde{y}$ on $]-\infty,b]$ by

$$\widetilde{y}(t)=\begin{cases}y(t) & \text{for } t\in[0,b],\\ \varphi(t) & \text{for } t\in]-\infty,0].\end{cases}$$

For each $z \in \mathscr{C}([0,b],X)$, we set $\widetilde{x}(t) = \widetilde{z}(t) + \widetilde{y}(t)$, where $\widetilde{z}$ is the extension by zero of the function z on $]-\infty, 0]$. Observe that x satisfies (3.1) if and only if $z(0) = 0$ and

$$z(t) = \int_0^t R(t-s)\left[f(s, \widetilde{z}_s + \widetilde{y}_s) + C(s)u(s)\right] ds \;\text{ for } t \in [0,b].$$

Since $\mathscr{U}_{ad}$ is bounded, let $\widetilde{N} > 0$ be such that $\|u\|_{L^2} \le \widetilde{K}$ for all $u \in \mathscr{U}_{ad}$

$$\begin{aligned}
\|z(t)\| &\le R_b \int_0^t \|f(s, \widetilde{z}_s + \widetilde{y}_s)\| \, ds + R_b \int_0^t \|C(s)u(s)\| \, ds \\
&\le R_b\, b\, a_f + R_b a_f \int_0^t \|\widetilde{z}_s + \widetilde{y}_s\| \, ds + R_b \|C\| \int_0^b \|u(s)\| \, ds \\
&\le R_b\, b\, a_f + R_b \sqrt{b}\, \|C\| \|u\|_{L^2} + R_b\, a_f \int_0^t \|\widetilde{z}_s + \widetilde{y}_s\|_{\mathscr{B}} \, ds \\
&\le R_b\, b\, a_f + R_b \sqrt{b}\, \|C\| \widetilde{N} + R_b\, a_f \int_0^t \|\widetilde{z}_s + \widetilde{y}_s\|_{\mathscr{B}} \, ds.
\end{aligned}$$

Thus

$$\|z(t)\| \le R_b\, b\, a_f + R_b \sqrt{b}\, \|C\| \widetilde{N} + R_b\, a_f \int_0^t \|\widetilde{z}_s + \widetilde{y}_s\|_{\mathscr{B}} \, ds, \tag{4.1}$$

$$\begin{aligned}
\|\widetilde{z}_s + \widetilde{y}_s\|_{\mathscr{B}} &\le \|\widetilde{z}_s\|_{\mathscr{B}} + \|\widetilde{y}_s\|_{\mathscr{B}} \\
&\le K(s) \sup_{0 \le \tau \le s} \|z(\tau)\| + M(s)\|z_0\|_{\mathscr{B}} + K(s) \sup_{0 \le \tau \le s} \|\widetilde{y}(\tau)\| + M(s)\|\widetilde{y}_0\|_{\mathscr{B}} \\
&\le K_b \sup_{\tau \in [0,s]} \|z(\tau)\| + K_b R_b \|\varphi\|_{\mathscr{B}} + M_b \|\varphi(0)\|.
\end{aligned}$$

This implies that (4.1) can be rewritten as follows

$$\begin{aligned}
\sup_{s \in [0,t]} \|z(s)\| &\le R_b\, b\, a_f + R_b \sqrt{b}\, \|C\| \widetilde{N} + R_b\, a_f\, b \Big(K_b R_b \|\varphi\|_{\mathscr{B}} + M_b \|\varphi(0)\| \Big) \\
&\quad + K_b R_b\, a_f \int_0^t \sup_{\tau \in [0,s]} \|z(\tau)\| \, d\tau \\
&= N^* + K_b R_b\, a_f \int_0^t \sup_{\tau \in [0,s]} \|z(\tau)\| \, d\tau
\end{aligned}$$

with $N^* = R_b\, b\, a_f + R_b \sqrt{b}\, \|C\| \widetilde{N} + R_b\, a_f\, b \Big(K_b R_b \|\varphi\|_{\mathscr{B}} + M_b \|\varphi(0)\| \Big)$. It follows from the Gronwall's inequality that

$$\|z(t)\| \le N^* e^{(b\, a_f R_b K_b)} =: \widetilde{N^*}.$$

As a result, for $t \in I$, we have

$$\begin{aligned}
\|x_u(t)\| &\le \|z(t)\| + \|R(t)\varphi(0)\| \\
&\le \widetilde{N^*} + R_b \|\varphi(0)\| := \rho,
\end{aligned}$$

that is, $\|x_u(t)\| \le \rho$ for all $t \in I$. This completes the proof of the Lemma.

We have the following continuous dependence result.

Theorem 4.2. *For all $r > 0$, there exists $\lambda^*(r) > 0$ such that for all $\varphi^1,\ \varphi^2 \in B(0,r)$,*

$$\|x^1(t) - x^2(t)\| \le \lambda^*(r) \Big(\|\varphi^1(0) - \varphi^2(0)\| + \|\varphi^1 - \varphi^2\|_{\mathscr{B}} + \|u^1 - u^2\|_{L^2} \Big) \;\; \textit{for} \;\; t \in [0,b],$$

where

$$x^i(t)=\begin{cases} R(t)\varphi^i(0)+\int_0^t R(t-s)\left[f(s,x_s^i)+C(s)u^i(s)\right]ds \;\; for\; t\in I,\\ \varphi^i(t)\;\; for\;\; -\infty\le t\le 0, \end{cases} \tag{4.2}$$

and $u^i\in\mathscr{U}_{ad}$, for $i=1,2$.

Proof. Let x^i, for $i=1,2$, be a mild solution of equation (1.1), corresponding to the controls $u^i\in\mathscr{U}_{ad}$ and initial conditions $\varphi^i\in B(0,r)$

$$x^i(t)=\begin{cases} R(t)\varphi^i(0)+\int_0^t R(t-s)\left[f(s,x_s^i)+C(s)u^i(s)\right]ds \;\; \text{for}\; t\in I,\\ \varphi^i(t)\;\; \text{for}\;\; -\infty\le t\le 0. \end{cases}$$

By Lemma 4.1, we have that there exists a constant $\rho_r=\widetilde{N^*}+R_b r>0$ such that $\|x_s^i\|\le\rho_r,\; i=1,2$. Now, for $t\in[0,b]$, we have

$$\begin{aligned} \|x^1(t)-x^2(t)\| &\le R_b\|\varphi^1(0)-\varphi^2(0)\|+R_b\int_0^t\|f(s,x_s^1)-f(s,x_s^2)\|ds\\ &+ R_b\int_0^t\|C(s)u^1(s)-C(s)u^2(s)\|ds\\ &\le R_b\|\varphi^1(0)-\varphi^2(0)\|+R_bL_f(\rho_r)\int_0^t\left(\|x_s^1-x_s^2\|_{\mathscr{B}}\right)ds\\ &+ R_b\int_0^t\|C(s)u^1(s)-C(s)u^2(s)\|ds\\ &\le R_b\|\varphi^1(0)-\varphi^2(0)\|+R_bL_f(\rho_r)\int_0^t\|x_s^1-x_s^2\|_{\mathscr{B}}ds\\ &+ R_bL_f(\rho_r)\sqrt{b}\,\|C\|\,\|u^1-u^2\|_{L^2}\\ &\le R_b\|\varphi^1(0)-\varphi^2(0)\|+R_bL_f(\rho_r)K_b\int_0^t\sup_{\tau\in[0,s]}\|x^1(\tau)-x^2(\tau)\|d\tau\\ &+ R_bL_f(\rho_r)M_b\,b\,\|x_0^1-x_0^2\|_{\mathscr{B}}+R_bL_f(\rho)\sqrt{b}\,\|C\|\,\|u^1-u^2\|_{L^2}\\ &\le R_b\|\varphi^1(0)-\varphi^2(0)\|+R_bL_f(\rho_r)K_b\int_0^t\sup_{\tau\in[0,s]}\|x^1(\tau)-x^2(\tau)\|d\tau\\ &+ R_bL_f(\rho_r)M_b\,b\,\|\varphi^1-\varphi^2\|_{\mathscr{B}}+R_bL_f(\rho_r)\sqrt{b}\,\|C\|\,\|u^1-u^2\|_{L^2}. \end{aligned}$$

It follows that

$$\begin{aligned} \sup_{s\in[0,t]}\|x^1(s)-x^2(s)\| &\le R_b\|\varphi^1(0)-\varphi^2(0)\|+R_bL_f(\rho_r)\sqrt{b}\,\|C\|\,\|u^1-u^2\|_{L^2}\\ &+ R_bL_f(\rho)M_b\,b\,\|\varphi^1-\varphi^2\|_{\mathscr{B}}+R_bL_f(\rho_r)K_b\int_0^t\sup_{\tau\in[0,s]}\|x^1(\tau)-x^2(\tau)\|d\tau. \end{aligned}$$

By Gronwall's inequality, we have that

$$\sup_{s\in[0,t]}\|x^1(s)-x^2(s)\| \le N^{**}e^{R_bL_f(\rho_r)K_b b},$$

where

$$N^{**}=\left[R_b\|\varphi^1(0)-\varphi^2(0)\|+R_bL_f(\rho_r)M_b\,b\,\|\varphi^1-\varphi^2\|_{\mathscr{B}}+R_bL_f(\rho_r)\sqrt{b}\,\|C\|\,\|u^1-u^2\|_{L^2}\right].$$

This implies that

$$\|x^1(t)-x^2(t)\| \le \Big[R_b\|\varphi^1(0)-\varphi^2(0)\|+R_bL_f(\rho_r)M_b\,b\,\|\varphi^1-\varphi^2\|_{\mathscr{B}} + R_bL_f(\rho_r)\sqrt{b}\,\|C\|\,\|u^1-u^2\|_{L^2}\Big]e^{R_bL_f(\rho_r)K_b\,b}.$$

Let

$$\lambda^*(r) := \max\Big\{R_b\,e^{R_bL_f(\rho_r)K_b\,b},\ R_bL_f(\rho_r)M_b\,b\,e^{R_bL_f(\rho)K_b\,b},\ R_bL_f(\rho_r)\sqrt{b}\,\|C\|\,e^{R_bL_f(\rho_r)K_b\,b}\Big\}.$$

Then

$$\|x^1(t)-x^2(t)\| \le \lambda^*(r)\Big(\|\varphi^1(0)-\varphi^2(0)\|+\|\varphi^1-\varphi^2\|_{\mathscr{B}}+\|u^1-u^2\|_{L^2}\Big)\ \text{ for } t\in[0,b].$$

The proof is complete.

We now study the existence of solutions to the following Lagrange problem

$$(\mathscr{LP})\begin{cases}\text{Find a control } u^0\in\mathscr{U}_{ad} \text{ such that}\\ \mathscr{J}(u^0)\le\mathscr{J}(u) \text{ for all } u\in\mathscr{U}_{ad},\end{cases}$$

where

$$\mathscr{J}(u) := \int_0^b \mathscr{L}\Big(t, x_t^u, x^u(t), u(t)\Big)\,dt,$$

and x^u denotes the mild solution of (1.1) corresponding to the control $u\in\mathscr{U}_{ad}$.

For the existence of solutions to problem $(\mathscr{LP})$, we make the following assumptions.

$(\mathbf{H_L})$

- **(i)** The functional $\mathscr{L}: I\times\mathscr{B}\times X\times U\to\mathbb{R}\cup\{\infty\}$ is Borel measurable.
- **(ii)** $\mathscr{L}(t,\cdot,\cdot,\cdot)$ is sequencially lower semicontinuous on $\mathscr{B}\times X\times U$ for almost all $t\in I$.
- **(iii)** $\mathscr{L}(t,\psi,y,\cdot)$ is convex on U for each $\psi\in\mathscr{B},\ y\in X$ and almost all $t\in I$.
- **(iv)** There exist constants $\nu,\ \beta\ge 0,\ \gamma>0$, and $\mu\in L^1(I)$ nonnegative such that

$$\mathscr{L}(t,\psi,y,u)\ge\mu(t)+\nu\|\psi\|_{\mathscr{B}}+\beta\|y\|+\gamma\|u\|_U.$$

We have the following result on the existence of optimal controls for problem $(\mathscr{LP})$.

Theorem 4.3. *Assume that hypotheses* $(\mathbf{H_1})-(\mathbf{H_5})$ *and* $(\mathbf{H_L})$ *hold. Then the Lagrange problem* $(\mathscr{LP})$ *admits at least one optimal pair, that is, there exists an admissible control pair* $(x^0,u^0)\in\mathscr{C}([0,b],X)\times\mathscr{U}_{ad}$ *such that*

$$\mathscr{J}(u^0)=\int_0^b\mathscr{L}\Big(t,x_t^0,x^0(t),u^0(t)\Big)\,dt\le\int_0^b\mathscr{L}\Big(t,x_t^u,x^u(t),u(t)\Big)\,dt=\mathscr{J}(u)\ \ \forall u\in\mathscr{U}_{ad}.$$

Proof. If $\inf\Big\{\mathscr{J}(u): u\in\mathscr{U}_{ad}\Big\}=\infty$, we are done. Without loss of generality, we may assume that $\inf\Big\{\mathscr{J}(u): u\in\mathscr{U}_{ad}\Big\}=\delta<\infty$. If $\delta=-\infty$, then, for each $n\in\mathbb{N}$, there exists $(u^n)_{n\ge1}\subset\mathscr{U}_{ad}$ such that

$$\mathscr{J}(u^n)<-n \tag{$*$}$$

Boundedness of $\mathscr{U}_{ad}$ implies that $(u^n)_{n\ge1}$ is bounded and so that there exists a subsequence $(u^{n_k})_{k\ge1}$ of $(u^n)_{n\ge1}$ that converges weakly to some u^0 in $L^2(I,U)$, since $L^2(I,U)$ is reflexive. But $\mathscr{U}_{ad}$ is closed and convex. By Mazur's Theorem, it is weakly closed. Therefore, $u^0 \in \mathscr{U}_{ad}$. By hypothesis $(\mathbf{H_L})$, $\mathscr{L}(t,x,y,\cdot)$ is weakly lower semicontinuous. It follows that

$$\mathscr{L}(t,\,\psi,\,y,\,u^0) \le \liminf_{k\to\infty} \mathscr{L}(t,\,\psi,\,y,\,u^{n_k}) < -\infty,$$

which implies from $(*)$ that $\mathscr{J}(u^0) < -\infty$. This is a contradiction since $\mathscr{J}(u^0) \in \mathbb{R}\cup\{\infty\}$. Hence $\delta \in \mathbb{R}$. Now, from the definition of δ, there exists a minimizing sequence, a feasible pair $((x^n,u^n))_{n\ge1} \subset \mathscr{S}_{ad}$ such that

$$\int_0^b \mathscr{L}\Big(t,\, x_t^n,\, x^n(t),\, u^n(t)\Big)\, dt \longrightarrow \delta \ \text{ as } n\to\infty,$$

where

$$\mathscr{S}_{ad} := \Big\{(x,u) \,:\, x \text{ is a mild solution of equation (1.1) corresponding to the control } u \in \mathscr{U}_{ad}\Big\}.$$

Boundedness of $\mathscr{U}_{ad}$ and the fact that $L^2(I,U)$ is reflexive imply that $(u^n)_{n\ge1}$ has a subsequence denoted by $(u^k)_{k\ge1}$. It converges weakly to some u^0 in $L^2(I,U)$. But $\mathscr{U}_{ad}$ is closed and convex. Using Mazur's Theorem, it is weakly closed, therefore, $u^0 \in \mathscr{U}_{ad}$. Let

$$x^k(t) = \begin{cases} R(t)\varphi(0) + \displaystyle\int_0^t R(t-s)\left[f(s,x_s^k) + C(s)u^k(s)\right] ds \ \text{ for } t \in I, \\ \varphi(t) \ \text{ for } \ -r \le t \le 0 \end{cases}$$

denote the subsequence of $(x^n)_{n\ge1}$ corresponding to the control sequence $(u^k)_{k\ge1}$ and let x^0 be the mild solution corresponding to the control $u^0 \in \mathscr{U}_{ad}$. We show that $x^k \to x^0$. For $t \in [0,b]$, we have

$$\begin{aligned} \left\|x^k(t) - x^0(t)\right\| &\le \int_0^t \left\|R(t-s)\left[f(s,x_s^k) - f(s,x_s^0)\right]\right\| ds \\ &+ \int_0^t \left\|R(t-s)\left[C(s)u^k(s) - C(s)u^0(s)\right]\right\| ds \\ &\le R_b L_f(\rho) \int_0^t \left\|x_s^k - x_s^0\right\| ds + R_b \int_0^t \left\|C(s)u^k(s) - C(s)u^0(s)\right\| ds \\ &\le R_b L_f(\rho) \int_0^t \left\|x_s^k - x_s^0\right\| ds + R_b\sqrt{b}\left(\int_0^t \left\|C(s)u^k(s) - C(s)u^0(s)\right\|^2 ds\right)^{\frac12} \\ &\le R_b L_f(\rho) K_b \int_0^t \sup_{\tau\in[0,s]} \left\|x^k(\tau) - x^0(\tau)\right\| d\tau + R_b\sqrt{b}\left\|Cu^k - Cu^0\right\|_{L^2(I,U)}. \end{aligned}$$

This implies that

$$\sup_{s\in[0,t]} \left\|x^k(s) - x^0(s)\right\| \le R_b\sqrt{b}\left\|Cu^k - Cu^0\right\|_{L^2(I,U)} + R_b L_f(\rho) K_b \int_0^t \sup_{\tau\in[0,s]} \left\|x^k(\tau) - x^0(\tau)\right\| d\tau.$$

It follows from Gronwall's inequality that

$$\left\|x^k(t) - x^0(t)\right\| \le \lambda^{**}\left\|Cu^k - Cu^0\right\|_{L^2(I,U)}, \ \text{ where } \ \lambda^{**} = R_b\sqrt{b}\, e^{R_b b L_f(\rho) K_b}. \tag{4.3}$$

We have the following lemma.

Lemma 4.4. [9] *Let $(u^n)_{n\ge1} \subset \mathscr{U}_{ad}$ and $u^0 \in \mathscr{U}_{ad}$ such that $(u^n)_{n\ge1}$ converges weakly to u^0. Then,*

$$\left\|Cu^k - Cu^0\right\|_{L^2(I,U)} \longrightarrow 0 \ \text{ as } k\to\infty, \text{ if } C \in L^\infty(I;\mathscr{L}(U,X)).$$

We find from (4.3) that

$$\left\|x^k - x^0\right\| \le \lambda^{**}\left\|Cu^k - Cu^0\right\|_{L^2(I,U)},$$

therefore, it follows by Lemma 4.4 that

$$x^k \longrightarrow x^0 \ \text{ as } k \to \infty.$$

We note that $(\mathbf{H_L})$ implies the assumptions of Balder's Theorem (see Theorem 2.5). Hence by Balder's Theorem, we can conclude that $(x_t, x, u) \mapsto \int_0^b \mathscr{L}\Big(t, x_t, x(t), u(t)\Big)\, dt$ is sequencially lower semicontinuous in the strong topology of $\mathscr{B} \times L^1(I,X) \times L^1(I,U)$.

Now, since $\mathscr{B} \times L^2(I,X) \times L^2(I,U) \subset \mathscr{B} \times L^1(I,X) \times L^1(I,U)$, $\mathscr{J}$ is also sequentially lower semicontinuous on $\mathscr{B} \times L^2(I,X) \times L^2(I,U)$, and in the strong topology of $L^1(I, \mathscr{B} \times X \times U)$. Hence, $\mathscr{J}$ is weakly lower semicontinuous on $L^2(I,U)$. From $(\mathbf{H_L}) - (\mathbf{iv})$, $\mathscr{J} > -\infty$, $\mathscr{J}$ attains its infimum at $u^0 \in \mathscr{U}_{ad}$, that is,

$$\delta = \lim_{k\to\infty} \int_0^b \mathscr{L}\Big(t, x_t^k, x^k(t), u^k(t)\Big)\, dt \ge \int_0^b \mathscr{L}\Big(t, x_t^0, x^0(t), u^0(t)\Big)\, dt = \mathscr{J}(u^0) \ge \delta.$$

Thus, $\delta = \mathscr{J}(u^0)$. Hence there exists an admissible control $u^0 \in \mathscr{U}_{ad}$ such that

$$\mathscr{J}(u^0) \le \mathscr{J}(u) \ \text{ for all } \ u \in \mathscr{U}_{ad}.$$

This completes the proof.

5. Examples

We now illustrate our main result by the following example.

Let Ω be bounded domain in $\mathbb{R}^n$ with smooth boundary and consider the following nonlinear integrodifferential equation.

$$\begin{cases} \frac{\partial v(t,\xi)}{\partial t} = \Delta v(t,\xi) + \int_0^t \zeta(t-s)\Delta v(s,\xi)\, ds + \int_{-\infty}^0 \alpha(\theta) g(t, v(t+\theta,\xi))\, d\theta + \beta(t)\omega(t,\xi) \\ \text{for } t \in I = [0,1] \text{ and } \xi \in \Omega, \\ v(t,\xi) = 0 \text{ for } t \in [0,1] \text{ and } \xi \in \partial\Omega, \\ v(\theta,\xi) = \phi(\theta,\xi) \text{ for } \theta \in]-\infty, 0] \text{ and } \xi \in \Omega, \end{cases} \tag{5.1}$$

where $\beta \in \mathscr{C}([0,1];\mathbb{R})$, $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and Lipschitzian with respect to the second variable, the initial data function $\phi : \mathbb{R}^- \times \Omega \to \mathbb{R}$ is a given function, $\omega : [0,1] \times \Omega \to \mathbb{R}$ continuous in t, $\alpha : \mathbb{R}^- \to \mathbb{R}$ is continuous, $\alpha \in L^1(\mathbb{R}^-, \mathbb{R})$ and $\zeta \in W^{1,1}(\mathbb{R}^+, \mathbb{R}^+)$.

Let $X = U = L^2(\Omega)$ and the phase space $\mathscr{B} = BUC(\mathbb{R}^-, X)$. The space of uniformly bounded continuous functions endowed with the following norm

$$\|\varphi\|_{\mathscr{B}} = \sup_{\theta \le 0} \|\varphi(\theta)\|.$$

Then, the space $BUC(\mathbb{R}^-, X)$ satisfies axioms $(\mathbf{A_1})$, $(\mathbf{A_2})$ and $(\mathbf{A_3})$. For $\eta > 0$, we define the set of admissible controls $\mathscr{U}_{ad}$ by

$$\mathscr{U}_{ad} := \Big\{ u : I \to U : u \text{ is measurable and } \|u\|_{L^2(I,U)} \le \eta \Big\},$$

where

$$\|u\|^2_{L^2(I,U)} = \int_0^1 \left(\int_\Omega u^2(s)(\xi)\, d\xi \right) ds.$$

We define $A:\mathscr{D}(A)\subset X\to X$ by

$$\begin{cases} \mathscr{D}(A)=H^2(\Omega)\cap H_0^1(\Omega), \\ Av = \Delta v \ \text{ for } v\in\mathscr{D}(A). \end{cases}$$

Theorem 5.1. *(**Theorem 4.1.2, p. 79 of** [27]) A is the infinitesimal generator of a C_0-semigroup on $L^2(\Omega)$.*

A generates a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ on $L^2(\Omega)$. Define

$$x(t)(\xi) \ = \ v(t,\xi), \quad x'(t)(\xi) \ = \ \frac{\partial v(t,\xi)}{\partial t}, \quad \omega(t,\xi)=u(t)(\xi),$$

$$\varphi(\theta)(\xi)=\phi(\theta,\xi) \ \text{ for } \theta\in]-\infty,0] \text{ and } \xi\in\Omega,$$

$$f(t,\psi)(\xi) \ = \ \int_{-\infty}^{0}\alpha(\theta)g(t,\psi(\theta)(\xi))\,d\theta \ \text{ for } \theta\in]-\infty,0] \text{ and } \xi\in\Omega.$$

Let $C(t):X\to X$ be defined by $\Big(C(t)u(t)\Big)(\xi) \ = \ C(t)u(t)(\xi)=\beta(t)\omega(t,\xi),$

$$(\gamma(t)x)(\xi) \ = \ \zeta(t)\Delta v(t,\xi) \ \text{ for } \ t\in[0,1], \ \ x\in\mathscr{D}(A) \text{ and } \ \xi\in\Omega.$$

We suppose that $\varphi\in BUC(\mathbb{R}^-,X)$. Equation (5.1) is then transformed into the following form

$$\begin{cases} x'(t)=Ax(t) \ + \ \displaystyle\int_0^t\gamma(t-s)x(s)ds \ + \ f(t,x_t)+C(t)u(t) \ \text{ for } t\in I=[0,1], \\ x_0=\varphi. \end{cases} \tag{5.2}$$

Suppose that there exist a continuous function $p\in L^1(I;\mathbb{R}^+)$ such that

$$|g(t,y)|\leq p(t)|y| \ \text{ for } t\in I \text{ and } y\in\mathbb{R}.$$

One can see that f satisfies $(\mathbf{H_3})$. Now we consider the following cost function:

$$\mathscr{J}(u):=\int_0^1\mathscr{L}\Big(t,x_t^u,x^u(t),u(t)\Big)\,dt,$$

where $\mathscr{L}\,:\,[0,1]\times\mathscr{B}\times L^2(\Omega)\times L^2(\Omega)\longrightarrow\mathbb{R}$ is defined by

$$\mathscr{L}\Big(t,\,\psi,\,x,\,u\Big) \ \ = \ \ \|\psi\|_{\mathscr{B}}+\|x\|+\|u\|.$$

$\mathscr{L}$ satisfies all the conditions of hypothesis $(\mathbf{H_L})$. Then,

$$\mathscr{J}(u)=\int_0^1\Big(\|x_t^u\|+\|x^u(t)\|+\|u(t)\|\Big)\,dt.$$

Hence, all the conditions of Theorem 4.3 are satisfied. Therefore, equation (5.2) has at least one optimal pair.

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