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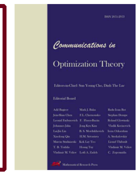

# INFINITELY MANY FAST HOMOCLINIC SOLUTIONS FOR DAMPED VIBRATION SYSTEMS WITH LOCALLY DEFINED POTENTIALS

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**Abstract.** In this paper, we study the existence of infinitely many fast homoclinic solutions for a class of damped vibration system $\ddot{u}(t)+q(t)\dot{u}(t)-L(t)u(t)+\nabla W(t,u(t))=0$, $t\in\mathbb{R}$, where $q\in C(\mathbb{R},\mathbb{R})$, $L\in C(\mathbb{R},\mathbb{R}^{N^2})$ is unnecessary coercive nor positive definite for all $t\in\mathbb{R}$ and $W(t,x)$ is only locally defined near the origin with respect to the second variable.

**Keywords.** Damped vibration systems; Fast homoclinic solution; Local condition; Variational method.

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## 1. INTRODUCTION

Consider the following second order non autonomous system:

$$(\mathscr{DV}) \qquad \ddot{u}(t)+q(t)\dot{u}(t)-L(t)u(t)+\nabla W(t,u(t))=0, \forall t\in\mathbb{R},$$

where $q:\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous function with $Q(t)=\int_0^t q(s)ds\longrightarrow+\infty$ as $|t|\longrightarrow\infty$, $L\in C(\mathbb{R},\mathbb{R}^{N^2})$ is a symmetric matrix-valued function and $W\in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$ with $\nabla W(t,x)=\frac{\partial W}{\partial x}(t,x)$. Here, as usual, we say that a solution $u$ of $(\mathscr{DV})$ is homoclinic (to 0) if $u\in C^2(\mathbb{R},\mathbb{R}^N)$, $u(t)\longrightarrow 0$ and $\dot{u}(t)\longrightarrow 0$ as $t\longrightarrow\pm\infty$. If $u(t)\neq 0$, $u$ is called a nontrivial homoclinic solution.

It is well known that homoclinic solutions play an important role in analysing the chaos of dynamical systems. If a system has the transversely intersected homoclinic orbits, then it must be chaotic. If it has the smoothly connected homoclinic orbits, then it cannot stand the perturbation, its perturbed system probably produce chaotic phenomena. Therefore, it is of practical importance and mathematical significance to consider the existence of homoclinic orbits of $(\mathscr{DV})$ emanating from zero. The aim of this paper is to look for the solutions $u$ of $(\mathscr{DV})$ which are fast homoclinic (to 0). In the following, . denotes the standard inner product in $\mathbb{R}^N$ and $|.|$ is the induced norm. If $q(t)=0$ for all $t\in\mathbb{R}$, $(\mathscr{DV})$ is just the following second order Hamiltonian system

$$(\mathscr{HS}) \qquad \ddot{u}(t)-L(t)u(t)+\nabla W(t,u(t))=0, \forall t\in\mathbb{R}.$$

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With the aid of variational methods and critical point theory, the existence and multiplicity of homoclinic solutions for $(\mathscr{HS})$ have been extensively investigated in the literature over the past several decades; see [1, 2, 3, 4, 5, 6, 7, 8] and the references therein. Many early results treated the periodic case where $L(t)$ and $W(t,x)$ are either independent of $t$ or periodic in $t$; see [9, 10, 11] and the references therein. Compared to the periodic case, the problem is quite different in nature for the nonperiodic case due to the lack of compactness of the Sobolev embedding. After the work of Rabinowitz and Tanaka [1], there are many papers concerning the nonperiodic case, see, e.g., [1, 2, 3, 5, 7, 9, 12, 13, 14, 15]. For this case, the function $L$ plays an important role. As far as the case $q(t) \neq 0$ is concerned, there are only a few results on the existence of homoclinic solutions; see [12, 13, 14, 15, 16, 17, 18]. In all these results, the function $L$ is either coercive or uniformly positively definite. In addition, we also note that, in all these papers, the potential $W(t,x)$ is required to satisfy some kinds of growth conditions at infinity with respect to $x$, such as superquadratic, asymptotically quadratic or subquadratic growth. Motivated by the above results, in the present work, we will study the existence of infinitely many fast homoclinic solutions for $(\mathscr{DV})$ in the case where the function $L$ is unnecessary coercive or positive definite and the potential $W$ is still only locally defined near the origin with respect to $x$. More precisely, let $W : \mathbb{R}\times B_\delta(0) \longrightarrow \mathbb{R}$ be a continuous function, differentiable in the second variable with continuous derivative, where $\delta$ is a positive constant and $B_\delta(0)$ is the open ball in $\mathbb{R}^N$ centered at zero with radius $\delta$. We make the following assumptions:
$(Q)$ There exists a constant $\sigma > 1$ such that

$$\|q\|_\infty < \infty,\ Q(t) = \int_0^t q(s)ds \longrightarrow +\infty \ as\ |t| \longrightarrow \infty \ and \int_{|t|\geq 1} e^{Q(t)}\,|t|^{-\sigma}\,dt < \infty;$$

$(L)$ the smallest eigenvalue of $L(t)$ is bounded from below;

$$(L_Q) \qquad meas_Q(\{t\in\mathbb{R}/\,|t|^{-\sigma}L(t) < bI_N\}) < \infty,\ \forall b > 0,$$

where $\sigma$ is introduced in $(Q)$ and $meas_Q$ denotes the Lebesgue's measure on $\mathbb{R}$ with density $e^{Q(t)}$, i.e., $meas_Q(A) = \int_A e^{Q(t)}dt$;
$(W_1)$ There exist constants $\nu \in ]0,1[$, $\alpha \in [\frac{1}{\nu},\infty[$, $b \geq 0$ and $a \in L^\alpha_Q(\mathbb{R},\mathbb{R}^+)$ such that

$$|\nabla W(t,x)| \leq a(t) + b\,|x|^\nu,\ \forall (t,x) \in \mathbb{R}\times B_\delta(0);$$

where $L^\alpha_Q(\mathbb{R},\mathbb{R}^+)$ is defined in Section 2;

$$(W_2) \qquad W(t,0) = 0 \ and\ W(t,-x) = W(t,x),\ \forall (t,x) \in \mathbb{R}\times B_\delta(0),$$

$$(W_3) \qquad \lim_{|x|\to 0} \frac{|W(t,x)|}{|x|^2} = +\infty,\ uniformly\ for\ all\ t \in \mathbb{R}.$$

Our main result reads as follows.

**Theorem 1.1.** *Assume that $(Q)$, $(L)$, $(L_Q)$ and $(W_1)-(W_3)$ are satisfied. Then $(\mathscr{DV})$ possesses infinitely many nontrivial fast homoclinic orbits $(u_k)$ such that $\max_{t\in\mathbb{R}}|u_k(t)| \longrightarrow 0$ as $k \longrightarrow \infty$.*

Let $\sigma > 1$ and $0 < \omega < \sigma - 1$ be some constants. Define $q(t) = \frac{\omega t}{t^2+1}$ for $t \in \mathbb{R}$. It is easy to verify that $q$ satisfies $(Q)$.

In Theorem 1.1, $L(t)$ is unnecessarily required to be either uniformly positive definite or coercive. For example $L(t) = (t^2 sin^2 t - 1)I_N$ satisfies $(L)$, $(L_Q)$ but it is neither positive definite nor coercive.

Let $0 < \nu < 1$ and $b \geq 0$ be some given constants and define

$$W(t,x) = (\frac{1}{t^2+1})^{\frac{\nu(\omega+2)}{4}} ln(1+|x|^2) + \frac{b}{\nu+1}|x|^{\nu+1},\ t \in \mathbb{R},\ |x| < 1,$$

where $\omega$ is defined as above. It is clear that $W$ satisfies $(W_2)$ and $(W_3)$ with $\delta = 1$. An easy computation shows that

$$|\nabla W(t,x)| \leq a(t) + b|x|^{\nu},\ \forall t \in \mathbb{R},\ |x| < 1,$$

where $a(t) = (\frac{1}{t^2+1})^{\frac{\nu(\omega+2)}{4}}$. Moreover, for $\alpha = \frac{2}{\nu} > \frac{1}{\nu}$, we have

$$\int_{\mathbb{R}} e^{Q(t)}(a(t))^{\alpha} = \int_{\mathbb{R}} \frac{1}{t^2+1} < \infty.$$

Hence assumption $(W_1)$ is satisfied. Therefore by Theorem 1.1, the corresponding system $(\mathscr{DV})$ possesses infinitely many homoclinic orbits.

## 2. Preliminaries

In order to introduce the concept of fast homoclinic solutions for $(\mathscr{DV})$ conveniently, we first describe some properties of the weighted Sobolev space $E$ on which the certain variational functional associated with $(\mathscr{DV})$ is defined and the fast homoclinic solutions of $(\mathscr{DV})$ are the critical points of such functional. We shall use $L^2_Q(\mathbb{R})$ to denote the Hilbert space of measurable functions from $\mathbb{R}$ into $\mathbb{R}^N$ under the inner product

$$< u,v >_{L^2_Q} = \int_{\mathbb{R}} e^{Q(t)} u(t).v(t)dt$$

and the induced norm

$$\|u\|_{L^2_Q} = (\int_{\mathbb{R}} e^{Q(t)}|u(t)|^2 dt)^{\frac{1}{2}}.$$

Similarly, $L^p_Q(\mathbb{R})$ $(1 \leq p < \infty)$ denotes the Banach space of functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the norm

$$\|u\|_{L^p_Q} = (\int_{\mathbb{R}} e^{Q(t)}|u(t)|^p dt)^{\frac{1}{p}}$$

and $L^\infty_Q(\mathbb{R})$ denotes the Banach space of functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the norm

$$\|u\|_{L^\infty_Q} = esssup\left\{e^{\frac{Q(t)}{2}}|u(t)| / t \in \mathbb{R}\right\}.$$

In this section, we assume that $L$ satisfies the following condition instead of condition $(L)$:

$$(L_0) \qquad l(t) = \inf_{|\xi|=1} L(t)\xi.\xi \geq 1,\ \forall t \in \mathbb{R}$$

and we introduce the Hilbert space

$$E = \left\{u \in H^1_Q(\mathbb{R},\mathbb{R}^N) / \int_{\mathbb{R}} e^{Q(t)} L(t)u(t).u(t)dt < \infty\right\}$$

equipped with the following inner product

$$< u,v > = \int_{\mathbb{R}} e^{Q(t)}[\dot{u}(t).\dot{v}(t) + L(t)u(t).v(t)]dt$$

and the induced norm $\|u\| = < u,u >^{\frac{1}{2}}$. Here, $H^1_Q(\mathbb{R},\mathbb{R}^N)$ denotes the Sobolev space

$$H^1_Q(\mathbb{R},\mathbb{R}^N) = \left\{u \in L^2_Q(\mathbb{R},\mathbb{R}^N) / \dot{u} \in L^2_Q(\mathbb{R},\mathbb{R}^N)\right\}.$$

**Definition 2.1.** A solution $u$ of $(\mathscr{D}\mathscr{V})$ is called a fast homoclinic orbit if $u \in E$.

**Lemma 2.2.** *Assume* $(Q)$, $(L_0)$ *and* $(L_Q)$ *are satisfied. Then* $E$ *is compactly embedded in* $L^p_Q(\mathbb{R})$ *for all* $1 \leq p \leq \infty$. *In particular, for all* $1 \leq p \leq \infty$, *there exists a constant* $\eta_p > 0$ *such that*

$$\|u\|_{L^p_Q} \leq \eta_p \|u\|, \ \forall u \in E. \tag{2.1}$$

*Proof.* For any $\varepsilon > 0$, by condition $(L_Q)$ we can choose $r_\varepsilon \geq 1$ such that $meas_Q(B_\varepsilon) \leq \varepsilon$, where

$$B_\varepsilon = \left\{ t \in \mathbb{R}\backslash ]-r_\varepsilon, r_\varepsilon[ / |t|^{-\sigma} L(t) < \frac{1}{\varepsilon} I_N \right\}.$$

Let

$$D_\varepsilon = \mathbb{R} \setminus (B_\varepsilon \cup ]-r_\varepsilon, r_\varepsilon[)$$

and

$$l_\varepsilon = \inf_{|\xi|=1, t \in D_\varepsilon} |t|^{-\sigma} L(t)\xi.\xi.$$

Then $\frac{1}{l_\varepsilon} \leq \varepsilon$. Let $(u_k) \subset E$ be a sequence such that $u_k \rightharpoonup u$ weakly in $E$. The Banach-Steinhaus Theorem implies that

$$M = \sup_{k \in \mathbb{N}} \|u_k - u\|_{X^\alpha} < \infty.$$

Since $E \subset H^1_Q(\mathbb{R}, \mathbb{R}^N) \subset L^p_Q(\mathbb{R}, \mathbb{R}^N)$ for all $p \in [2, \infty]$ with continuous embedding, it holds

$$M_p = \sup_{k \in \mathbb{N}} \|u_k - u\|_{L^p_Q} < \infty.$$

Sobolev's Theorem implies that $u_k \longrightarrow u$ uniformly in $\bar{I}_\varepsilon = [-r_\varepsilon, r_\varepsilon]$.

Step 1. We claim that $E$ is compactly embedded in $L^\infty_Q(\mathbb{R})$. In fact, let us remark that for any $n \in \mathbb{N}$ and $t \in \mathbb{R}$, $|t| \geq r_\varepsilon$, one has (with $v(t) = e^{\frac{Q(t)}{2}} u(t)$)

$$v(t) = \int_t^{t+1} [-\dot{v}(s)(t+1-s)^{n+1} + v(s)(n+1)(t+1-s)^n] ds.$$

So by Hölder's inequality, one has

$$\begin{aligned}
|v(t)| &\leq \frac{1}{\sqrt{2n+3}} (\int_t^{t+1} |\dot{v}(s)|^2 ds)^{\frac{1}{2}} + \frac{n+1}{\sqrt{2n+1}} (\int_t^{t+1} |v(s)|^2 ds)^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{2n+3}} (\int_t^{t+1} e^{Q(s)} \left| \frac{q(s)}{2} u(s) + \dot{u}(s) \right|^2 ds)^{\frac{1}{2}} + \frac{n+1}{\sqrt{2n+1}} (\int_t^{t+1} e^{Q(s)} |u(s)|^2 ds)^{\frac{1}{2}} \\
&\leq \frac{1}{\sqrt{2n+3}} (\int_t^{t+1} e^{Q(s)} |\dot{u}(s)|^2 ds)^{\frac{1}{2}} + \frac{\|q\|_\infty}{2\sqrt{2n+3}} (\int_t^{t+1} e^{Q(s)} |u(s)|^2 ds)^{\frac{1}{2}} \\
&\quad + \frac{n+1}{\sqrt{2n+1}} (\int_t^{t+1} e^{Q(s)} |u(s)|^2 ds)^{\frac{1}{2}}
\end{aligned}$$

for any $n\in\mathbb{N}$ and $t\in\mathbb{R}$. Let $c_0=\inf_{t\in\mathbb{R}} e^{Q(t)}$. Then for any $k\in\mathbb{N}$ and $|t|\geq r_\varepsilon$, one has

$$\begin{aligned}
& e^{\frac{Q(t)}{2}}|u_k(t)-u(t)| \\
\leq & \frac{1}{\sqrt{2n+3}}(\int_{|s|\geq r_\varepsilon} e^{Q(s)}|\dot{u}_k(s)-\dot{u}(s)|^2 ds)^{\frac{1}{2}} \\
& +\frac{\|q\|_\infty}{2\sqrt{2n+3}}(\int_{|s|\geq r_\varepsilon} e^{Q(s)}|u_k(s)-u(s)|^2 ds)^{\frac{1}{2}}+\frac{n+1}{\sqrt{2n+1}}(\int_{|s|\geq r_\varepsilon} e^{Q(s)}|u_k(s)-u(s)|^2 ds)^{\frac{1}{2}} \\
\leq & \frac{1+\frac{1}{2}\|q\|_\infty}{\sqrt{2n+3}}\|u_k-u\|+\frac{n+1}{\sqrt{2n+1}}(\int_{|s|\geq r_\varepsilon} e^{Q(s)}|u_k(s)-u(s)|^2 ds)^{\frac{1}{2}} \\
\leq & \frac{M(1+\frac{1}{2}\|q\|_\infty)}{\sqrt{2n+3}}+\frac{n+1}{\sqrt{2n+1}}[(\int_{B_\varepsilon} e^{Q(s)}|u_k(s)-u(s)|^2 ds)^{\frac{1}{2}}+(\int_{D_\varepsilon} e^{Q(s)}|u_k(s)-u(s)|^2 ds)^{\frac{1}{2}}] \\
\leq & \frac{M(1+\frac{1}{2}\|q\|_\infty)}{\sqrt{2n+3}}+\frac{n+1}{\sqrt{2n+1}}(meas(B_\varepsilon))^{\frac{1}{2}}\|u_k-u\|_{L^\infty_Q(\mathbb{R})} \\
& +\frac{n+1}{\sqrt{2n+1}}(\frac{1}{l_\varepsilon}\int_{D_\varepsilon} e^{Q(s)}L(s)(u_k(s)-u(s))(u_k(s)-u(s))ds)^{\frac{1}{2}} \\
\leq & \frac{M(1+\frac{1}{2}\|q\|_\infty)}{\sqrt{2n+3}}+\frac{n+1}{\sqrt{2n+1}}(\frac{1}{c_0}meas_Q(B_\varepsilon))^{\frac{1}{2}}M_\infty \\
& +\frac{n+1}{\sqrt{2n+1}}\varepsilon^{\frac{1}{2}}(\int_{D_\varepsilon} e^{Q(s)}L(s)(u_k(s)-u(s))(u_k(s)-u(s))ds)^{\frac{1}{2}} \\
\leq & \frac{M(1+\frac{1}{2}\|q\|_\infty)}{\sqrt{2n+3}}+\frac{n+1}{\sqrt{2n+1}}\frac{1}{\sqrt{c_0}}\sqrt{\varepsilon}M_\infty+\frac{n+1}{\sqrt{2n+1}}\sqrt{\varepsilon}M \\
\leq & \frac{M(1+\frac{1}{2}\|q\|_\infty)}{\sqrt{2}\sqrt{n+1}}+\sqrt{2}(\frac{M_\infty}{\sqrt{c_0}}+M)\sqrt{n}\sqrt{\varepsilon}.
\end{aligned}$$

Choosing $n=[\frac{1}{\sqrt{\varepsilon}}]$ (the integer part of $\frac{1}{\sqrt{\varepsilon}}$), we have $n\leq\frac{1}{\sqrt{\varepsilon}}<n+1$. Hence

$$e^{\frac{Q(t)}{2}}|u_k(t)-u(t)|\leq[\frac{M(3+\frac{1}{2}\|q\|_\infty)}{\sqrt{2}}+M_\infty\sqrt{\frac{2}{c_0}}]\varepsilon^{\frac{1}{4}},\ \forall|t|\geq r_\varepsilon. \tag{2.2}$$

On the other hand, since $u_k\longrightarrow u$ uniformly on $\bar{I}_\varepsilon$, there exits $k_0\in\mathbb{N}$ such that for all $k\geq k_0$

$$\left\|e^{\frac{Q(t)}{2}}(u_k(t)-u(t))\right\|_{L^\infty(I_\varepsilon)}=\|u_k-u\|_{L^\infty_Q(I_\varepsilon)}<\varepsilon. \tag{2.3}$$

Combining (2.2) and (2.3), we get $u_k\longrightarrow u$ in $L^\infty_Q(\mathbb{R})$.

Step 2. We claim that $E$ is compactly embedded in $L^2_Q(\mathbb{R})$. In fact, we have

$$\begin{aligned}
\int_{|t|\geq r_\varepsilon} e^{Q(t)}|u_k-u|^2 dt &= \int_{B_\varepsilon} e^{Q(t)}|u_k-u|^2 dt+\int_{D_\varepsilon} e^{Q(t)}|u_k-u|^2 dt \\
&\leq meas(B_\varepsilon)\|u_k-u\|^2_{L^\infty_Q}+\int_{D_\varepsilon} e^{Q(t)}|t|^\sigma|u_k-u|^2 dt \\
&\leq \frac{1}{c_0}meas_Q(B_\varepsilon)M^2_\infty+\frac{1}{l_\varepsilon}\int_{D_\varepsilon} L(t)(u_k-u).(u_k-u)dt \\
&\leq \frac{1}{c_0}\varepsilon M^2_\infty+\varepsilon\|u_k-u\|^2\leq\varepsilon(\frac{M^2_\infty}{c_0}+M^2).
\end{aligned}$$

Since $u_k\longrightarrow u$ uniformly on $\bar{I}_\varepsilon$, we get $u_k\longrightarrow u$ in $L^2_Q(\mathbb{R},\mathbb{R}^N)$ as $k\longrightarrow\infty$.

Step 3. $p \in ]2,\infty[$. We claim that $E$ is compactly embedded in $L^p_Q(\mathbb{R})$. In fact, we have

$$\begin{aligned}\|u_k-u\|^p_{L^p_Q} &= \int_{\mathbb{R}} e^{Q(t)}\,|u_k-u|^p\,dt\\ &\le \|u_k-u\|^{p-2}_{L^\infty}\int_{\mathbb{R}} e^{Q(t)}\,|u_k-u|^2\,dt\\ &\le c_0^{-\frac{p-2}{2}}\,\|u_k-u\|^{2-p}_{L^\infty_Q(\mathbb{R})}\,\|u_k-u\|^2_{L^2_Q(\mathbb{R})}.\end{aligned}$$

By Step 1 and Step 2, we deduce that $u_k \longrightarrow u$ in $L^p_Q(\mathbb{R},\mathbb{R}^N)$.

Step 4. $p\in[1,2[$. We claim that $u_k \longrightarrow u$ in $L^p_Q(\mathbb{R})$. Let $s=\frac{\sigma}{2-p}$. Then $p>\frac{2}{1+\sigma}$ and $sp>1$. For $v\in L^p_Q(\mathbb{R})$, we have

$$\begin{aligned}&\int_{|t|\ge r_\varepsilon} e^{Q(t)}\,|v|^p\,dt\\ &=\int_{B_\varepsilon} e^{Q(t)}\,|v|^p\,dt+\int_{D_\varepsilon} e^{Q(t)}\,|v|^p\,dt\\ &=\int_{B_\varepsilon} e^{Q(t)}\,|v|^p\,dt+\int_{\{t\in D_\varepsilon/|t|^s|v(t)|\le 1\}} e^{Q(t)}\,|v|^p\,dt+\int_{\{t\in D_\varepsilon/|t|^s|v(t)|\ge 1\}} e^{Q(t)}\,|v|^p\,dt\\ &\le \Big(\int_{B_\varepsilon} e^{Q(t)}dt\Big)^{\frac12}\Big(\int_{B_\varepsilon} e^{Q(t)}\,|v|^{2p}\,dt\Big)^{\frac12}+\int_{D_\varepsilon} e^{Q(t)}\,|t|^{-sp}\,dt+\int_{\{t\in D_\varepsilon/|t|^s|v(t)|\ge 1\}} e^{Q(t)}(|t|^s\,|v|)^p\,|t|^{-sp}\,dt\\ &\le (meas_Q(B_\varepsilon))^{\frac12}\,\|v\|^p_{L^{2p}_Q}+\int_{|t|\ge r_\varepsilon} e^{Q(t)}\,|t|^{-\sigma}\,dt+\int_{D_\varepsilon} e^{Q(t)}(|t|^s\,|v|)^2\,|t|^{-sp}\,dt\\ &\le \sqrt{\varepsilon}\,\|v\|^p_{L^{2p}_Q}+\int_{|t|\ge r_\varepsilon} e^{Q(t)}\,|t|^{-\sigma}\,dt+\int_{\{t\in D_\varepsilon/|t|^s|v(t)|\ge 1\}} e^{Q(t)}\,|t|^{(2-p))s}\,|v|^2\,dt\\ &\le \sqrt{\varepsilon}\,\|v\|^p_{L^{2p}_Q}+\int_{|t|\ge r_\varepsilon} e^{Q(t)}\,|t|^{-\sigma}\,dt+\int_{\{t\in D_\varepsilon/|t|^s|v(t)|\ge 1\}} e^{Q(t)}\,|t|^{\sigma}\,|v|^2\,dt\\ &\le \sqrt{\varepsilon}\,\|v\|^p_{L^{2p}_Q}+\int_{|t|\ge r_\varepsilon} e^{Q(t)}\,|t|^{-\sigma}\,dt+\frac{1}{l_\varepsilon}\int_{|t|\ge r_\varepsilon} e^{Q(t)}L(t)v.vdt\\ &\le \sqrt{\varepsilon}\,\|v\|^p_{L^{2p}_Q}+\int_{|t|\ge r_\varepsilon} e^{Q(t)}\,|t|^{-\sigma}\,dt+\frac{1}{l_\varepsilon}\,\|v\|^2.\end{aligned}$$

Choose $r_\varepsilon$ large enough such that $\int_{|t|\ge r_\varepsilon} e^{Q(t)}\,|t|^{-\sigma}\,dt\le\sqrt{\varepsilon}$, we obtain

$$\int_{|t|\ge r_\varepsilon} e^{Q(t)}\,|v|^p\,dt\le\sqrt{\varepsilon}[\|v\|^p_{L^{2p}_Q}+1+\|v\|^2].$$

Hence, we have

$$\int_{|t|\ge r_\varepsilon} e^{Q(t)}\,|v|^p\,dt\le\sqrt{\varepsilon}[\|v\|^p_{L^{2p}_Q}+1+\|v\|^2].$$

Since $2p\ge 2$, we deduce that $\|u_k-u\|_{L^{2p}}\le M_{2p}$ for all $k\in\mathbb{N}$ and

$$\int_{|t|\ge r_\varepsilon} e^{Q(t)}\,|u_k-u|^p\,dt\le\sqrt{\varepsilon}(M^p_{2p}+1+M^{2p}),\ \forall k\in\mathbb{N}.$$

As above, we have $\int_{\bar{I}_\varepsilon} e^{Q}\,|u_k-u|^p\,dt\longrightarrow 0$ as $k\longrightarrow\infty$. Hence $u_k\longrightarrow u$ in $L^p_Q(\mathbb{R})$. The proof of Lemma 2.2 is finished. □

Now, we use the following variant symmetric mountain pass lemma due to Kajikiya [19] to prove our result. We will recall the notion of genus. Let $E$ be a Banach space and let $A$ be a subset of $E$. $A$ is said to be symmetric if $u\in A$ implies $-u\in A$. For a closed symmetric set $A$ which does not contain the origin, we define the genus $\gamma(A)$ of $A$ by the smallest integer $k$ for which there exists an odd continuous mapping from $\mathbb{R}$ to $\mathbb{R}^k \setminus \{0\}$. If such a $k$ does not exist, we define $\gamma(A) = +\infty$. Moreover, we set $\gamma(\phi) = 0$. Let

$$\Gamma_k = \{A \subset E/A \text{ is a closed symmetric subset}, \ 0 \notin A, \ \gamma(A) \geq k\}.$$

The properties of genus used in the proof of our main result are summarized as follows.

**Lemma 2.3.** [19] *Let $A$ and $B$ be closed symmetric subsets of $E$ that do not contain the origin. Then the following hold.*
*a) If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.*
*b) The $n-$dimensional sphere $S^n$ has a genus of $n+1$ by the Borsuk-Ulam theorem.*

**Lemma 2.4.** [19] *Let $E$ be an infinite-dimensional Banach space and $f \in C^1(E,\mathbb{R})$ satisfies the following*
*$(f_1)$ $f(0) = 0$, $f$ is even and bounded from below and $f$ satisfies the $(PS)-$condition;*
*$(f_2)$ For each $k \in \mathbb{N}$, there exists $A_k \subset \Gamma_k$ such that*

$$\sup_{u \in A_k} f(u) < 0.$$

*Then $f$ possesses a sequence of critical points $(u_k)$ such that*

$$f(u_k) \leq 0, \ u_k \neq 0, \ \forall k \in \mathbb{N} \text{ and } \lim_{k \longrightarrow \infty} u_k = 0.$$

## 3. Proof of Theorem 1.1

From $(L)$, we know that there exists a positive constant $b_0$ such that $L(t) + 2b_0 I_N \geq I_N$ for all $t \in \mathbb{R}$. Let $\overline{L}(t) = L(t) + 2b_0 I_N$ and $\overline{W}(t,x) = W(t,x) + b_0 |x|^2$. Consider the following damped vibration system

$$(\overline{\mathscr{DV}}). \qquad \ddot{u}(t) + q(t)\dot{u}(t) - \overline{L}(t)u(t) + \nabla\overline{W}(t,u(t)) = 0, \ \forall t \in \mathbb{R},$$

Then $(\overline{\mathscr{DV}})$ is equivalent to $(\mathscr{DV})$. Moreover, it is easy to check that the hypothesis $(W_1) - (W_3)$ still hold for $\overline{W}$, with $b$ in $(W_1)$ is replaced by $\bar{b} = b + 2b_0\delta^{1-\nu}$, provided that those hold for $W$, and $\overline{L}$ satisfies the conditions $(L_0)$ and $(L_Q)$. Hence, in what follows, we will always assume without loss of generality that $L$ satisfies $(L_0)$ and $(L_Q)$.

In order to prove our main result via critical point theory, we need to modify $W(t,x)$ for $x$ outside a neighborhood of the origin to get $\widetilde{W}(t,x)$ as follows. Choose a constant $r \in ]0, \frac{\delta}{2}[$ and define a cut-off function $\chi \in C^1(\mathbb{R},\mathbb{R})$ such that $\chi(s) = 1$ for $0 \leq s \leq r$, $\chi(s) = 0$ for $s \geq 2r$ and $-\frac{2}{r} \leq \chi'(s) < 0$ for $r < s < 2r$. Let

$$\widetilde{W}(t,x) = \chi(|x|)W(t,x), \ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^{\mathbb{N}}. \tag{3.1}$$

Combining $(W_1)$, $(W_2)$ and the definition of $\chi$ yields

$$\left|\widetilde{W}(t,x)\right| \leq a(t)\,|x| + b\,|x|^{\nu+1}, \ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N \tag{3.2}$$

and

$$\left|\nabla\widetilde{W}(t,x)\right| \leq 5(a(t) + b\,|x|^{\nu}), \ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N. \tag{3.3}$$

Now, we introduce the following modified damped vibration system

$$(\widetilde{\mathscr{DV}}) \qquad \ddot{u}(t) + q(t)\dot{u}(t) - L(t)u(t) + \nabla\widetilde{W}(t,u(t)) = 0, \ \forall t \in \mathbb{R},$$

and define the variational functional $f$ associated with $(\widetilde{\mathscr{DV}})$ by

$$f(u)=\frac{1}{2}\int_{\mathbb{R}}e^{Q(t)}[|\dot{u}(t)|^2+L(t)u(t).u(t)]dt-\int_{\mathbb{R}}e^{Q(t)}\widetilde{W}(t,u(t))dt,\ u\in E$$
$$=\frac{1}{2}\|u\|^2-g(u), \tag{3.4}$$

where

$$g(u)=\int_{\mathbb{R}}e^{Q(t)}\widetilde{W}(t,u)dt,\ u\in E.$$

To prove our result, we will apply Lemma 2.4 to the functional $f$ on $E$.

**Lemma 3.1.** *Assume $(Q)$, $(L_0)$, $(L_Q)$ and $(W_1)$ are satisfied. If $u_n\rightharpoonup u$ in $E$, then $\nabla\widetilde{W}(t,u_n)\longrightarrow\nabla\widetilde{W}(t,u)$ in $L_Q^{\alpha}(\mathbb{R})$.*

*Proof.* Arguing indirectly, by Lemma 2.2, we may assume that there exists a subsequence $(u_{n_k})$ such that

$$u_{n_k}\longrightarrow u \text{ in } L_Q^{\nu\alpha}(\mathbb{R}) \text{ and } u_{n_k}\longrightarrow u \text{ a.e. in } \mathbb{R} \text{ as } k\longrightarrow\infty \tag{3.5}$$

and

$$\int_{\mathbb{R}}e^{Q(t)}\left|\nabla\widetilde{W}(t,u_{n_k})-\nabla\widetilde{W}(t,u)\right|^{\alpha}dt\geq\varepsilon_0,\ \forall k\in\mathbb{N} \tag{3.6}$$

for some positive constant $\varepsilon_0$. By (3.5) and up to a subsequence if necessary, we can assume that $\sum_{k=1}^{\infty}\|u_{n_k}-u\|_{L_Q^{\nu\alpha}}<\infty$. Let $w(t)=\sum_{k=1}^{\infty}|u_{n_k}(t)-u(t)|$ for all $t\in\mathbb{R}$, then $w\in L_Q^{\nu\alpha}(\mathbb{R})$. By $(W_1)$, there holds for all $k\in\mathbb{N}$ and $t\in\mathbb{R}$

$$\begin{aligned}
&\left|\nabla\widetilde{W}(t,u_{n_k}(t))-\nabla\widetilde{W}(t,u(t))\right|^{\alpha}\\
&\leq(\left|\nabla\widetilde{W}(t,u_{n_k}(t))\right|+\left|\nabla\widetilde{W}(t,u(t))\right|)^{\alpha}\\
&\leq 2^{\alpha-1}(|\nabla W(t,u_{n_k}(t))|^{\alpha}+|\nabla W(t,u(t))|^{\alpha})\\
&\leq 2^{\alpha-1}[(a(t)+b|u_{n_k}(t)|^{\nu})^{\alpha}+(a(t)+b|u(t)|^{\nu})^{\alpha}]\\
&\leq 2^{2(\alpha-1)}[2(a(t))^{\alpha}+b^{\alpha}|u_{n_k}(t)|^{\nu\alpha}+b^{\alpha}|u(t)|^{\nu\alpha}]\\
&\leq 2^{2(\alpha-1)}[2(a(t))^{\alpha}+b^{\alpha}(|u_{n_k}(t)-u(t)|+|u(t)|)^{\nu\alpha}+b^{\alpha}|u(t)|^{\nu\alpha}]\\
&\leq 2^{2(\alpha-1)}[2(a(t))^{\alpha}+b^{\alpha}2^{\nu\alpha-1}(|u_{n_k}(t)-u(t)|^{\nu\alpha}+|u(t)|^{\nu\alpha})+b^{\alpha}|u(t)|^{\nu\alpha}]\\
&\leq c_1[(a(t))^{\alpha}+|w(t)|^{\nu\alpha}+|u(t)|^{\nu\alpha}],
\end{aligned}$$

where $c_1$ is a positive constant. Combining this with (3.5), the Lebesgue's Dominated Convergence Theorem implies

$$\lim_{k\longrightarrow\infty}\int_{\mathbb{R}}e^{Q(t)}\left|\nabla\widetilde{W}(t,u_{n_k}(t))-\nabla\widetilde{W}(t,u(t))\right|^{\alpha}dt=0,$$

which contradicts to (3.6). Hence $\nabla\widetilde{W}(t,u_{n_k})\longrightarrow\nabla\widetilde{W}(t,u)$ in $L_Q^{\alpha}(\mathbb{R})$. The proof of Lemma 3.1 is completed. □

**Lemma 3.2.** *Under assumptions $(Q)$, $(L_0)$, $(L_Q)$ and $(W_1)$, the functional $g$ is continuously differentiable on $E$ and*

$$g'(u)v=\int_{\mathbb{R}}e^{Q(t)}\nabla\widetilde{W}(t,u(t)).v(t)dt,\ \forall u,v\in E. \tag{3.7}$$

*Proof.* For any $u \in E$, define an associated linear operator $K(u) : E \longrightarrow \mathbb{R}$ by

$$< K(u), v >= \int_{\mathbb{R}} e^{Q(t)} \nabla \widetilde{W}(t,u).vdt,\ v \in E. \tag{3.8}$$

By the Hölder's inequality, one has

$$\begin{aligned}
|< K(u), v >| &\leq \int_{\mathbb{R}} e^{Q(t)} \left|\nabla \widetilde{W}(t,u)\right| |v|\, dt \\
&\leq 5 \int_{\mathbb{R}} e^{Q(t)} [a(t) + b\,|u|^{\nu}]\, |v|\, dt \\
&\leq 5 \int_{\mathbb{R}} e^{Q(t)} a(t)\, |v|\, dt + 5b \int_{\mathbb{R}} e^{Q(t)} |u|^{\nu} |v|\, dt \\
&\leq 5 (\int_{\mathbb{R}} e^{Q(t)} (a(t))^{\alpha} dt)^{\frac{1}{\alpha}} (\int_{\mathbb{R}} e^{Q(t)} |v(t)|^{\frac{\alpha}{\alpha-1}} dt)^{\frac{\alpha-1}{\alpha}} \\
&\quad + 5b (\int_{\mathbb{R}} e^{Q(t)} |u(t)|^{\nu+1} dt)^{\frac{\nu}{\nu+1}} (\int_{\mathbb{R}} e^{Q(t)} |v(t)|^{\nu+1} dt)^{\frac{1}{\nu+1}} \\
&\leq 5 \|a\|_{L^{\alpha}_Q} \|v\|_{L_Q^{\frac{\alpha}{\alpha-1}}} + 5b \|u\|^{\nu}_{L^{\nu+1}_Q} \|v\|_{L^{\nu+1}_Q} \\
&\leq 5 [\eta_{\frac{\alpha}{\alpha-1}} \|a\|_{L^{\alpha}_Q} + b \eta^{\nu+1}_{\nu+1} \|u\|^{\nu}]\, \|v\|,\ \forall v \in E.
\end{aligned}$$

Hence $K(u)$ is bounded. By (3.3), for any $s \in [0,1]$, $t \in \mathbb{R}$ and $u, v \in E$, there holds

$$\begin{aligned}
e^{Q(t)} \left|\nabla \widetilde{W}(t, u+sv) v\right| &\leq 5 e^{Q(t)} [a(t)\,|v| + b\,|u+sv|^{\nu} |v|] \\
&\leq 5 e^{Q(t)} [a(t)\,|v| + b(|u^{\nu}|\,|v| + |v|^{\nu+1})],
\end{aligned}$$

which is integrable in $\mathbb{R}$. Consequently, for all $u, v \in E$, by the Mean Value Theorem and Lebesgue's Dominated Convergence Theorem, there holds

$$\begin{aligned}
\lim_{s \longrightarrow 0} \frac{g(u+sv) - g(u)}{s} &= \lim_{s \longrightarrow 0} \int_{\mathbb{R}} \int_0^1 e^{Q(t)} \nabla \widetilde{W}(t, u+\theta sv).vd\theta dt \\
&= \int_{\mathbb{R}} e^{Q(t)} \nabla \widetilde{W}(t,u).vdt =< K(u), v > .
\end{aligned} \tag{3.9}$$

This implies that $g$ is Gâteaux differentiable on $E$ and the Gâteaux derivative of $g$ at $u \in E$ is $K(u)$.
Next, we prove that $K$ is weakly continuous. Suppose $u_n \rightharpoonup u$ in $E$. By Lemma 3.1, $\nabla \widetilde{W}(t,u_n) \longrightarrow \nabla \widetilde{W}(t,u)$ in $L^{\alpha}_Q(\mathbb{R})$. By Hölder's inequality and (2.1), it holds

$$\begin{aligned}
\|K(u_n) - K(u)\|_{E'} &= \sup_{\|v\|=1} \int_{\mathbb{R}} e^{Q(t)} (\nabla \widetilde{W}(t,u_n) - \nabla \widetilde{W}(t,u)).vdt \\
&\leq \sup_{\|v\|=1} (\int_{\mathbb{R}} e^{Q(t)} \left|\nabla \widetilde{W}(t,u_n) - \nabla \widetilde{W}(t,u)\right|^{\alpha} dt)^{\frac{1}{\alpha}} (\int_{\mathbb{R}} e^{Q(t)} |v|^{\frac{\alpha}{\alpha-1}} dt)^{\frac{\alpha-1}{\alpha}} \\
&\leq \eta_{\frac{\alpha}{\alpha-1}} (\int_{\mathbb{R}} e^{Q(t)} \left|\nabla \widetilde{W}(t,u_n) - \nabla \widetilde{W}(t,u)\right|^{\alpha} dt)^{\frac{1}{\alpha}} \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned}$$

This means that $K$ is weakly continuous and then continuous. Thus $g \in C^1(E, \mathbb{R})$ and (3.7) holds with $g' = K$. In addition, due to the form of $f$, it is clear to see that $f \in C^1(E, \mathbb{R})$ and

$$f'(u)v = \int_{\mathbb{R}} e^{Q(t)} [\dot{u}(t).\dot{v}(t) + L(t)u(t).v(t)]dt - \int_{\mathbb{R}} e^{Q(t)} \nabla \widetilde{W}(t,u)vdt,\ \forall u, v \in E.$$

Finally, a standard argument shows that nontrivial critical points of $f$ on $E$ are fast homoclinic solutions of $(\widetilde{\mathscr{DV}})$. The proof of Lemma 3.2 is completed. □

**Lemma 3.3.** *Assume $(Q)$, $(L_0)$, $(L_Q)$, $(W_1)$ and $(W_2)$ are satisfied. Then $f$ is bounded from below and satisfies the $(PS)-$condition.*

*Proof.* First, we prove that $f$ is bounded from below. By (2.1), (3.2) and the Hölder's inequality, it holds for all $u \in E$

$$\begin{aligned}\int_{\mathbb{R}} e^{Q(t)} \left|\widetilde{W}(t,u)\right| dt &\leq \int_{\mathbb{R}} e^{Q(t)} (a(t)\,|u| + |u|^{\nu+1}) dt \\ &\leq \|a\|_{L_Q^\alpha} \|u\|_{L_Q^{\frac{\alpha}{\alpha-1}}} + b\,\|u\|_{L_Q^{\nu+1}}^{\nu+1} \\ &\leq \eta_{\frac{\alpha}{\alpha-1}} \|a\|_{L_Q^\alpha} \|u\| + b\eta_{\nu+1}^{\nu+1} \|u\|^{\nu+1}.\end{aligned}$$

Thus

$$\begin{aligned}f(u) &\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} e^{Q(t)} \left|\widetilde{W}(t,u)\right| dt \\ &\geq \frac{1}{2}\|u\|^2 - \eta_{\frac{\alpha}{\alpha-1}} \|a\|_{L_Q^\alpha} \|u\| - b\eta_{\nu+1}^{\nu+1} \|u\|^{\nu+1},\ \forall u \in E.\end{aligned} \tag{3.10}$$

Since $\nu < 1$, it follows that $f$ is bounded from below.

Next, we show that $f$ satisfies the $(PS)-$condition. Let $(u_n) \subset E$ be a $(PS)-$sequence, that is

$$|f(u_n)| \leq M,\ \forall n \in \mathbb{N},\ f'(u_n) \longrightarrow 0 \text{ as } n \longrightarrow \infty \tag{3.11}$$

for some constant $M > 0$. By (3.10) and (3.11), we get

$$M \geq \frac{1}{2}\|u_n\|^2 - \eta_{\frac{\alpha}{\alpha-1}} \|a\|_{L_Q^\alpha} \|u_n\| - b\eta_{\nu+1}^{\nu+1} \|u_n\|^{\nu+1},$$

which implies that $(u_n)$ is bounded in $E$ since $\nu < 1$. Hence, up to a subsequence if necessary, we can assume that

$$u_n \rightharpoonup u \text{ in } E \text{ and } u_n \longrightarrow u \text{ in } L_Q^{\frac{\alpha}{\alpha-1}} \text{ as } n \longrightarrow \infty, \tag{3.12}$$

for some $u \in E$. Next, we have

$$\|u_n - u\|^2 = (f'(u_n) - f'(u))(u_n - u) + \int_{\mathbb{R}} e^{Q(t)} (\nabla\widetilde{W}(t,u_n) - \nabla\widetilde{W}(t,u)).(u_n - u) dt. \tag{3.13}$$

It is clear that

$$(f'(u_n) - f'(u))(u_n - u) \longrightarrow 0 \text{ as as } n \longrightarrow \infty. \tag{3.14}$$

By Hölder's inequality, (2.1) and Lemma 3.1, one has

$$\begin{aligned}&\left|\int_{\mathbb{R}} e^{Q(t)} (\nabla\widetilde{W}(t,u_n) - \nabla\widetilde{W}(t,u)).(u_n - u) dt\right| \\ &\leq \left|\nabla\widetilde{W}(t,u_{n_k}) - \nabla\widetilde{W}(t,u)\right|\Big\|_{L_Q^\alpha} \|u_n - u\|_{L_Q^{\frac{\alpha}{\alpha-1}}} \longrightarrow 0 \text{ as } n \longrightarrow \infty.\end{aligned} \tag{3.15}$$

Combining (3.13)-(3.15), we deduce that $u_n \longrightarrow u$ in $E$. The proof of Lemma 3.3 is completed. □

**Lemma 3.4.** *Suppose $(Q)$, $(L_0)$, $(L_Q)$ and $(W_3)$ are satisfied. Then for each $k \in \mathbb{N}$, there exists an $A_k \subset E$ with genus $\gamma(A_k) = k$ such that $\sup_{u \in A_k} f(u) < 0$.*

*Proof.* Let $(e_n)$ be an orthonormal basis of $E$. For any $k \in \mathbb{N}$, let

$$E_k = \oplus_{m=1}^k X_m,\ X_m = span\{e_m\}.$$

Since $E_k$ is with finite-dimensional, there exists a constant $\beta_k > 0$ such that

$$\|u\| \leq \beta_k \|u\|_{L_Q^2},\ \forall u \in E_k. \tag{3.16}$$

By $(W_3)$, there exists a constant $R>0$ such that

$$\widetilde{W}(t,x) \geq \beta_k^2 |x|^2, \ \forall t \in \mathbb{R}, \ |x| \leq R. \tag{3.17}$$

Let $u \in E$ be such that $\|u\| \leq \frac{R\sqrt{c_0}}{\eta_\infty}$. By (2.1), we know that $|u(t)| \leq R$ for all $t \in \mathbb{R}$. Thus, by (3.17), it holds

$$\widetilde{W}(t,u(t)) \geq \beta_k^2 |u(t)|^2, \ \forall t \in \mathbb{R}. \tag{3.18}$$

Therefore, by (3.16) and (3.18), for all $u \in E_k$ with $0 < \|u\| = \tau_k = \min\{r,R\} \frac{\sqrt{c_0}}{\eta_\infty}$, one has

$$\begin{aligned} f(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} e^{Q(t)} \widetilde{W}(t,u)dt \\ &\leq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} e^{Q(t)} \beta_k^2 |u(t)|^2 dt \\ &\leq \frac{1}{2}\|u\|^2 - \|u\|^2 = -\frac{1}{2}\tau_k^2, \end{aligned}$$

which implies

$$\{u \in E_k / \|u\| = \tau_k\} \subset A_k = \left\{u \in E / f(u) \leq -\frac{1}{2}\tau_k^2\right\}. \tag{3.19}$$

It follows that

$$\gamma(A_k) \geq \gamma(\{u \in E_k / \|u\| = \tau_k\}) \geq k.$$

Hence, by the definition of $\Gamma_k$, we have $A_k \subset \Gamma_k$. Moreover, the definition of $\Gamma_k$ implies

$$\sup_{u \in A_k} f(u) \leq -\frac{1}{2}\tau_k^2 < 0,$$

which ends the proof of Lemma 3.4. □

Finally, assumption $(W_2)$ implies that $f(0)=0$ and $f$ is even. It follows from this and Lemma 3.3 that the condition $(f_1)$ of Lemma 2.4 is satisfied. Lemma 3.4 shows that $f$ satisfies $(f_2)$ of Lemma 2.4. Consequently, by Lemma 2.4, there exists a sequence of nontrivial critical points $(u_k)$ for $f$ satisfying $u_k \longrightarrow 0$ in $E$ as $k \longrightarrow \infty$. By vitue of Lemma 3.2, $(u_k)$ is a sequence of homoclinic solutions of $(\widetilde{\mathscr{DV}})$. By (2.1), it follows that $\sup_{t\in\mathbb{R}} |u_k(t)| \longrightarrow 0$ as $k \longrightarrow \infty$. Hence, there exists a positive constant $k_0$ such that for all $k \geq k_0$, $\sup_{t\in\mathbb{R}} |u_k(t)| \leq r$, where $r$ is defined above. Therefore for all $k \geq k_0$, $u_k$ is a fast homoclinic solution of $(\mathscr{DV})$. This completes the proof of Theorem 1.1.

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