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ON THE UPPER C-CONTINUITY ON THE SOLUTION SET OF GENERAL VECTOR ALPHA OPTIMIZATION PROBLEMS IN INFINITE-DIMENSIONAL SPACES

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Abstract. In this paper, we present some important properties on the upper C-continuity as well as the C-boundedness of set-valued mappings on the solutions set of general vector alpha optimization problems in Hausdorff locally convex topological vector spaces. As applications, we provide some important properties on the upper $(-C)$-continuity as well as the $(-C)$-boundedness of set-valued mappings on the solutions set of dual general vector alpha optimization problems in Hausdorff locally convex topological vector spaces.

Keywords. Equilibrium problem; Optimization problem; Set-valued mapping; Topological vector space.

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1. Introduction

General vector alpha optimization problem plays an important role in set-valued analysis. They was introduced by Tan [1] in 2004, and by Lin and Tan [2] in 2006. Recently, it has attracted much attention. We known that the upper and lower C-continuities of a set-valued mapping in the Hausdorff locally convex topological vector spaces are one of the efficient tools and very useful to establish theorems on the existence of solutions of quasivariational inclusion problems, quasi-equilibrium problems, quasi-optimization problems, general vector alpha optimization problems, general vector alpha quasi-optimization problems and the many other problems; see, e.g., [1, 2, 3, 4, 5, 6] and the references therein.

There are a lot of results on the solution existence of general vector alpha optimization problems and some other related problems; see, e.g., [2, 3, 7, 8, 9, 10] and references therein. Tan and Hoa [6] derived the results on the solution existence of quasi-equilibrium problems by using the lower and upper semicontinuous mappings. They also obtained the existence results of quasi-equilibrium problems by using the lower semicontinuous mappings; see [4]. Tan [5] obtained the existence results of the solution of quasi-equilibrium problems involving the fixed point theorems of separately l.s.c and u.s.c

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mappings. However, the upper (lower) C-continuity of a set-valued mapping on the solution set of the preceding said problems are not considered. Therefore, the main purpose of this paper is for the upper C-continuity as well as the upper $(-C)$-continuity of a set-valued mapping on the solution set of general vector alpha optimization problems and dual general vector alpha optimization problems. It should be noted here that set-valued analysis theory, which was introduced in thirties of the 20 century, has emerged as an interesting and fascinating branch of applicable mathematics with a wide range of applications in optimization, variational inequalities, control theory, multi-objective optimization, differential equation, economics, etc.

The organization of this paper is as follows. In Section 2, we recall some basic concepts and introduce the properties of efficient points. In Section 3, we provide some important properties for the upper C- and $(-C)$-continuities of a set-valued mapping on the solution set of general vector alpha quasi-optimization problems and dual general vector alpha quasi-optimization problems.

2. Preliminaries

From now on, if not otherwise stated, we always assume that X and Y are Hausdorff locally convex topological vector spaces in which Y be partially ordered by C with C be a closed convex cone, D is a nonempty subset of X, and the set-valued mapping F from D into 2^Y is denoted as $F : D \rightrightarrows Y$. The domain, graph, epi-graph and hypo-graph of F are defined respectively by

$$\begin{aligned}
&\mathrm{dom}\, F = \{x \in D : F(x) \neq \emptyset\},\\
&\mathrm{graph}\, F = \{(x,y) \in D \times Y : x \in \mathrm{dom}\, F,\, y \in F(x)\},\\
&\mathrm{epi}\, F = \{(x,y) \in D \times Y : F(x) \subset y + C\},\\
&\mathrm{hyp}\, F = \{(x,y) \in D \times Y : F(x) \subset y - C\}.
\end{aligned}$$

For $A \subset X$, as usual, we denote $\mathrm{int}A, clA$ instead of the interior and the closure of A, respectively. The set of Ideal, Pareto, Proper and Weak minimal points of A with respect to C is denoted respectively as $IMin(A)$, $PMin(A)$, $PrMin(A)$ and $WMin(A)$. The set of Ideal, Pareto, Proper and Weak maximal points of A with respect to C is denoted dually respectively as $IMax(A)$, $PMax(A)$, $PrMax(A)$ and $WMax(A)$. It should be noted here that (for more details, see [11]), we denote $\alpha Min(A)$ (resp., $\alpha Max(A)$) instead of $\alpha Min(A|C)$ (resp., $\alpha Max(A|C)$) for all $\alpha \in \{I, P, Pr, W\}$, and moreover, these minimal points can be illustrated as follows.

(1) $IMin(A) = \{a \in A : A \subset a + C\}$;
(2) $PMin(A) = \{a \in A : A \cap (a - C) \subset a + l(C)\}$, where $l(C) = C \cap (-C)$;
(3) If $l(C) = \{0\}$, then $PMin(A) = \{a \in A : A \cap (a - C) = \{a\}\}$;
(4) $PrMin(A) \subset PMin(A) \subset WMin(A)$;
(5) If $IMin(A) \neq \emptyset$, then $PMin(A) = IMin(A)$ and it is a point whenever C is pointed (it means that $l(C) = \{0\}$);
(6) When C is not the whole space, $WMin(A) = \{a \in A : A \cap (a - intC) = \emptyset\}$.
(7) If $intC \neq \emptyset$, then $WMin(A) = \{a \in A : a \in PMin(A|\{0\} \cup intC)\}$.

Similarly to the maximal points, we also obtain the following results.

(1') $IMax(A) = \{a \in A : A \subset a - C\}$;
(2') $PMax(A) = \{a \in A : A \cap (a + C) \subset a + l(C)\}$;

(3') If $l(C) = \{0\}$, then $PMax(A) = \{a \in A : A \cap (a+C) = \{a\}\}$;
(4') $PrMax(A) \subset PMax(A) \subset WMax(A)$;
(5') If $IMax(A) \neq \emptyset$, then $PMax(A) = IMax(A)$ and it is a point whenever C is pointed;
(6') When C is not the whole space, $WMax(A) = \{a \in A : A \cap (a + intC) = \emptyset\}$.
(7') If $intC \neq \emptyset$, then $WMax(A) = \{a \in A : a \in PMax(A|\{0\} \cup intC)\}$.

In fact, we set $C_1 = -C$,

$$IMax(A|C_1) = IMin(A|C),$$
$$IMin(A|C_1) = IMax(A|C).$$

because

$$\begin{aligned} IMin(A|C) &= \{a \in A : A \subseteq a + C\} \\ &= \{a \in A : A \subseteq a - C_1\} \\ &= IMax(A|C_1), \end{aligned}$$

and

$$\begin{aligned} IMin(A|C_1) &= \{a \in A : A \subseteq a + C_1\} \\ &= \{a \in A : A \subseteq a - C\} \\ &= IMax(A|C). \end{aligned}$$

In this paper, we consider the general vector alpha optimization problems corresponding to D, F and C (to short, $(GVOP)_{\alpha,min}$) as follows: Find $\bar{x} \in D$ such that

$$F(\bar{x}) \cap \alpha Min(F(D)|C) \neq \emptyset.$$

The set of such points $\bar{x}$ is said to be a solutions set of $(GVOP)_{\alpha,min}$ which is denoted by $\alpha S_{min}(D,F,C)$. The elements of $\alpha Min(F(D)|C)$ are called alpha optimal values of $(GVOP)_{\alpha,min}$. The dual problem of $(GVOP)_{\alpha,min}$ is denoted as $(GVOP)_{\alpha,max}$: Find $\bar{x} \in D$ such that

$$F(\bar{x}) \cap \alpha Max(F(D)|C) \neq \emptyset.$$

The set of such points $\bar{x}$ is said to be a solutions set of $(GVOP)_{\alpha,max}$, which is denoted by $\alpha S_{max}(D,F,C)$. The elements of $\alpha Max(F(D)|C)$ are called alpha optimal values of $(GVOP)_{\alpha,max}$.

We next recall the following concepts of C-continuities and C-boundedness which will be used in the paper.

Definition 2.1 ([2, 4, 5, 6, 11])**.** Let $F : D \rightrightarrows Y$ be a set-valued mapping.

(i) F is said to be upper C-continuous in $\bar{x} \in \mathrm{dom}F$ if for any neighborhood V of the origin in Y, there exists a neighborhood U of $\bar{x}$ such that

$$F(x) \subset F(\bar{x}) + V + C \;\; \forall x \in U \cap \mathrm{dom}F.$$

(ii) F is said to be lower C-continuous in $\bar{x} \in \mathrm{dom}F$ if for any neighborhood V of the origin in Y, there exists a neighborhood U of $\bar{x}$ such that

$$F(\bar{x}) \subset F(x) + V - C \;\; \forall x \in U \cap \mathrm{dom}F.$$

(iii) If F is upper C-continuous and lower C-continuous in $\bar{x} \in \mathrm{dom}F$ simultaneously, we say that F is C-continuous in $\bar{x}$.

(iv) If F is upper (resp. lower) C-continuous in any points of $\bar{x} \in \mathrm{dom}F$, we say that F is upper (resp., lower) C-continuous on D.

Definition 2.2 ([2, 4, 5, 6, 11])**.** Let $F : D \rightrightarrows Y$ be a set-valued mapping.

(i) F is said to be a closed mapping if the graph of F is a closed subset in the product space $X \times Y$, means that
$$cl\,\mathrm{graph}F = \mathrm{graph}F.$$

(ii) F is said to be C-bounded on $M \subseteq X$ if for any neighborhood W of the origin in Y, there exists a positive real number t such that $F(M) \subseteq tW + C$.

3. MAIN RESULTS

We first provide a sufficent condition about the upper C-continuous set-valued mapping on the solutions set of general vector alpha optimization problems $(GVOP)_{\alpha,min}$ with $\alpha \in \{I, P, W, Pr\}$, and besides also give the C-boundedness of a set-valued mapping is given in these problems on D.

Theorem 3.1. *Let X, Y be Hausdorff locally convex topological vector spaces. Let D be an open subset of X and let C be a closed convex cone in Y. Let $F : D \rightrightarrows Y$ be a set-valued mapping with nonempty closed values. Assume, in addition, that $intC \neq \emptyset$ and $int\,epiF \neq \emptyset$. Then we have the following statements hold.*

(i) *F is upper C-continuous on $IS_{min}(D, F, C)$.*

(ii) *If $IMin(F(D)|C) \neq \emptyset$, then F is upper C-continuous on $\alpha S_{min}(D, F, C)$, where $\alpha \in \{Pr, P\}$.*

(iii) *If $IS_{min}(D, F, C) \neq \emptyset$, then F is C-bounded on D.*

(iv) *If $IMin(F(D)|\{0\} \cup intC) \neq \emptyset$, then F is upper C-continuous on $WS_{min}(D, F, C)$. If, in addition, $WS_{min}(D, F, C) \neq \emptyset$, then F is also C-bounded on D.*

Proof. It can easily be seen that
$$\mathrm{dom}\,F = D = intD \neq \emptyset.$$
Case (i). If $IS_{min}(D, F, C) = \emptyset$, then we have nothing to prove. Otherwise, we arbitrarily take $\bar{x} \in IS_{min}(D, F, C)$, which is equivalent to $\bar{x} \in D$ satisfying
$$F(\bar{x}) \cap IMin(F(D)|C) \neq \emptyset.$$
Then there exists $\bar{y} \in F(\bar{x})$ such that $F(D) \subset \bar{y} + C$. For any neighborhood V of the origin in Y, we always have
$$F(D) \subset F(\bar{x}) + V + C.$$
Because $D = intD$, one finds that D is also an open neighborhood of $\bar{x}$ in X. So, F is upper C-continuous at any $\bar{x} \in IS_{min}(D, F, C)$. It means that F is upper C-continuous on $IS_{min}(D, F, C)$.

Case (ii). We consider the case $\alpha = P$. It follows from assumption that
$$IMin(F(D)|C) = PMin(F(D)|C).$$
Therefore,
$$PS_{min}(D, F, C) = IS_{min}(D, F, C),$$
which yields that F is upper C-continuous on $PS_{min}(D, F, C)$.

The case $\alpha = Pr$. It is plain that
$$PrS_{min}(D, F, C) \subset PS_{min}(D, F, C),$$
and the claim follows.

Case (iii). As $IS_{min}(D,F,C) \neq \emptyset$, there exists $\hat{x} \in D$ such that

$$F(\hat{x}) \cap IMin(F(D)|C) \neq \emptyset.$$

It is not difficult to verify that

$$F(D) \subset \hat{y} + C$$

for some $\hat{y} \in F(\hat{x})$. For any V neighborhood of the origin in Y, we can find a positive real number t sufficent large such that

$$\hat{y} \in tW.$$

This leads to the following inclusion holds

$$F(D) \subset tW + C.$$

Thus, F is $C-$ bounded on D.

Case (iv). If $IMin(F(D)|\{0\} \cup intC) \neq \emptyset$, then

$$PMin(F(D)|\{0\} \cup intC) = IMin(F(D)|\{0\} \cup intC).$$

Consequently,

$$\begin{aligned} WMin(F(D)|C) &= \{y \in F(D) : y \in IMin(F(D)|\{0\} \cup intC)\} \\ &= \{y \in F(D) : y \in IMin(F(D)|C)\} \end{aligned} \tag{3.1}$$

(see Luc ([11], Proposition 2.4 (1))). Now we take $x_0 \in WS_{min}(D,F,C)$. By definitions, one gets $x_0 \in D$ such that

$$F(x_0) \cap WMin(F(D)|C) \neq \emptyset.$$

Take $y_0 \in F(x_0) \subset F(D)$ such that $y_0 \in WMin(F(D)|C)$. By virtue of (3.1), it follows that $y_0 \in IMin(F(D)|C)$. Thus, $y_0 \in F(x_0) \cap IMin(F(D)|C)$. Consequently, $x_0 \in IS_{min}(D,F,C)$. Making use of the result of case (i), one sees that F is $C-$ continuous at x_0. So, F is C-continuous on $WS_{min}(D,F,C)$.

On the other hand, one finds that $IMin(F(D)|C) \subset F(D)$, which combines with (3.1) yielding that

$$IS_{min}(D,F,C) = WS_{min}(D,F,C).$$

Using the obtained results of case (iii), F is C-bounded on D, and the conclusion follows. □

Note that the results of Theorem 3.1 are still true if $int\,\mathrm{epi}F \neq \emptyset$ is replaced by $int\,\mathrm{hyp}F \neq \emptyset$. In fact, from the hypotheses $int\,\mathrm{hyp}F \neq \emptyset$, it follow that there exists $x_0 \in D$ and a neighborhood U of $\bar{x}$ such that $U \subset D$. In this case, $\mathrm{dom}F = D = intD \neq \emptyset$.

From Theorem 3.1, we can obtain the following corollaries.

Corollary 3.2. *Let X,Y be Hausdorff locally convex topological vector spaces. Let D be an open subset of X and let C be a closed convex cone in Y. Let $F : D \rightrightarrows Y$ be a set-valued mapping with nonempty closed values. Assume, in addition, that $intC \neq \emptyset$ and the problem $(GVOP)_{I,min}$ has solution. Then the following assertions are equivalent.*

(i) *$int\, epiF \neq \emptyset$.*

(ii) *F is upper C-continuous on $\alpha S_{min}(D,F,C)$ with $\alpha \in \{I,P,Pr\}$.*

Proof. If (i) holds, we taking account of Theorem 3.1 (i) get that F is upper C-continuous on $IS_{min}(D,F,C)$. Because problem $(GVOP)_{I,min}$ has a solution, we find that there exists $\bar{x} \in D$ such that

$$F(\bar{x}) \cap IMin(F(D)|C) \neq \emptyset.$$

This leads to

$$IMin(F(D)|C) \neq \emptyset.$$

Making use of Theorem (ii), we deduce that F is upper C-continuous on $\alpha S_{min}(D,F,C)$ with $\alpha \in \{P, Pr\}$. So, the implication $(i) \Longrightarrow (ii)$ holds.

We next prove the implication $(ii) \Longrightarrow (i)$ holds. Indeed, we assume that F is upper C-continuous on $IS_{min}(D,F,C)$. By hypotheses, there is $\bar{x} \in D$ such that F is upper C-continuous at $\bar{x}$ and satisfies

$$F(\bar{x}) \cap IMin(F(D)|C) \neq \emptyset.$$

Taking $\bar{y} \in F(\bar{x})$ such that

$$F(\bar{x}) \subset F(D) \subset \bar{y} + C.$$

For any neighborhood V of the origin in Y, there exists a positive real number sufficently large $\bar{t}$ such that

$$\bar{y} \in \bar{t}V.$$

For the preceding neighborhood V, making use of the upper C-continuity of F at $\bar{x}$, we find that there is an open neighborhood $\overline{U}$ of $\bar{x}$ in D satisfying

$$F(\overline{U}) \subset F(\bar{x}) + \bar{t}V + C.$$

Because $F(\bar{x})$ is closed set, one sees that C is closed cone and V is arbitrary taken. Thus

$$F(\overline{U}) \subset F(\bar{x}) + C.$$

It is clear that

$$F(\bar{x}) \subset F(\overline{U}) \subset F(D) \subset \bar{y} + C,$$

which in turn yields that

$$F(\bar{x}) \subset \bar{y} + C,$$

and

$$F(\overline{U}) \subset \bar{y} + C.$$

We set

$$\text{epi}F^0 = \{(x,y) \in D \times Y \,:\, x \in \overline{U},\, \bar{y} \in y + intC\}.$$

Next, we prove

$$\text{epi}F^0 \subset \text{epi}F.$$

In fact, take arbitrary $(x,y) \in \text{epi}F^0$. By definition, we find that $x \in \overline{U}$ and $\bar{y} \in y + intC$. Because $intC + C = intC$, we see that a consequence from here is

$$F(x) \subset F(\overline{U}) \subset \bar{y} + C \subset y + intC + C = y + intC \subset y + C,$$

which means that $(x,y) \in \text{epi}F$.

Next we check that $\text{epi}F^0$ is open and nonempty. In fact, we pick $c \in intC$ and obtain

$$F(\bar{x}) \subset \bar{y} + C = \bar{y} - c + c + C \subset (\bar{y} - c) + intC + C = (\bar{y} - c) + intC.$$

It means that $(\bar{x}, \bar{y}-c) \in \text{epi}F^0$, which is equivalent to $\text{epi}F^0 \neq \emptyset$. The openness of $\text{epi}F^0$ is easy to check. So, we obtain $int\, \text{epi}F \neq \emptyset$. The proof is complete. □

Corollary 3.3. *Let X,Y be Hausdorff locally convex topological vector spaces. Let D be an open subset of X and let C be a closed convex cone in Y. Let $F : D \rightrightarrows Y$ be a set-valued mapping with nonempty closed values. Assume, in addition, that $intC \neq \emptyset$ and there exists $\bar{x} \in D$ such that $F(\bar{x}) \cap IMin(F(D)|\{0\} \cup intC) \neq \emptyset$. Then the following assertions are equivalent.*

(i) *$int\, epiF \neq \emptyset$.*

(ii) *F is upper $C-$ continuous on $\alpha S_{min}(D,F,C)$ with $\alpha \in \{I,P,Pr,W\}$.*

Proof. We assume that (i) holds. Because $F(\bar{x}) \cap IMin(F(D)|\{0\} \cup intC) \neq \emptyset$, and due to Luc ([11], Proposition 2.4), we deduce that $F(\bar{x}) \cap IMin(F(D)|C) \neq \emptyset$. Therefore, problem $(GVOP)_{I,min}$ has at least a solution $\bar{x}$. According to Corollary 3.2, F is upper C-continuous on $\alpha S_{min}(D,F,C)$ with $\alpha \in \{I,P,Pr\}$. From Theorem 3 (iv), we find that F is upper $C-$ continuous on $WS_{min}(D,F,C)$, which means that (ii) holds. Argue similarly as for the proof of Corollary 3.2, we deduce that the implication $(ii) \Longrightarrow (i)$ holds. Hence we get desired conclusion. □

By using the obtained results in Theorem 3, we also provide here a sufficent condition on the upper $(-C)$-continuous set-valued mapping on the solutions set of dual general vector alpha optimization problems $(GVOP)_{\alpha,max}$ with $\alpha \in \{I,P,W,Pr\}$, and give the $(-C)$-boundedness of a set-valued mapping is given in these problems on D.

Theorem 3.4. *Let X,Y be Hausdorff locally convex topological vector spaces. Let D be an open subset of X and let C be a closed convex cone in Y. Let $F : D \rightrightarrows Y$ be a set-valued mapping with nonempty closed values. Assume, in addition, that $intC \neq \emptyset$ and $int\, hypF \neq \emptyset$. Then we have the following statements hold.*

(i) *F is upper $(-C)-$ continuous on $IS_{max}(D,F,C)$.*

(ii) *If $IMax(F(D)|C) \neq \emptyset$ then F is upper $(-C)$-continuous on $\alpha S_{max}(D,F,C)$, where $\alpha \in \{Pr,P\}$.*

(iii) *If $IS_{max}(D,F,C) \neq \emptyset$ then F is $(-C)$-bounded on D.*

(iv) *If $IMax(F(D)|\{0\} \cup intC) \neq \emptyset$ then F is upper $(-C)$-continuous on $WS_{max}(D,F,C)$. If, in addition, $WS_{max}(D,F,C) \neq \emptyset$ then F is also $(-C)$-bounded on D.*

Proof. By the initial assumption $int\, \text{hyp}F \neq \emptyset$, we are easy to find that

$$D = intD = \text{dom}F \neq \emptyset.$$

We set $C_1 := -C$. Then C_1 is also a closed convex cone in Y satisfying

$$intC_1 = -intC,$$
$$IMax(F(D)|C) = IMin(F(D)|C_1).$$

Thus, $intC_1 \neq \emptyset$. Moreover,

$$IS_{max}(D,F,C) = IS_{min}(D,F,C_1).$$

So, the cases (i) and (iii) are obvious true. We next prove case (ii). In fact, from the initial assumption, we find that

$$PMin(F(D)|C_1) = IMin(F(D)|C_1) \neq \emptyset.$$

Hence the result of (ii) holds for $\alpha = P$, and this also valid for $\alpha = Pr$.

Finally, we proof case (iv). Indeed, it is obvious that

$$IMin(F(D)|\{0\}\cup intC_1) = IMax(F(D)|\{0\}\cup intC).$$

Hence $IMin(F(D)|\{0\}\cup intC_1) \neq \emptyset$. Indeed, we also have

$$\begin{aligned} WMax(F(D)|C) &= \{y \in F(D) : y \in IMax(F(D)|\{0\}\cup intC)\} \\ &= \{y \in F(D) : y \in IMin(F(D)|\{0\}\cup intC_1)\} \\ &= \{y \in F(D) : y \in IMin(F(D)|C_1)\} \\ &= WMin(F(D)|C_1). \end{aligned} \tag{3.2}$$

(see Luc ([11], Proposition 2.4 (1))). A consequence is

$$WS_{max}(D,F,C) = WS_{min}(D,F,C_1).$$

Adapting the obtained result in case (iv) of Theorem 3, it follows that the case (iv) is also satisfied. This theorem is proved completely. □

Similarly as in the remark at the end of Theorem 3, the obtained results in Theorem 3.4 are still true if $int\,\mathrm{hyp}F \neq \emptyset$ is replaced by $int\,\mathrm{epi}F \neq \emptyset$.

From Theorem 3.4, we can also obtain the following corollaries.

Corollary 3.5. *Let X,Y be Hausdorff locally convex topological vector spaces. Let D be an open subset of X and let C be a closed convex cone in Y. Let $F : D \rightrightarrows Y$ be a set-valued mapping with nonempty closed values. Assume, in addition, that $intC \neq \emptyset$ and the problem $(GVOP)_{I,max}$ has solution. Then the following assertions are equivalent.*

(i) *$int\,hypF \neq \emptyset$.*
(ii) *F is upper $(-C)$-continuous on $\alpha S_{max}(D,F,C)$ with $\alpha \in \{I,P,Pr\}$.*

Proof. If (i) holds, making use of Theorem 3.4 (i), we have that F is upper $(-C)$-continuous on $IS_{max}(D,F,C)$. Since $(GVOP)_{I,max}$ has a solution, we find that there exists $\bar{x} \in D$ such that

$$F(\bar{x}) \cap IMax(F(D)|C) \neq \emptyset,$$

which yields that

$$IMax(F(D)|C) \neq \emptyset.$$

In view of Theorem 3.4 (ii), it follows that the set-valued F is upper $(-C)$-continuous on $\alpha S_{max}(D,F,C)$ with $\alpha \in \{P,Pr\}$. Therefore the implication $(i) \Longrightarrow (ii)$ holds.

We next proof the implication $(ii) \Longrightarrow (i)$ holds. In fact, we suppose that F is upper $(-C)$-continuous on $IS_{max}(D,F,C)$. By hypotheses, there is $\bar{x} \in D$ such that F is upper $(-C)$-continuous at $\bar{x}$ such that

$$F(\bar{x}) \cap IMax(F(D)|C) \neq \emptyset.$$

Taking $\bar{y} \in F(\bar{x})$ such that

$$F(\bar{x}) \subset F(D) \subset \bar{y} - C.$$

For any neighborhood V of the origin in Y, there exists a positive real number sufficently large $\bar{t}$ such that

$$\bar{y} \in \bar{t}V.$$

For the preceding neighborhood V, and making use of the upper $(-C)$-continuity of F at $\bar{x}$, we deduce that there is an open neighborhood $\overline{U}$ of $\bar{x}$ in D satisfying

$$F(\overline{U}) \subset F(\bar{x}) + \bar{t}V - C.$$

Because $F(\bar{x})$ is closed set, we find that $(-C)$ is closed cone and V is arbitrary taken. Thus

$$F(\overline{U}) \subset F(\bar{x}) - C.$$

It is clear that

$$F(\bar{x}) \subset F(\overline{U}) \subset F(D) \subset \bar{y} - C,$$

which yields that

$$F(\bar{x}) \subset \bar{y} - C,$$

and

$$F(\overline{U}) \subset \bar{y} - C.$$

We set

$$\text{hyp}F^0 = \{(x,y) \in D \times Y \,:\, x \in \overline{U},\, \bar{y} \in y - intC\}.$$

Next, we show

$$\text{hyp}F^0 \subset \text{hyp}F.$$

In fact, take arbitrary $(x,y) \in \text{hyp}F^0$. By definition, we find that $x \in \overline{U}$ and $\bar{y} \in y - intC$. Because $intC + C = intC$, a consequence from here is that

$$F(x) \subset F(\overline{U}) \subset \bar{y} - C \subset y - intC - C = y - intC \subset y - C,$$

which means that $(x,y) \in \text{hyp}F$.

Next we check that $\text{hyp}F^0$ is open and nonempty. In fact, we pick $c \in intC$ and obtain

$$F(\bar{x}) \subset \bar{y} - C = \bar{y} + c - c - C \subset (\bar{y} + c) - intC - C = (\bar{y} + c) - intC,$$

which means that $(\bar{x}, \bar{y} + c) \in \text{hyp}F^0$. This is equivalent to $\text{hyp}F^0 \neq \emptyset$. The openness of $\text{hyp}F^0$ is easy to check. So, we obtain $int\,\text{hyp}F \neq \emptyset$. This completes the proof. □

Corollary 3.6. *Let X,Y be Hausdorff locally convex topological vector spaces. Let D be an open subset of X and let C be a closed convex cone in Y. Let $F : D \rightrightarrows Y$ be a set-valued mapping with nonempty closed values. Assume, in addition, that $intC \neq \emptyset$ and there exists $\bar{x} \in D$ such that $F(\bar{x}) \cap IMax(F(D)|\{0\} \cup intC) \neq \emptyset$. Then the following statements are equivalent.*

(i) *$int\,hypF \neq \emptyset$.*
(ii) *F is upper $(-C)$-continuous on $\alpha S_{max}(D,F,C)$ with $\alpha \in \{I,P,Pr,W\}$.*

Proof. We assume that (i) holds. Because $F(\bar{x}) \cap IMax(F(D)|\{0\} \cup intC) \neq \emptyset$, and due to Luc ([11], Proposition 2.4), we deduce that $F(\bar{x}) \cap IMax(F(D)|C) \neq \emptyset$. Therefore, $(GVOP)_{I,max}$ has at least a solution $\bar{x}$. According to Corollary 3.5, F is upper $(-C)$-continuous on $\alpha S_{max}(D,F,C)$ with $\alpha \in \{I,P,Pr\}$. According to Theorem 3 (iv), F is upper $(-C)$-continuous on $WS_{max}(D,F,C)$, which means that (ii) holds. Argue similarly as for the proof of Corollary 3.5, we deduce that the implication $(ii) \Longrightarrow (i)$ holds and we get desired conclusion. □

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