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EXISTENCE OF POSITIVE PERIODIC SOLUTIONS OF NEUTRAL DIFFERENCE EQUATIONS WITH VARIABLE COEFFICIENTS

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Abstract. In this paper, we study the existence of positive periodic solutions of the nonlinear neutral difference equation with variable coefficients $x(n+1)=a(n)x(n)+\Delta(c(n)x(n-\tau))+f(n,x(n-\tau))$. The main tool in this paper is the Krasnoselskii's hybrid fixed point theorem which deals with a sum of two mappings, one is a contraction and the other is completely continuous.
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1. INTRODUCTION

Due to their importance in numerous applications, for example, physics, population dynamics, industrial robotics, and other areas, many authors are studying the existence, uniqueness, stability and positivity of solutions for delay differential and difference equations, see [1]-[12] and references therein. In this paper, we are interested in the analysis of qualitative theory of positive periodic solutions of delay difference equations. Motivated by the papers [1]-[9], we concentrate on the existence of positive periodic solutions for the nonlinear neutral difference equation with variable coefficients

$$x(n+1)=a(n)x(n)+\triangle(c(n)x(n-\tau))+f(n,x(n-\tau)), \tag{1.1}$$

where τ is positive integer and

$$f:\mathbb{Z}\times\mathbb{R}\to\mathbb{R},$$

with $\mathbb{Z}$ is the set of integers and $\mathbb{R}$ is the set of real numbers. Throughout this paper, $\triangle$ denotes the forward difference operator $\triangle x(n)=x(n+1)-x(n)$ for any sequence $\{x(n),\ n\in\mathbb{Z}\}$. Also, we define the operator E by $Ex(n)=x(n+1)$. For more on the calculus of difference equations, we refer the reader to [13].

The purpose of this paper is to use the Krasnoselskii's fixed point theorem to show the existence of positive periodic solutions for equation (1.1). To apply the Krasnoselaskii's fixed point theorem, we need to construct two mappings, one is a contraction and the other is completely continuous. In the

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case that $c(n)=c_0$, where $0\leq c_0<1$ and $-1<c_0\leq 0$, Raffoul and Yankson in [9] showed that (1.1) has a positive periodic solutions by using the Krasnoselskii's fixed point theorem. In this paper, we have two main contributions comparing with the existing results. First, instead of constant c_0, we take variable coefficient c. Second, in addition to $0\leq c(n)<1$ and $-1<c(n)\leq 0$, we consider the ranges $1<c(n)<\infty$ and $-\infty<c(n)<-1$ for c, which are new in the literature.

The organization of this paper is as follows. In Section 2, we present the inversion of difference equation (1.1) and the Krasnoselskii's fixed point theorem. For details on Krasnoselskii's theorem, we refer the reader to [14]. In Section 3 and Section 4, we present our main results on existence of positive periodic solutions of (1.1).

2. Preliminaries

Let T be an integer such that $T\geq 1$. Define $P_T=\{\varphi\in C(\mathbb{Z},\mathbb{R}):\varphi(n+T)=\varphi(n)\}$, where $C(\mathbb{Z},\mathbb{R})$ is the space of all real valued functions. Then $(P_T,\|.\|)$ is a Banach space with the maximum norm

$$\|x\|=\sup_{n\in[0,T-1]\cap\mathbb{Z}}|x(n)|.$$

Since we are searching for the existence of periodic solutions for equation (1.1), it is natural to assume that

$$a(n+T)=a(n),\ c(n+T)=c(n). \tag{2.1}$$

Assume

$$0<a(n)<1. \tag{2.2}$$

We also assume that the function $f(n,x)$ is continuous in x and periodic in n with period T, that is,

$$f(n+T,x)=f(n,x). \tag{2.3}$$

The following lemma is crucial to our main results.

Lemma 2.1. *Suppose that (2.1)-(2.3) hold. If $x\in P_T$, then x is a solution of equation (1.1) if and only if*

$$x(n)=c(n)x(n-\tau)+\sum_{u=n}^{n+T-1}G(n,u)\left[f(u,x(u-\tau))-(1-a(u))c(u)x(u-\tau)\right], \tag{2.4}$$

where

$$G(n,u)=\frac{\prod_{s=u+1}^{n+T-1}a(s)}{1-\prod_{s=n}^{n+T-1}a(s)}. \tag{2.5}$$

Proof. We consider two cases: $n\geq 1$ and $n\leq 0$. Let $x\in P_T$ be a solution of (1.1). For $n\geq 1$, equation (1.1) is equivalent to

$$\triangle\left[x(n)\prod_{s=0}^{n-1}a^{-1}(s)\right]=\left[\triangle(c(n)x(n-\tau))+f(n,x(n-\tau))\right]\prod_{s=0}^{n}a^{-1}(s). \tag{2.6}$$

By summing (2.6) from n to $n+T-1$, we obtain

$$\sum_{u=n}^{n+T-1}\triangle\left[x(u)\prod_{s=0}^{u-1}a^{-1}(s)\right]=\sum_{u=n}^{n+T-1}\left[\triangle(c(u)x(u-\tau))+f(u,x(u-\tau))\right]\prod_{s=0}^{u}a^{-1}(s).$$

It follows that

$$\begin{aligned} & x(n+T)\prod_{s=0}^{n+T-1} a^{-1}(s) - x(n)\prod_{s=0}^{n-1} a^{-1}(s) \\ & = \sum_{u=n}^{n+T-1} \left[\triangle (c(u)x(u-\tau)) + f(u,x(u-\tau))\right]\prod_{s=0}^{u} a^{-1}(s). \end{aligned}$$

Since $x(n+T) = x(n)$, we obtain

$$x(n)\left[\prod_{s=0}^{n+T-1} a^{-1}(s) - \prod_{s=0}^{n-1} a^{-1}(s)\right] = \sum_{u=n}^{n+T-1} \left[\triangle (c(u)x(u-\tau)) + f(u,x(u-\tau))\right]\prod_{s=0}^{u} a^{-1}(s). \tag{2.7}$$

Rewrite

$$\sum_{u=n}^{n+T-1} \triangle (c(u)x(u-\tau))\prod_{s=0}^{u} a^{-1}(s) = \sum_{u=n}^{n+T-1} E\left[\prod_{s=0}^{u-1} a^{-1}(s)\right]\triangle (c(u)x(u-\tau)).$$

Performing a summation by parts on the on the above equation, we get

$$\begin{aligned} \sum_{u=n}^{n+T-1} \triangle (c(u)x(u-\tau))\prod_{s=0}^{u} a^{-1}(s) &= c(n)x(n-\tau)\left[\prod_{s=0}^{n+T-1} a^{-1}(s) - \prod_{s=0}^{n-1} a^{-1}(s)\right] \\ &\quad - \sum_{u=n}^{n+T-1} c(u)x(u-\tau)\triangle\left[\prod_{s=0}^{u-1} a^{-1}(s)\right] \\ &= c(n)x(n-\tau)\left[\prod_{s=0}^{n+T-1} a^{-1}(s) - \prod_{s=0}^{n-1} a^{-1}(s)\right] \\ &\quad - \sum_{u=n}^{n+T-1} c(u)x(u-\tau)\left[1-a(u)\right]\prod_{s=0}^{u} a^{-1}(s). \end{aligned} \tag{2.8}$$

Substituting (2.8) into (2.7), we obtain

$$\begin{aligned} x(n)\left[\prod_{s=0}^{n+T-1} a^{-1}(s) - \prod_{s=0}^{n-1} a^{-1}(s)\right] &= c(n)x(n-\tau)\left[\prod_{s=0}^{n+T-1} a^{-1}(s) - \prod_{s=0}^{n-1} a^{-1}(s)\right] \\ &\quad - \sum_{u=n}^{n+T-1} c(u)x(u-\tau)\left[1-a(u)\right]\prod_{s=0}^{u} a^{-1}(s) \\ &\quad + \sum_{u=n}^{n+T-1} f(u,x(u-\tau))\prod_{s=0}^{u} a^{-1}(s). \end{aligned}$$

Dividing both sides of the above equation by $\prod_{s=0}^{n+T-1} a^{-1}(s) - \prod_{s=0}^{n-1} a^{-1}(s)$, we obtain (2.4).

Now for $n \leq 0$, equation (1.1) is equivalent to

$$\triangle\left[x(n)\prod_{s=n}^{0} a^{-1}(s)\right] = \left[\triangle (c(n)x(n-\tau)) + f(n,x(n-\tau))\right]\prod_{s=n+1}^{0} a^{-1}(s).$$

Summing the above expression from n to $n+T-1$, we obtain (2.4) by a similar argument. This completes the proof. □

Corollary 2.2. *Suppose that $c(n) \neq 0$ for all $n \in \mathbb{Z}$ and (2.1)-(2.3) hold. If $x \in P_T$, then x is a solution of equation (1.1) if and only if*

$$x(n) = \frac{x(n+\tau)}{c(n+\tau)} + \frac{1}{c(n+\tau)} \sum_{u=n+\tau}^{n+\tau+T-1} G(n+\tau,u)\left[(1-a(u))c(u)x(u-\tau) - f(u,x(u-\tau))\right], \quad (2.9)$$

where G is given by (2.5).

It is easy to see that for all $n,u \in \mathbb{Z}$, we have

$$G(n+T,u+T) = G(n,u), \; G(n+\tau+T,u+T) = G(n+\tau,u), \quad (2.10)$$

and

$$\sum_{u=n}^{n+T-1} G(n,u)(1-a(u)) = 1, \; \sum_{u=n+\tau}^{n+\tau+T-1} G(n+\tau,u)(1-a(u)) = 1. \quad (2.11)$$

Finally, we state the Krasnoselskii's fixed point theorem which enables us to prove the existence of positive periodic solutions to (1.1). For its proof we refer the reader to [14].

Theorem 2.3 (Krasnoselskii)**.** *Let $\mathbb{D}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B},\|.\|)$. Suppose that $\mathscr{A}$ and $\mathscr{B}$ map $\mathbb{D}$ into $\mathbb{B}$ such that*

(i) $x,y \in \mathbb{D}$, implies $\mathscr{A}x + \mathscr{B}y \in \mathbb{D}$,

(ii) $\mathscr{A}$ is completely continuous,

(iii) $\mathscr{B}$ is a contraction mapping.

Then there exists $z \in \mathbb{D}$ with $z = \mathscr{A}z + \mathscr{B}z$.

3. Existence of positive periodic solutions in the case $|c(n)| > 1$

To apply Theorem 2.3, we need to define a Banach space $\mathbb{B}$, a closed convex subset $\mathbb{D}$ of $\mathbb{B}$ and construct two mappings: one is a contraction and the other is compact. So, we let $(\mathbb{B},\|.\|) = (P_T,\|.\|)$ and $\mathbb{D} = \{\varphi \in \mathbb{B} : L \leq \varphi \leq K\}$, where L is non-negative constant and K is positive constant. We express equation (2.9) as

$$\varphi(n) = (\mathscr{B}_1\varphi)(n) + (\mathscr{A}_1\varphi)(n) := (H_1\varphi)(n),$$

where $\mathscr{A}_1, \mathscr{B}_1 : \mathbb{D} \to \mathbb{B}$ are defined by

$$(\mathscr{A}_1\varphi)(n) = \frac{1}{c(n+\tau)} \sum_{u=n+\tau}^{n+\tau+T-1} G(n+\tau,u)\left[(1-a(u))c(u)\varphi(u-\tau) - f(u,\varphi(u-\tau))\right], \quad (3.1)$$

and

$$(\mathscr{B}_1\varphi)(n) = \frac{\varphi(n+\tau)}{c(n+\tau)}. \quad (3.2)$$

In this section, we obtain the existence of a positive periodic solution of (1.1) by considering the two cases: $1 < c(n) < \infty$ and $-\infty < c(n) < -1$ for all $n \in \mathbb{Z}$.

Denote

$$F(n,x) = c(n)x - \frac{f(n,x)}{1-a(n)}.$$

In the case, $1 < c(n) < \infty$, we assume that there exist positive constants c_1 and c_2 such that

$$c_1 \leq c(n) \leq c_2, \text{ for all } n \in [0,T-1] \cap \mathbb{Z}, \quad (3.3)$$

$$c_1 > 1, \quad (3.4)$$

and for all $n \in [0, T-1] \cap \mathbb{Z}$, $x \in \mathbb{D}$

$$(c_2 - 1) L \leq F(n, x) \leq (c_1 - 1) K. \tag{3.5}$$

Lemma 3.1. *Suppose that (2.1)-(2.3) and (3.3)-(3.5) hold. Then $\mathscr{A}_1 : \mathbb{D} \to \mathbb{B}$ is completely continuous.*

Proof. We first show that $(\mathscr{A}_1 \varphi)(n+T) = (\mathscr{A}_1 \varphi)(n)$.

Let $\varphi \in \mathbb{D}$. Using (3.1) we arrive at

$$(\mathscr{A}_1 \varphi)(n+T) = \frac{1}{c(n+\tau+T)} \sum_{u=n+\tau+T}^{n+\tau+2T-1} G(n+\tau+T, u) \left[(1-a(u)) c(u) \varphi(u-\tau) - f(u, \varphi(u-\tau)) \right].$$

Let $j = u - T$. Using (2.1), (2.3) and (2.10), we have

$$\begin{aligned}
&(\mathscr{A}_1 \varphi)(n+T) \\
&= \frac{1}{c(n+\tau)} \sum_{j=n+\tau}^{n+\tau+T-1} G(n+\tau+T, j+T) \\
&\times \left[(1-a(j+T)) c(j+T) \varphi(j+T-\tau) - f(j+T, \varphi(j+T-\tau)) \right] \\
&= \frac{1}{c(n+\tau)} \sum_{j=n+\tau}^{n+\tau+T-1} G(n+\tau, j) \left[(1-a(j)) c(j) \varphi(j-\tau) - f(j, \varphi(j-\tau)) \right] \\
&= (\mathscr{A}_1 \varphi)(n).
\end{aligned}$$

To see that $\mathscr{A}_1(\mathbb{D})$ is uniformly bounded, we let $n \in [0, T-1] \cap \mathbb{Z}$ and for $\varphi \in \mathbb{D}$, we have by (3.5) that

$$\begin{aligned}
&|(\mathscr{A}_1 \varphi)(n)| \\
&\leq \left| \frac{1}{c(n+\tau)} \sum_{u=n+\tau}^{n+\tau+T-1} G(n+\tau, u) \left[(1-a(u)) c(u) \varphi(u-\tau) - f(u, \varphi(u-\tau)) \right] \right| \\
&\leq \frac{1}{c_1} \sum_{u=n+\tau}^{n+\tau+T-1} G(n+\tau, u) (1-a(u)) \left[c(u) \varphi(u-\tau) - \frac{f(u, \varphi(u-\tau))}{1-a(u)} \right] \\
&\leq \frac{(c_1 - 1) K}{c_1}.
\end{aligned}$$

From the estimation of $|(\mathscr{A}_1 \varphi)(n)|$, it follows that

$$\|\mathscr{A}_1 \varphi\| \leq \frac{(c_1 - 1) K}{c_1}.$$

This shows that $\mathscr{A}_1(\mathbb{D})$ is uniformly bounded.

Next, we show that $\mathscr{A}_1$ maps bounded subsets into compact sets. As $\mathscr{A}_1(\mathbb{D})$ is uniformly bounded in $\mathbb{R}^T$, we see that $\mathscr{A}_1(\mathbb{D})$ is contained in a compact subset of $\mathbb{B}$. Therefore $\mathscr{A}_1$ is completely continuous. This completes the proof. □

Lemma 3.2. *Suppose that (2.1), (3.3) and (3.4) hold. Then $\mathscr{B}_1 : \mathbb{D} \to \mathbb{B}$ is a contraction.*

Proof. Let $\mathscr{B}_1$ be defined by (3.2). Obviously, $(\mathscr{B}_1\varphi)(n+T) = (\mathscr{B}_1\varphi)(n)$. So, for any $\varphi, \psi \in \mathbb{D}$, we have

$$\begin{aligned}
|(\mathscr{B}_1\varphi)(n) - (\mathscr{B}_1\psi)(n)| &\leq \left| \frac{\varphi(n+\tau)}{c(n+\tau)} - \frac{\psi(n+\tau)}{c(n+\tau)} \right| \\
&\leq \frac{1}{c_1} |\varphi(n+\tau) - \psi(n+\tau)| \\
&\leq \frac{1}{c_1} \|\varphi - \psi\|.
\end{aligned}$$

Then $\|\mathscr{B}_1\varphi - \mathscr{B}_1\psi\| \leq \frac{1}{c_1}\|\varphi - \psi\|$. Thus $\mathscr{B}_1 : \mathbb{D} \to \mathbb{B}$ is a contraction by (3.4). □

Theorem 3.3. *Suppose (2.1)-(2.3) and (3.3)-(3.5) hold and there exists a n_0 such that $(c_2 - 1)L < F(n_0, x)$ for any $x \in \mathbb{D}$. Then equation (1.1) has a positive T-periodic solution x in the subset $\mathbb{D}_1 = \{\varphi \in \mathbb{B} : L < \varphi \leq K\}$.*

Proof. By Lemma 3.1, the operator $\mathscr{A}_1 : \mathbb{D} \to \mathbb{B}$ is completely continuous. Also, from Lemma 3.2, the operator $\mathscr{B}_1 : \mathbb{D} \to \mathbb{B}$ is a contraction. Moreover, if $\varphi, \psi \in \mathbb{D}$, we see that

$$\begin{aligned}
&(\mathscr{B}_1\psi)(n) + (\mathscr{A}_1\varphi)(n) \\
&= \frac{\psi(n+\tau)}{c(n+\tau)} + \frac{1}{c(n+\tau)} \sum_{u=n+\tau}^{n+\tau+T-1} G(n+\tau,u)\left[(1-a(u))c(u)\varphi(u-\tau) - f(u,\varphi(u-\tau))\right] \\
&\leq \frac{1}{c_1}K + \frac{1}{c_1} \sum_{u=n+\tau}^{n+\tau+T-1} G(n+\tau,u)(1-a(u))\left[c(u)\varphi(u-\tau) - \frac{f(u,\varphi(u-\tau))}{1-a(u)}\right] \\
&\leq \frac{1}{c_1}K + \frac{(c_1-1)K}{c_1} = K.
\end{aligned}$$

On the other hand, one has

$$\begin{aligned}
&(\mathscr{B}_1\psi)(n) + (\mathscr{A}_1\varphi)(n) \\
&= \frac{\psi(n+\tau)}{c(n+\tau)} + \frac{1}{c(n+\tau)} \sum_{u=n+\tau}^{n+\tau+T-1} G(n+\tau,u)\left[(1-a(u))c(u)\varphi(u-\tau) - f(u,\varphi(u-\tau))\right] \\
&\geq \frac{1}{c_2}L + \frac{1}{c_2} \sum_{u=n+\tau}^{n+\tau+T-1} G(n+\tau,u)(1-a(u))\left[c(u)\varphi(u-\tau) - \frac{f(u,\varphi(u-\tau))}{1-a(u)}\right] \\
&\geq \frac{1}{c_2}L + \frac{(c_2-1)L}{c_2} = L.
\end{aligned}$$

This shows that $\mathscr{B}_1\psi + \mathscr{A}_1\varphi \in \mathbb{D}$. Clearly, all the hypotheses of the Krasnoselskii theorem are satisfied. Thus there exists a fixed point $x \in \mathbb{D}$ such that $x = \mathscr{A}_1 x + \mathscr{B}_1 x$. Using Lemma 2.1, this fixed point is a solution of (1.1).

Next, we prove that $x \in \mathbb{D}_1$. We just need to prove that for all $n \in [0, T-1] \cap \mathbb{Z}$, $x(n) > L$. Otherwise, there exists $n^* \in [0, T-1] \cap \mathbb{Z}$ satisfying $x(n^*) = L$. From (2.9), we have

$$\begin{aligned}
L &= \frac{x(n^*+\tau)}{c(n^*+\tau)} + \frac{1}{c(n^*+\tau)} \sum_{u=n^*+\tau}^{n^*+\tau+T-1} G(n^*+\tau,u)\left[(1-a(u))c(u)x(u-\tau) - f(u,x(u-\tau))\right] \\
&\geq \frac{1}{c_2}L + \frac{1}{c_2} \sum_{u=n^*+\tau}^{n^*+\tau+T-1} G(n^*+\tau,u)(1-a(u))\left\{c(u)x(u-\tau) - \frac{f(u,x(u-\tau))}{1-a(u)}\right\}.
\end{aligned}$$

From $\sum_{u=n^*+\tau}^{n^*+\tau+T-1} G(n^*+\tau,u)(1-a(u)) = 1$, it follows that

$$\sum_{u=n^*+\tau}^{n^*+\tau+T-1} G(n^*+\tau,u)(1-a(u))[F(u,x)-(c_2-1)L] \le 0.$$

Noting that $F(u,x) \ge (c_2-1)L$ and $F(n_0,x) > (c_2-1)L$, $n_0 \in [0,T-1] \cap \mathbb{Z}$, we obtain

$$\sum_{u=n^*+\tau}^{n^*+\tau+T-1} G(n^*+\tau,u)(1-a(u))[F(u,x)-(c_2-1)L] > 0.$$

This is a contraction. So, $x \in \mathbb{D}_1$. The proof is complete. □

In the case that $-\infty < c(n) < -1$, we substitute conditions (3.3)-(3.5) with the following conditions respectively. We assume that there exist negative constants c_3 and c_4 such that

$$c_3 \le c(n) \le c_4, \text{ for all } n \in [0,T-1] \cap \mathbb{Z}, \tag{3.6}$$

$$c_4 < -1, \tag{3.7}$$

and for all $n \in [0,T-1] \cap \mathbb{Z}$, $x \in \mathbb{D}$

$$K - c_3 L \le -F(n,x) \le L - c_4 K. \tag{3.8}$$

Theorem 3.4. *Suppose (2.1)-(2.3) and (3.6)-(3.8) hold and there exists a n_0 such that $K - c_3 L < -F(n_0,x)$ for any $x \in \mathbb{D}$. Then equation (1.1) has a positive T-periodic solution x in the subset $\mathbb{D}_1$.*

The proof follows along the lines of Theorem 3.3, and hence we omit it here.

4. Existence of positive periodic solutions in the case $|c(n)| < 1$

We express equation (2.4) as

$$\varphi(n) = (\mathscr{B}_2\varphi)(n) + (\mathscr{A}_2\varphi)(n) := (H_2\varphi)(n),$$

where $\mathscr{A}_2, \mathscr{B}_2 : \mathbb{D} \to \mathbb{B}$ are defined by

$$(\mathscr{A}_2\varphi)(n) = \sum_{u=n}^{n+T-1} G(n,u)[f(u,\varphi(u-\tau)) - (1-a(u))c(u)\varphi(u-\tau)], \tag{4.1}$$

and

$$(\mathscr{B}_2\varphi)(n) = c(n)\varphi(n-\tau). \tag{4.2}$$

In this section, we obtain the existence of a positive periodic solution of (1.1) by considering the two cases: $0 \le c(n) < 1$ and $-1 < c(n) \le 0$ for all $n \in \mathbb{Z}$.

Denote

$$H(n,x) = \frac{f(n,x)}{1-a(n)} - c(n)x.$$

In the case that $0 \le c(n) < 1$, we assume that there exists positive constant c_1 such that

$$0 \le c(n) \le c_1, \text{ for all } n \in [0,T-1] \cap \mathbb{Z}, \tag{4.3}$$

$$c_1 < 1, \tag{4.4}$$

and for all $n \in [0,T-1] \cap \mathbb{Z}$, $x \in \mathbb{D}$

$$L \le H(n,x) \le (1-c_1)K. \tag{4.5}$$

In the case that $-1 < c(n) \leq 0$, we assume that there exists negative constant c_2 such that

$$c_2 \leq c(n) \leq 0, \text{ for all } n \in [0, T-1] \cap \mathbb{Z}, \tag{4.6}$$

$$c_2 > -1, \tag{4.7}$$

and for all $n \in [0, T-1] \cap \mathbb{Z}$, $x \in \mathbb{D}$

$$L - c_2 K \leq H(n, x) \leq K. \tag{4.8}$$

Lemma 4.1. *Suppose that (2.1)-(2.3) and (4.3)-(4.5) hold. Then $\mathscr{A}_2 : \mathbb{D} \to \mathbb{B}$ is completely continuous.*

Proof. We first show that $(\mathscr{A}_2\varphi)(n+T) = (\mathscr{A}_2\varphi)(n)$.

Let $\varphi \in \mathbb{D}$. Using (4.1), we arrive at

$$(\mathscr{A}_2\varphi)(n+T) = \sum_{u=n+T}^{n+2T-1} G(n+T, u)\left[f(u, \varphi(u-\tau)) - (1-a(u))c(u)\varphi(u-\tau)\right].$$

Let $j = u - T$. Using (2.1), (2.3) and (2.10), one has

$$\begin{aligned}
(\mathscr{A}_2\varphi)(n+T) &= \sum_{j=n}^{n+T-1} G(n+T, j+T)\left[f(j+T, \varphi(j+T-\tau)) - (1-a(j+T))c(j+T)\varphi(j+T-\tau)\right] \\
&= \sum_{j=n}^{n+T-1} G(n, j)\left[f(j, \varphi(j-\tau)) - (1-a(j))c(j)\varphi(j-\tau)\right] \\
&= (\mathscr{A}_2\varphi)(n).
\end{aligned}$$

To see that $\mathscr{A}_2(\mathbb{D})$ is uniformly bounded, we let $n \in [0, T-1] \cap \mathbb{Z}$ and for $\varphi \in \mathbb{D}$, we have by (4.5) that

$$\begin{aligned}
|(\mathscr{A}_2\varphi)(n)| &\leq \left|\sum_{u=n}^{n+T-1} G(n, u)\left[f(u, \varphi(u-\tau)) - (1-a(u))c(u)\varphi(u-\tau)\right]\right| \\
&\leq \sum_{u=n}^{n+T-1} G(n, u)(1-a(u))\left[\frac{f(u, \varphi(u-\tau))}{1-a(u)} - c(u)\varphi(u-\tau)\right] \\
&\leq (1-c_1)K.
\end{aligned}$$

From the estimation of $|(\mathscr{A}_2\varphi)(n)|$, it follows that

$$\|\mathscr{A}_2\varphi\| \leq (1-c_1)K.$$

This shows that $\mathscr{A}_2(\mathbb{D})$ is uniformly bounded.

Next, we show that $\mathscr{A}_2$ maps bounded subsets into compact sets. As $\mathscr{A}_2(\mathbb{D})$ is uniformly bounded in $\mathbb{R}^T$, we obtain that $\mathscr{A}_2(\mathbb{D})$ is contained in a compact subset of $\mathbb{B}$. Therefore $\mathscr{A}_2$ is completely continuous. This completes the proof. □

Lemma 4.2. *Suppose that (2.1), (4.3) and (4.4) hold. Then $\mathscr{B}_2 : \mathbb{D} \to \mathbb{B}$ is a contraction.*

Proof. Let $\mathscr{B}_2$ be defined by (4.2). Obviously, $(\mathscr{B}_2\varphi)(n+T) = (\mathscr{B}_2\varphi)(n)$. So, for any $\varphi, \psi \in \mathbb{D}$, we have

$$\begin{aligned}
|(\mathscr{B}_2\varphi)(n) - (\mathscr{B}_2\psi)(n)| &\leq |c(n)\varphi(n-\tau) - c(n)\psi(n-\tau)| \\
&\leq c_1|\varphi(n-\tau) - \psi(n-\tau)| \\
&\leq c_1\|\varphi - \psi\|.
\end{aligned}$$

Then $\|\mathscr{B}_2\varphi-\mathscr{B}_2\psi\|\leq c_1\|\varphi-\psi\|$. Thus $\mathscr{B}_2:\mathbb{D}\to\mathbb{B}$ is a contraction by (4.4). □

Similar to the results in Section 3, we have the following results.

Theorem 4.3. *Suppose (2.1)-(2.3) and (4.3)-(4.5) hold and there exists a n_0 such that $L<H(n_0,x)$ for any $x\in\mathbb{D}$. Then equation (1.1) has a positive T-periodic solution x in the subset $\mathbb{D}_1$.*

Theorem 4.4. *Suppose (2.1)-(2.3) and (4.6)-(4.8) hold and there exists a n_0 such that $L-c_2K<H(n_0,x)$ for any $x\in\mathbb{D}$. Then equation (1.1) has a positive T-periodic solution x in the subset $\mathbb{D}_1$.*

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