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ON THE GENERALIZED ULAM-HYERS-RASSIAS STABILITY FOR COUPLED FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we investigate a generalized Ulam-Hyers-Rassias stability for coupled differential equations via the Hilfer-Katugampola fractional derivative. The uniqueness is based on the Banach contraction principle and the existence is also analyzed.

Keywords. Coupled differential equation; Fractional derivative; Uniqueness; Stability; Existence.

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1. INTRODUCTION

Recently, the fractional differential equations have obtained more and more attention. The physical and natural phenomena can be modelled through differential equations of fractional order which provide better results than integer order differential equations. Fractional differential equations have numerous applications in various disciplines such as chemical technology, viscoelasticity, mathematical economy, fractals theory, ecology, economics, electromagnetic theory, biology, signal and image processing, control theory and aerodynamics, etc. In the recent years, the theory of fractional differential has been analytically investigated; see [1, 2, 3, 4, 5, 6] and the references therein.

The study of a coupled system of fractional order is very important because this type of system can often exist in various models such as Brine Tank, Chemical Kinetics, Predator Prey, Love Affairs, Chemostats and Microorganism Culturing, Irregular Heartbeats and Lidocaine and Pesticides in Soil and Trees, etc. The development of coupled fractional differential equations and their dynamical behaviors were studied in [7, 8, 9, 10, 11, 12] recently. The fractional derivative is widely used, namely, Riemann-Liouville, Caputo, Hadamard and Hilfer etc. Later Katugampola introduced a new fractional derivative which is the generalization of Riemann-Liouville and Hadamard fractional derivatives, see [13, 14, 15]. Oliveira constructed a new fractional derivative called as the Hilfer-Katugampola fractional derivative;

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see [16]. The Hilfer-Katugampola fractional derivative is the generalization of Hilfer, Hilfer- Hadamard, Hadamard, Caputo, Riemann-Liouville fractional derivatives.

In this paper, we introduce the existence and uniqueness results by using the Schaefer fixed point theorem and the Banach contraction principle. The stability is analyzed by the concept of Ulam. The detailed note on the Ulam type stability can be seen in [17, 18, 19, 20, 21, 22]. We consider the coupled differential equations of fractional-order are given by

$$ {}^{\rho}\mathfrak{D}^{\alpha_1,\beta_1}\mathfrak{h}(t) = \mathfrak{g}_1(t,\mathfrak{h}(t),\mathfrak{y}(t)), \quad t \in J := [a,b], \tag{1.1} $$

$$ {}^{\rho}\mathfrak{D}^{\alpha_2,\beta_2}\mathfrak{y}(t) = \mathfrak{g}_2(t,\mathfrak{h}(t),\mathfrak{y}(t)), \tag{1.2} $$

$$ {}^{\rho}\mathfrak{J}^{1-\gamma_1}\mathfrak{h}(a) = \mathfrak{c}, \ {}^{\rho}\mathfrak{J}^{1-\gamma_2}\mathfrak{y}(a) = \mathfrak{d}, \tag{1.3} $$

where ${}^{\rho}\mathfrak{D}^{\alpha_1,\beta_1}$ and ${}^{\rho}\mathfrak{D}^{\alpha_2,\beta_2}$ are the Hilfer-Katugampola fractional derivatives of orders $\alpha_1, \ \alpha_2 \in (0,1)$ and type $\beta_1, \ \beta_2 \in [0,1]$ and ${}^{\rho}\mathfrak{J}^{1-\gamma_1}, \ {}^{\rho}\mathfrak{J}^{1-\gamma_2}$ are the generalized fractional integral of orders $1-\gamma_1(\gamma_1 = \alpha_1+\beta_1-\alpha_1\beta_1)$ and $1-\gamma_2(\gamma_2 = \alpha_2+\beta_2-\alpha_2\beta_2)$ respectively. Let R be a Banach space, $\mathfrak{g}_i : J\times R\times R \to R$, $i = 1,2$ are given continuous functions.

The paper is constructed as follows: In Section 2, we present some definitions and interesting results on the Hilfer-Katugampola fractional derivative. In Section 3, the main result is given. In Section 4, an application of the main results is provided.

2. Preliminaries

In this section, we recall some definitions and results from fractional calculus. The following observations are taken from [5, 16]. Let $C(J,R)$ be the Banach space of all continuous functions and let C_γ be denoted the weighted spaces of continuous function defined by

$$ X := C_{\gamma_1}(J,R) = \left\{ \mathfrak{h} | \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma_1} \mathfrak{h}(t) \in C(J,R) \right\}, $$

$$ Y := C_{\gamma_2}(J,R) = \left\{ \mathfrak{y} | \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma_2} \mathfrak{y}(t) \in C(J,R) \right\}, $$

equipped with the norm

$$ \|\mathfrak{h}\|_X = \sup_{t\in J}\left| \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma_1} \mathfrak{h}(t) \right|, \ \ \|\mathfrak{y}\|_Y = \sup_{t\in J}\left| \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma_2} \mathfrak{y}(t) \right|. $$

For all $\mathfrak{h} \in X, \ \mathfrak{y} \in Y$ is denoted by

$$ \|(\mathfrak{h},\mathfrak{y})\|_{X\times Y} = \max_{t\in J}\left\{\|\mathfrak{h}\|_X, \|\mathfrak{y}\|_Y\right\}. $$

Further, we denote $K = \{(\mathfrak{h},\mathfrak{y}) \in X\times Y\}$, where $K \in X\times Y$.

Definition 2.1. The generalized left-sided fractional integral ${}^{\rho}\mathfrak{J}^{\alpha}f$ of order $\alpha \in C(\Re(\alpha))$ is defined by

$$ \left({}^{\rho}\mathfrak{J}^{\alpha}f\right)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t (t^\rho - s^\rho)^{\alpha-1}s^{\rho-1}f(s)ds, \ t > a. \tag{2.1} $$

The generalized fractional derivative, corresponding to the generalized fractional integral (2.1) is defined, for $0 \le a < t$, by

$$ \left({}^{\rho}\mathfrak{D}^{\alpha}f\right)(t) = \frac{\rho^{\alpha-n-1}}{\Gamma(n-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t (t^\rho - s^\rho)^{n-\alpha+1}s^{\rho-1}f(s)ds. \tag{2.2} $$

Definition 2.2. The Hilfer-Katugampola fractional derivative with respect to t and $\rho > 0$ is realized by

$$\begin{aligned}\left({}^{\rho}\mathfrak{D}^{\alpha,\beta} f\right)(t) &= \left({}^{\rho}\mathfrak{I}^{\alpha}\left(t^{\rho-1}\frac{d}{dt}\right){}^{\rho}\mathfrak{I}^{(1-\beta)(1-\alpha)} f\right)(t) \\ &= \left({}^{\rho}\mathfrak{I}^{\beta(1-\alpha)}\delta_{\rho}{}^{\rho}\mathfrak{I}^{(1-\beta)(1-\alpha)} f\right)(t).\end{aligned} \tag{2.3}$$

Remark 2.3. The operator ${}^{\rho}\mathfrak{D}^{\alpha,\beta}$ can be written as

$${}^{\rho}\mathfrak{D}^{\alpha,\beta} = {}^{\rho}\mathfrak{I}^{\beta(1-\alpha)}\delta_{\rho}{}^{\rho}\mathfrak{I}^{1-\gamma} = {}^{\rho}\mathfrak{I}^{\beta(1-\alpha)\rho}\mathfrak{D}^{\gamma}, \;\; \gamma = \alpha+\beta-\alpha\beta.$$

Lemma 2.4. *Let $\alpha,\beta > 0$. Then we obtain the following semigroup property*

$${}^{\rho}\mathfrak{I}^{\alpha\rho}\mathfrak{I}^{\beta}\mathfrak{g}(t) = {}^{\rho}\mathfrak{I}^{\alpha+\beta}\mathfrak{g}(t),$$

and

$${}^{\rho}\mathfrak{D}^{\alpha\rho}\mathfrak{I}^{\alpha}\mathfrak{g}(t) = \mathfrak{g}(t).$$

Lemma 2.5. *Let $\alpha,\beta > 0$.*

(1) If $\mathfrak{g}(t) = \left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\beta-1}$, then

$${}^{\rho}\mathfrak{I}^{\alpha}\mathfrak{g}(t) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha+\beta-1}.$$

(2) If $g(t) = \left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha-1}$, then

$${}^{\rho}\mathfrak{D}^{\alpha}g(t) = 0.$$

Lemma 2.6. *Let $0 < \alpha < 1$. If $\mathfrak{g} \in C_{\gamma}(J,R)$, then*

$$\left({}^{\rho}\mathfrak{I}^{\alpha}\;{}^{\rho}\mathfrak{D}^{\alpha}\mathfrak{g}\right)(t) = \mathfrak{g}(t) - \frac{\left({}^{\rho}\mathfrak{I}^{1-\alpha}_{a+}\mathfrak{g}\right)(a)}{\Gamma(\alpha)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha-1},$$

for all $t \in (a,b]$.

Let $\varepsilon_i > 0$ and $\varphi_i(t) : J \to [0,\infty)$ be continuous functions for $i = 1,2$. Consider the following inequalities for all $t \in J$:

$$\begin{aligned}\left|{}^{\rho}\mathfrak{D}^{\alpha_1,\beta_1}\mathfrak{z}_1(t) - \mathfrak{g}_1(t,\mathfrak{z}_1(t),\mathfrak{z}_2(t))\right| &\le \varepsilon_1, \\ \left|{}^{\rho}\mathfrak{D}^{\alpha_2,\beta_2}\mathfrak{z}_2(t) - \mathfrak{g}_2(t,\mathfrak{z}_1(t),\mathfrak{z}_2(t))\right| &\le \varepsilon_2.\end{aligned} \tag{2.4}$$

$$\begin{aligned}\left|{}^{\rho}\mathfrak{D}^{\alpha_1,\beta_1}\mathfrak{z}_1(t) - \mathfrak{g}_1(t,\mathfrak{z}_1(t),\mathfrak{z}_2(t))\right| &\le \varepsilon_1\varphi_1(t), \\ \left|{}^{\rho}\mathfrak{D}^{\alpha_2,\beta_2}\mathfrak{z}_2(t) - \mathfrak{g}_2(t,\mathfrak{z}_1(t),\mathfrak{z}_2(t))\right| &\le \varepsilon_2\varphi_2(t).\end{aligned} \tag{2.5}$$

$$\begin{aligned}\left|{}^{\rho}\mathfrak{D}^{\alpha_1,\beta_1}\mathfrak{z}_1(t) - \mathfrak{g}_1(t,\mathfrak{z}_1(t),\mathfrak{z}_2(t))\right| &\le \varphi_1(t), \\ \left|{}^{\rho}\mathfrak{D}^{\alpha_2,\beta_2}\mathfrak{z}_2(t) - \mathfrak{g}_2(t,\mathfrak{z}_1(t),\mathfrak{z}_2(t))\right| &\le \varphi_2(t).\end{aligned} \tag{2.6}$$

Definition 2.7. Equation (1.1) is called an Ulam-Hyers stable if there occurs a real number $C_f > 0$ such that for each $\varepsilon > 0$ and for each outcome $\mathfrak{z}_1, \mathfrak{z}_2 \in (X,Y)$ of the inequalities (2.4) there exists a solution $\mathfrak{h},\ \mathfrak{y} \in (X,Y)$ of equation (1.1) with

$$|(\mathfrak{z}_1, \mathfrak{z}_2)(t) - (\mathfrak{u}, \mathfrak{v})(t)| \leq C_f \varepsilon.$$

Definition 2.8. Equation (1.1) is known as a generalized Ulam-Hyers stable if there exists $\psi_f \in C(R^+, R^+),\ \psi_f(0) = 0$ such that for each solution $\mathfrak{z}_1, \mathfrak{z}_2 \in (X,Y)$ of the inequalities (2.4) there exists a solution $\mathfrak{h},\ \mathfrak{y} \in (X,Y)$ of equation (1.1) with

$$|(\mathfrak{z}_1, \mathfrak{z}_2)(t) - (\mathfrak{u}, \mathfrak{v})(t)| \leq \psi_f \varepsilon.$$

Definition 2.9. Equation (1.1) is called a Ulam-Hyers-Rassias stable with respect to $\varphi \in C(J,R)$ if there exists a real number $C_f > 0$ such that for each $\varepsilon > 0$ and for each solution $\mathfrak{z}_1, \mathfrak{z}_2 \in (X,Y)$ of the inequalities (2.5) there occurs a solution $\mathfrak{h},\ \mathfrak{y} \in (X,Y)$ of equation (1.1) with

$$|(\mathfrak{z}_1, \mathfrak{z}_2)(t) - (\mathfrak{u}, \mathfrak{v})(t)| \leq C_f \varepsilon \varphi(t).$$

Definition 2.10. Equation (1.1) is a generalized Ulam-Hyers-Rassias stable with respect to $\varphi \in C(J,R)$ if there exists a real number $C_{f,\varphi} > 0$ such that for each solution $\mathfrak{z}_1, \mathfrak{z}_2 \in (X,Y)$ of the inequalities (2.6) there exists a solution $\mathfrak{h},\ \mathfrak{y} \in (X,Y)$ of equation (1.1) with

$$|(\mathfrak{z}_1, \mathfrak{z}_2)(t) - (\mathfrak{u}, \mathfrak{v})(t)| \leq C_{f,\varphi} \varphi(t).$$

Lemma 2.11. *Let $\alpha > 0$, $a(t)$ be a nonnegative function locally integrable on $a \leq t < b$ (some $b \leq \infty$), and let $g(t)$ be a nonnegative, nondecreasing continuous function defined on $a \leq t < b$, such that $g(t) \leq K$ for some constant K. Further, let $\mathfrak{h}(t)$ be a nonnegative locally integrable on $a \leq t < b$ function with*

$$|\mathfrak{h}(t)| \leq a(t) + g(t) \int_a^t \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha - 1} s^{\rho - 1} \mathfrak{h}(s) ds, \quad t \in J$$

with some $\alpha > 0$. Then

$$|\mathfrak{h}(t)| \leq a(t) + \int_a^t \left[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{n\alpha - 1} s^{\rho - 1} \right] \mathfrak{h}(s) ds, \quad a \leq t < b.$$

Theorem 2.12. *(Schaefer's fixed point theorem) Let $N : K \to K$ be completely continuous operator. If set $E[N] = \{x \in K : x = \delta(Nx), \text{for some } \delta \in [0,T]\}$ is bounded, then N has fixed point.*

Lemma 2.13. *A function $\mathfrak{h}$ is the solution of fractional initial value problem*

$$\begin{cases} {}^\rho \mathfrak{D}^{\alpha,\beta} \mathfrak{h}(t) = f(t), \\ {}^\rho \mathfrak{J}^{1-\gamma} \mathfrak{h}(a) = \mathfrak{h}_a, \end{cases}$$

if and only if $\mathfrak{h}$ satisfies the following integral equation

$$\mathfrak{h}(t) = \frac{\mathfrak{h}_a}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\gamma - 1} + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha - 1} s^{\rho - 1} f(s) ds.$$

3. MAIN RESULTS

We list the following hypotheses:

(H1) There exist constants $L_i > 0$ such that

$$|\mathfrak{g}_{\mathfrak{i}}(t,\mathfrak{h},\mathfrak{y})| \leq L_i, \ i = 1,2$$

for any $\mathfrak{h},\mathfrak{y} \in R$, and $t \in J$.

(H2)

$$\eta_1 = \frac{\mathfrak{c}}{\Gamma(\gamma_1)} + \frac{L_1}{\Gamma(\alpha_1+1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_1-\gamma_1+1},$$

$$\eta_2 = \frac{\mathfrak{d}}{\Gamma(\gamma_2)} + \frac{L_2}{\Gamma(\alpha_2+1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_2-\gamma_2+1}.$$

(H3) There exist constants $L_i > 0$ such that

$$\left|\mathfrak{g}_{\mathfrak{i}}(t,\mathfrak{h},\mathfrak{y}) - \mathfrak{g}_{\mathfrak{i}}(t,\overline{\mathfrak{h}},\overline{\mathfrak{y}})\right| \leq L_i\left(\left|\mathfrak{h}-\overline{\mathfrak{h}}\right| + |\mathfrak{y}-\overline{\mathfrak{y}}|\right), \ i = 1,2$$

for any $\mathfrak{h},\mathfrak{y},\overline{\mathfrak{h}},\overline{\mathfrak{y}} \in R$, and $t \in J$.

(H4)

$$G := \left(\frac{L_1}{\Gamma(\alpha_1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_1} B(\gamma_1,\alpha_1) + \frac{L_2}{\Gamma(\alpha_2)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_2} B(\gamma_2,\alpha_2)\right) < 1.$$

(H5) There exists two increasing functions $\varphi_1,\varphi_2 \in C(J,R^+)$ and there exist $\lambda_{\varphi_1},\lambda_{\varphi_2} > 0$ such that, for any $t \in J$,

$${}^\rho\mathfrak{I}^{\alpha_1}\varphi_1(t) \leq \lambda_{\varphi_1}\varphi_1(t), \quad {}^\rho\mathfrak{I}^{\alpha_2}\varphi_2(t) \leq \lambda_{\varphi_2}\varphi_2(t).$$

Theorem 3.1. *Assume that (H3) and (H4) hold. Then the problem* (1.1)-(1.3) *has a unique solution.*

Proof. In view of Lemma 2.13, the solution $(u,v) \in X \times Y$ of system (1.1)-(1.3) can be written as follows:

$$\mathfrak{h}(t) = \frac{\mathfrak{c}}{\Gamma(\gamma_1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma_1-1} + \frac{1}{\Gamma(\alpha_1)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_1-1} s^{\rho-1}\mathfrak{g}_1(s,\mathfrak{h}(s),\mathfrak{y}(s))ds,$$
$$\mathfrak{y}(t) = \frac{\mathfrak{d}}{\Gamma(\gamma_2)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma_2-1} + \frac{1}{\Gamma(\alpha_2)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_2-1} s^{\rho-1}\mathfrak{g}_2(s,\mathfrak{h}(s),\mathfrak{y}(s))ds.$$

Let $\mathfrak{R} : X \times Y \to X \times Y$ be an operator defined by

$$\mathfrak{R}(\mathfrak{h},\mathfrak{y}) = \left(\mathfrak{R}_1(\mathfrak{h},\mathfrak{y}),\mathfrak{R}_2(\mathfrak{h},\mathfrak{y})\right),$$

where

$$\mathfrak{R}_1(\mathfrak{h},\mathfrak{y})(t) = \frac{\mathfrak{c}}{\Gamma(\gamma_1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma_1-1} + \frac{1}{\Gamma(\alpha_1)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_1-1} s^{\rho-1}\mathfrak{g}_1(s,\mathfrak{h}(s),\mathfrak{y}(s))ds,$$
$$\mathfrak{R}_2(\mathfrak{h},\mathfrak{y})(t) = \frac{\mathfrak{d}}{\Gamma(\gamma_2)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma_2-1} + \frac{1}{\Gamma(\alpha_2)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_2-1} s^{\rho-1}\mathfrak{g}_2(s,\mathfrak{h}(s),\mathfrak{y}(s))ds.$$

It is obvious that $\mathfrak{R}$ is well defined and continuous on $X \times Y$. For every $\mathfrak{u}_1, \mathfrak{v}_1, \mathfrak{u}_2, \mathfrak{v}_2 \in K$, we have

$$\begin{aligned}
&\left| \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma_1} \mathfrak{R}_1(\mathfrak{u}_1, \mathfrak{v}_1)(t) - \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma_1} \mathfrak{R}_1(\mathfrak{u}_2, \mathfrak{v}_2)(t) \right| \\
&= \left| \frac{1}{\Gamma(\alpha_1)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma_1} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha_1 - 1} s^{\rho-1} \mathfrak{g}_1(s, \mathfrak{u}_1(s), \mathfrak{v}_1(s)) ds \right. \\
&\quad \left. - \frac{1}{\Gamma(\alpha_1)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma_1} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha_1 - 1} s^{\rho-1} \mathfrak{g}_1(s, \mathfrak{u}_2(s), \mathfrak{v}_2(s)) ds \right| \\
&\leq \frac{1}{\Gamma(\alpha_1)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma_1} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha_1 - 1} s^{\rho-1} \left| \mathfrak{g}_1(s, \mathfrak{u}_1(s), \mathfrak{v}_1(s)) - \mathfrak{g}_1(s, \mathfrak{u}_2(s), \mathfrak{v}_2(s)) \right| ds \\
&\leq \frac{1}{\Gamma(\alpha_1)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma_1} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha_1 - 1} s^{\rho-1} L_1 \left(|\mathfrak{u}_1(s) - \mathfrak{v}_1(s)| - |\mathfrak{u}_2(s) - \mathfrak{v}_2(s)| \right) ds \\
&\leq \frac{L_1}{\Gamma(\alpha_1)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma_1} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha_1 + \gamma_1 - 1} B(\gamma_1, \alpha_1) \left(\|\mathfrak{u}_1 - \mathfrak{v}_1\|_{X \times Y} - \|\mathfrak{u}_2 - \mathfrak{v}_2\|_{X \times Y} \right) \\
&\leq \frac{L_1}{\Gamma(\alpha_1)} \left(\frac{b^\rho - a^\rho}{\rho} \right)^{\alpha_1} B(\gamma_1, \alpha_1) \left(\|\mathfrak{u}_1 - \mathfrak{v}_1\|_{X \times Y} - \|\mathfrak{u}_2 - \mathfrak{v}_2\|_{X \times Y} \right),
\end{aligned}$$

which implies that

$$\|\mathfrak{R}_1(\mathfrak{u}_1, \mathfrak{v}_1)(t) - \mathfrak{R}_1(\mathfrak{u}_2, \mathfrak{v}_2)(t)\|_{X \times Y} \tag{3.1}$$

$$\leq \frac{L_1}{\Gamma(\alpha_1)} \left(\frac{b^\rho - a^\rho}{\rho} \right)^{\alpha_1} B(\gamma_1, \alpha_1) \left(\|\mathfrak{u}_1 - \mathfrak{v}_1\|_{X \times Y} - \|\mathfrak{u}_2 - \mathfrak{v}_2\|_{X \times Y} \right). \tag{3.2}$$

Similarly, we can obtain

$$\begin{aligned}
&\left| \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma_2} \mathfrak{R}_2(\mathfrak{u}_1, \mathfrak{v}_1)(t) - \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma_2} \mathfrak{R}_2(\mathfrak{u}_2, \mathfrak{v}_2)(t) \right| \\
&= \left| \frac{1}{\Gamma(\alpha_2)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma_2} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha_2 - 1} s^{\rho-1} \mathfrak{g}_1(s, \mathfrak{u}_1(s), \mathfrak{v}_1(s)) ds \right. \\
&\quad \left. - \frac{1}{\Gamma(\alpha_2)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma_2} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha_2 - 1} s^{\rho-1} \mathfrak{g}_2(s, \mathfrak{u}_2(s), \mathfrak{v}_2(s)) ds \right| \\
&\leq \frac{1}{\Gamma(\alpha_2)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma_2} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha_2 - 1} s^{\rho-1} \left| \mathfrak{g}_2(s, \mathfrak{u}_1(s), \mathfrak{v}_1(s)) - \mathfrak{g}_2(s, \mathfrak{u}_2(s), \mathfrak{v}_2(s)) \right| ds \\
&\leq \frac{1}{\Gamma(\alpha_2)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma_2} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha_2 - 1} s^{\rho-1} L_2 \left(|\mathfrak{u}_1(s) - \mathfrak{v}_1(s)| - |\mathfrak{u}_2(s) - \mathfrak{v}_2(s)| \right) ds \\
&\leq \frac{L_2}{\Gamma(\alpha_2)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma_2} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha_2 + \gamma_2 - 1} B(\gamma_2, \alpha_2) \left(\|\mathfrak{u}_1 - \mathfrak{v}_1\|_{X \times Y} - \|\mathfrak{u}_2 - \mathfrak{v}_2\|_{X \times Y} \right) \\
&\leq \frac{L_2}{\Gamma(\alpha_2)} \left(\frac{b^\rho - a^\rho}{\rho} \right)^{\alpha_2} B(\gamma_2, \alpha_2) \left(\|\mathfrak{u}_1 - \mathfrak{v}_1\|_{X \times Y} - \|\mathfrak{u}_2 - \mathfrak{v}_2\|_{X \times Y} \right),
\end{aligned}$$

which implies that

$$\|\mathfrak{R}_2(\mathfrak{u}_1,\mathfrak{v}_1)(t)-\mathfrak{R}_2(\mathfrak{u}_2,\mathfrak{v}_2)(t)\|_{X\times Y} \tag{3.3}$$

$$\leq \frac{L_2}{\Gamma(\alpha_2)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha_2} B(\gamma_2,\alpha_2)\left(\|\mathfrak{u}_1-\mathfrak{v}_1\|_{X\times Y}-\|\mathfrak{u}_2-\mathfrak{v}_2\|_{X\times Y}\right). \tag{3.4}$$

By (3.1) and (3.3), we get

$$\|\mathfrak{R}(\mathfrak{u}_1,\mathfrak{v}_1)(t)-\mathfrak{R}(\mathfrak{u}_2,\mathfrak{v}_2)(t)\|_{X\times Y}\leq G\left(\|\mathfrak{u}_1-\mathfrak{v}_1\|_{X\times Y}-\|\mathfrak{u}_2-\mathfrak{v}_2\|_{X\times Y}\right).$$

This means that $\mathfrak{R}$ is a contraction. Hence, by Banach contraction principle, $\mathfrak{R}$ has a unique fixed point in $X\times Y$, which is the unique solution of problem (1.1)-(1.3). □

Remark 3.2. Functions $\mathfrak{z}_1,\mathfrak{z}_2\in(X,Y)$ are solutions of the inequality for all $t\in J$

$$\left|{}^\rho\mathfrak{D}^{\alpha_1,\beta_1}\mathfrak{z}_1(t)-\mathfrak{g}_1(t,\mathfrak{z}_1(t),\mathfrak{z}_2(t))\right|\leq\varepsilon_1,$$

$$\left|{}^\rho\mathfrak{D}^{\alpha_2,\beta_2}\mathfrak{z}_2(t)-\mathfrak{g}_2(t,\mathfrak{z}_1(t),\mathfrak{z}_2(t))\right|\leq\varepsilon_2,$$

if there exist functions $\delta_1,\delta_2\in(X,Y)$ which depends upon $\mathfrak{u},\mathfrak{v}$ respectively such that

(1) $|\delta_1(t)|\leq\varepsilon_1,\quad |\delta_2(t)|\leq\varepsilon_2$.
(2) ${}^\rho\mathfrak{D}^{\alpha_1,\beta_1}\mathfrak{z}_1(t)=\mathfrak{g}_1(t,\mathfrak{z}_1(t),\mathfrak{z}_2(t))+\delta_1(t)$,
${}^\rho\mathfrak{D}^{\alpha_2,\beta_2}\mathfrak{z}_2(t)=\mathfrak{g}_2(t,\mathfrak{z}_1(t),\mathfrak{z}_2(t))+\delta_2(t)$.

Remark 3.3. Let $\mathfrak{z}_1,\mathfrak{z}_2$ be the solutions of (2.4). Then $\mathfrak{z}_1,\mathfrak{z}_2$ are the solutions of following integral inequalities hold:

$$\left|\mathfrak{z}_1(t)-\frac{p}{\Gamma(\gamma_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma_1-1}-\frac{1}{\Gamma(\alpha_1)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_1-1}s^{\rho-1}\mathfrak{g}_1(s,\mathfrak{z}_1(s),\mathfrak{z}_2(s))ds\right|\leq K_1\varepsilon_1,$$
$$\left|\mathfrak{z}_2(t)-\frac{q}{\Gamma(\gamma_2)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma_2-1}-\frac{1}{\Gamma(\alpha_2)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_2-1}s^{\rho-1}\mathfrak{g}_2(s,\mathfrak{z}_1(s),\mathfrak{z}_2(s))ds\right|\leq K_2\varepsilon_2,$$

where

$$\ell_1=\frac{1}{\Gamma(\alpha_1+1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha_1},\quad \ell_2=\frac{1}{\Gamma(\alpha_2+1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha_2}.$$

Indeed, by Remark 3.2, we have that

$${}^\rho\mathfrak{D}^{\alpha_1,\beta_1}\mathfrak{z}_1(t)=\mathfrak{g}_1(t,\mathfrak{z}_1(t),\mathfrak{z}_2(t))+\delta_1(t), \tag{3.5}$$
$${}^\rho\mathfrak{D}^{\alpha_2,\beta_2}\mathfrak{z}_2(t)=\mathfrak{g}_2(t,\mathfrak{z}_1(t),\mathfrak{z}_2(t))+\delta_2(t). \tag{3.6}$$

Then the solution of (3.5) is given by

$$\mathfrak{z}_1(t)=\frac{\mathfrak{c}}{\Gamma(\gamma_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma_1-1}-\frac{1}{\Gamma(\alpha_1)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_1-1}s^{\rho-1}\left(\mathfrak{g}_1(s,\mathfrak{z}_1(s),\mathfrak{z}_2(s))+\delta_1(s)\right)ds,$$

From which it follows that

$$\begin{aligned}
&\left|\mathfrak{z}_1(t)-\frac{\mathfrak{c}}{\Gamma(\gamma_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma_1-1}-\frac{1}{\Gamma(\alpha_1)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_1-1}s^{\rho-1}\mathfrak{g}_1(s,\mathfrak{z}_1(s),\mathfrak{z}_2(s))ds\right|\\
&=\left|\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_1-1}s^{\rho-1}\delta_1(s)ds\right|\\
&\leq\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_1-1}s^{\rho-1}|\delta_1(s)|ds\\
&\leq\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_1-1}s^{\rho-1}ds\,\varepsilon_1\\
&=\frac{1}{\Gamma(\alpha_1+1)}\left(\frac{B^\rho-a^\rho}{\rho}\right)^{\alpha_1}\varepsilon_1\\
&=\ell_1\varepsilon_1.
\end{aligned}$$

Similarly for Eq. (3.6), we obtain

$$\left|\mathfrak{z}_2(t)-\frac{\mathfrak{d}}{\Gamma(\gamma_2)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma_2-1}-\frac{1}{\Gamma(\alpha_2)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_2-1}s^{\rho-1}\mathfrak{g}_2(s,\mathfrak{z}_1(s),\mathfrak{z}_2(s))ds\right|\leq\ell_2\varepsilon_2.$$

Theorem 3.4. *The hypothesis [H2] is satisfied. Then problem* (1.1) - (1.3) *is generalized Ulam-Hyers-Rassias stable.*

Proof. Let $\mathfrak{z}_1,\mathfrak{z}_2$ be the solutions of (2.6) and let $\mathfrak{u},\mathfrak{v}$ be unique solutions of the problem

$$\begin{aligned}
{}^\rho\mathfrak{D}^{\alpha_1,\beta_1}\mathfrak{h}(t)&=\mathfrak{g}_1(t,\mathfrak{h}(t),\mathfrak{y}(t)),\\
{}^\rho\mathfrak{D}^{\alpha_2,\beta_2}\mathfrak{y}(t)&=\mathfrak{g}_2(t,\mathfrak{h}(t),\mathfrak{y}(t)),\\
{}^\rho\mathfrak{J}^{1-\gamma_1}\mathfrak{h}(a)&=\mathfrak{c}={}^\rho\mathfrak{J}^{1-\gamma_1}\mathfrak{z}_1(a),\\
{}^\rho\mathfrak{J}^{1-\gamma_2}\mathfrak{y}(a)&=\mathfrak{d}={}^\rho\mathfrak{J}^{1-\gamma_1}\mathfrak{z}_2(a),
\end{aligned}$$

Then, we write the solutions as

$$\begin{aligned}
\mathfrak{h}(t)&=\frac{\mathfrak{c}}{\Gamma(\gamma_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma_1-1}+\frac{1}{\Gamma(\alpha_1)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_1-1}s^{\rho-1}\mathfrak{g}_1(s,\mathfrak{h}(s),\mathfrak{y}(s))ds\\
\mathfrak{y}(t)&=\frac{\mathfrak{d}}{\Gamma(\gamma_2)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma_2-1}+\frac{1}{\Gamma(\alpha_2)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_2-1}s^{\rho-1}\mathfrak{g}_2(s,\mathfrak{h}(s),\mathfrak{y}(s))ds.
\end{aligned}$$

By differentiating inequalities (2.6), we have

$$\begin{aligned}
&\left|\mathfrak{z}_1(t)-\frac{\mathfrak{c}}{\Gamma(\gamma_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma_1-1}-\frac{1}{\Gamma(\alpha_1)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_1-1}s^{\rho-1}\mathfrak{g}_1(s,\mathfrak{z}_1(s),\mathfrak{z}_2(s))ds\right|\leq\lambda_{\varphi_1}\varphi_1(t),\\
&\left|\mathfrak{z}_2(t)-\frac{\mathfrak{d}}{\Gamma(\gamma_2)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma_2-1}-\frac{1}{\Gamma(\alpha_2)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_2-1}s^{\rho-1}\mathfrak{g}_2(s,\mathfrak{z}_1(s),\mathfrak{z}_2(s))ds\right|\leq\lambda_{\varphi_2}\varphi_2(t).
\end{aligned}$$

Hence it follows that

$$\begin{aligned}
&|\mathfrak{z}_1(t)-\mathfrak{h}(t)| \\
&\leq \left|\mathfrak{z}_1(t)-\frac{\mathfrak{c}}{\Gamma(\gamma_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma_1-1}-\frac{1}{\Gamma(\alpha_1)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_1-1}s^{\rho-1}\mathfrak{g}_1(s,\mathfrak{h}(s),\mathfrak{y}(s))ds\right| \\
&\leq \left|\mathfrak{z}_1(t)-\frac{\mathfrak{c}}{\Gamma(\gamma_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma_1-1}-\frac{1}{\Gamma(\alpha_1)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_1-1}s^{\rho-1}\mathfrak{g}_1(s,\mathfrak{h}(s),\mathfrak{y}(s))ds\right| \\
&+\left|\frac{1}{\Gamma(\alpha_1)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_1-1}s^{\rho-1}\mathfrak{g}_1(s,\mathfrak{z}_1(s),\mathfrak{z}_2(s))ds\right. \\
&\left.-\frac{1}{\Gamma(\alpha_1)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_1-1}s^{\rho-1}\mathfrak{g}_1(s,\mathfrak{z}_1(s),\mathfrak{z}_2(s))ds\right| \\
&\leq \left|\mathfrak{z}_1(t)-\frac{\mathfrak{c}}{\Gamma(\gamma_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma_1-1}-\frac{1}{\Gamma(\alpha_1)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_1-1}s^{\rho-1}\mathfrak{g}_1(s,\mathfrak{z}_1(s),\mathfrak{z}_2(s))ds\right| \\
&+\left|\frac{1}{\Gamma(\alpha_1)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_1-1}s^{\rho-1}\mathfrak{g}_1(s,\mathfrak{z}_1(s),\mathfrak{z}_2(s))ds\right. \\
&\left.-\frac{1}{\Gamma(\alpha_1)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_1-1}s^{\rho-1}\mathfrak{g}_1(s,\mathfrak{h}(s),\mathfrak{y}(s))ds\right| \\
&\leq \lambda_{\varphi_1}\varphi_1(t)+\frac{1}{\Gamma(\alpha_1)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_1-1}s^{\rho-1}\left|\mathfrak{g}_1(s,\mathfrak{z}_1(s),\mathfrak{z}_2(s))-\mathfrak{g}_1(s,\mathfrak{h}(s),\mathfrak{y}(s))\right|ds.
\end{aligned}$$

It follows that

$$|\mathfrak{z}_1(t)-\mathfrak{h}(t)|\leq\lambda_{\varphi_1}\varphi_1(t)+\frac{L_1}{\Gamma(\alpha_1)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_1-1}s^{\rho-1}(|\mathfrak{z}_1(\mathfrak{s})-\mathfrak{u}(\mathfrak{s})|+|\mathfrak{z}_2(\mathfrak{s})-\mathfrak{v}(\mathfrak{s})|)ds. \tag{3.7}$$

Similarly, we have

$$|\mathfrak{z}_2(t)-\mathfrak{y}(t)|\leq\lambda_{\varphi_2}\varphi_2(t)+\frac{L_2}{\Gamma(\alpha_2)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_2-1}s^{\rho-1}(|\mathfrak{z}_1(\mathfrak{s})-\mathfrak{u}(\mathfrak{s})|+|\mathfrak{z}_2(\mathfrak{s})-\mathfrak{v}(\mathfrak{s})|)ds. \tag{3.8}$$

By (3.7), (3.8) and letting $\max_{t\in J}\left\{\lambda_{\varphi_1}\varphi_1(t),\lambda_{\varphi_2}\varphi_2(t)\right\}=\lambda_\varphi\varphi(t)$, we obtain

$$\begin{aligned}
&|\mathfrak{z}_1(t)-\mathfrak{h}(t)|+|\mathfrak{z}_2(t)-\mathfrak{y}(t)| \\
&\leq\lambda_\varphi\varphi(t)+\frac{L_1}{\Gamma(\alpha_1)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_1-1}s^{\rho-1}(|\mathfrak{z}_1(\mathfrak{s})-\mathfrak{u}(\mathfrak{s})|+|\mathfrak{z}_2(\mathfrak{s})-\mathfrak{v}(\mathfrak{s})|)ds \\
&+\frac{L_2}{\Gamma(\alpha_2)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_2-1}s^{\rho-1}(|\mathfrak{z}_1(\mathfrak{s})-\mathfrak{u}(\mathfrak{s})|+|\mathfrak{z}_2(\mathfrak{s})-\mathfrak{v}(\mathfrak{s})|)ds \\
&\leq\left(\lambda_\varphi+\frac{K(\alpha)L_1\lambda_\varphi^2}{\Gamma(\alpha_1)}+\frac{K(\alpha)L_2\lambda_\varphi^2}{\Gamma(\alpha_2)}\right)\varphi(t) \\
&:=C_{f,\varphi}\varphi(t)
\end{aligned}$$

Thus, Eq.(1.1) - (1.3) is generalized Ulam-Hyers-Rassias stable. □

Theorem 3.5. *Assume that (H1) and (H2) hold. Then* (1.1)-(1.3) *has at least one solution.*

Proof. Let $\mathfrak{R} : X \times Y \to X \times Y$ be an operator defined by

$$\mathfrak{R}(\mathfrak{h},\mathfrak{y}) = (\mathfrak{R}_1(\mathfrak{h},\mathfrak{y}),\mathfrak{R}_2(\mathfrak{h},\mathfrak{y})),$$

where

$$\mathfrak{R}_1(\mathfrak{h},\mathfrak{y})(t) = \frac{\mathfrak{c}}{\Gamma(\gamma_1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma_1-1} + \frac{1}{\Gamma(\alpha_1)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_1-1} s^{\rho-1}\mathfrak{g}_1(s,\mathfrak{h}(s),\mathfrak{y}(s))ds$$
$$\mathfrak{R}_2(\mathfrak{h},\mathfrak{y})(t) = \frac{\mathfrak{d}}{\Gamma(\gamma_2)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma_2-1} + \frac{1}{\Gamma(\alpha_2)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_2-1} s^{\rho-1}\mathfrak{g}_2(s,\mathfrak{h}(s),\mathfrak{y}(s))ds.$$

It is obvious that the operator $\mathfrak{R}$ is well defined and continuous on $X \times Y$. We show that the operator $\mathfrak{R}$ is completely continuous. Thus by assumption (H1), $\mathfrak{R}$ is continuous by the continuity of functions g_1, g_2. Define $B_r = \left\{(\mathfrak{h},\mathfrak{y}) \in X \times Y; \|(\mathfrak{h},\mathfrak{y})\|_{X\times Y} \leq r, r > 0\right\}$. For $(\mathfrak{h},\mathfrak{y}) \in B_r$

$$\begin{aligned}
&\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma_1}\mathfrak{R}_1(\mathfrak{h},\mathfrak{y})\right| \\
&\leq \frac{\mathfrak{c}}{\Gamma(\gamma_1)} + \frac{1}{\Gamma(\alpha_1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma_1}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_1-1} s^{\rho-1}\left|\mathfrak{g}_1(s,\mathfrak{h}(s),\mathfrak{y}(s))\right|ds \\
&\leq \frac{\mathfrak{c}}{\Gamma(\gamma_1)} + \frac{L_1}{\Gamma(\alpha_1+1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_1-\gamma_1+1}.
\end{aligned}$$

And by assumption we have

$$\|\mathfrak{R}_1(\mathfrak{h},\mathfrak{y})\|_X \leq \frac{\mathfrak{c}}{\Gamma(\gamma_1)} + \frac{L_1}{\Gamma(\alpha_1+1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_1-\gamma_1+1} = \eta_1. \tag{3.9}$$

By the similar argument, we get

$$\|\mathfrak{R}_2(\mathfrak{h},\mathfrak{y})\|_Y \leq \frac{\mathfrak{d}}{\Gamma(\gamma_2)} + \frac{L_2}{\Gamma(\alpha_2+1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_2-\gamma_2+1} = \eta_2. \tag{3.10}$$

By (3.9) and (3.10), we get

$$\|\mathfrak{R}_2(\mathfrak{h},\mathfrak{y})\|_{X\times Y} \leq \eta_1 + \eta_2 = \eta.$$

Next, letting $t_1,t_2 \in J$, such that $t_1 < t_2$, we show the equicontinuous of operator $\mathfrak{R}$:

$$\begin{aligned}
&\left|\left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma_1}\mathfrak{R}_1(\mathfrak{h},\mathfrak{y})(t_2) - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{1-\gamma_1}\mathfrak{R}_1(\mathfrak{h},\mathfrak{y})(t_1)\right| \\
&\le \frac{1}{\Gamma(\alpha_1)}\left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma_1}\int_a^{t_2}\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha_1-1} s^{\rho-1}\left|\mathfrak{g}_1(s,\mathfrak{h}(s),\mathfrak{y}(s))\right| ds \\
&\quad - \frac{1}{\Gamma(\alpha_1)}\left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{1-\gamma_1}\int_a^{t_1}\left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha_1-1} s^{\rho-1}\left|\mathfrak{g}_1(s,\mathfrak{h}(s),\mathfrak{y}(s))\right| ds \\
&\le \frac{1}{\Gamma(\alpha_1)}\left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma_1}\int_{t_1}^{t_2}\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha_1-1} s^{\rho-1}\left|\mathfrak{g}_1(s,\mathfrak{h}(s),\mathfrak{y}(s))\right| ds \\
&\quad + \frac{1}{\Gamma(\alpha_1)}\int_a^{t_1}\left(\left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma_1}\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha_1-1}\right. \\
&\quad \left. - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{1-\gamma_1}\left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha_1-1}\right) s^{\rho-1}\left|\mathfrak{g}_1(s,\mathfrak{h}(s),\mathfrak{y}(s))\right| ds.
\end{aligned}$$

This implies

$$\begin{aligned}
&\|\mathfrak{R}_1(\mathfrak{h},\mathfrak{y})(t_2) - \mathfrak{R}_1(\mathfrak{h},\mathfrak{y})(t_1)\|_X \\
&\le \frac{1}{\Gamma(\alpha_1)}\left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma_1}\int_{t_1}^{t_2}\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha_1-1} s^{\rho-1}\left|\mathfrak{g}_1(s,\mathfrak{h}(s),\mathfrak{y}(s))\right| ds \\
&\quad + \frac{1}{\Gamma(\alpha_1)}\int_a^{t_1}\left(\left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma_1}\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha_1-1}\right. \\
&\quad \left. - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{1-\gamma_1}\left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{\alpha_1-1}\right) s^{\rho-1}\left|\mathfrak{g}_1(s,\mathfrak{h}(s),\mathfrak{y}(s))\right| ds.
\end{aligned}$$

Using the same method, we can write

$$\begin{aligned}
&\|\mathfrak{R}_2(\mathfrak{h},\mathfrak{y})(t_2) - \mathfrak{R}_2(\mathfrak{h},\mathfrak{y})(t_1)\|_Y \\
&\le \frac{1}{\Gamma(\alpha_2)}\left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma_2}\int_{t_1}^{t_2}\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha_2-1} s^{\rho-1}\left|\mathfrak{g}_2(s,\mathfrak{h}(s),\mathfrak{y}(s))\right| ds \\
&\quad + \frac{1}{\Gamma(\alpha_1)}\int_a^{t_1}\left(\left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma_2}\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha_2-1}\right. \\
&\quad \left. - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{1-\gamma_2}\left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{\alpha_2-1}\right) s^{\rho-1}\left|\mathfrak{g}_2(s,\mathfrak{h}(s),\mathfrak{y}(s))\right| ds.
\end{aligned}$$

As $t_2 \to t_1$, the right hand sides of the above inequalities approximate to zero. Then, the Arzela-Ascoli theorem implies that $\mathfrak{R}$ is completely continuous. We proceed to show that

$$\omega = \{(\mathfrak{h},\mathfrak{y}) \in X \times Y, (\mathfrak{h},\mathfrak{y}) = \lambda\mathfrak{R}(\mathfrak{h},\mathfrak{y}), 0 < \lambda < 1\}$$

is bounded. Hence, we obtain for $t \in J$

$$\mathfrak{h}(t) = \lambda\mathfrak{R}_1(\mathfrak{h},\mathfrak{y}), \quad \mathfrak{y}(t) = \lambda\mathfrak{R}_2(\mathfrak{h},\mathfrak{y}).$$

Utilizing the same conclusion, we arrive at

$$\|\mathfrak{h}\|_X \leq \lambda \left(\frac{\mathfrak{c}}{\Gamma(\gamma_1)} + \frac{L_1}{\Gamma(\alpha_1+1)} \left(\frac{b^\rho - a^\rho}{\rho} \right)^{\alpha_1-\gamma_1+1} \right) = \lambda \eta_1. \tag{3.11}$$

$$\|\mathfrak{y}\|_Y \leq \lambda \left(\frac{\mathfrak{d}}{\Gamma(\gamma_2)} + \frac{L_2}{\Gamma(\alpha_2+1)} \left(\frac{b^\rho - a^\rho}{\rho} \right)^{\alpha_2-\gamma_2+1} \right) = \lambda \eta_2. \tag{3.12}$$

Hence from (3.11) and (3.12), we get

$$\|(\mathfrak{h},\mathfrak{y})\|_{X\times Y} \leq \lambda(\eta_1+\eta_2).$$

This shows that the set ω is bounded. Then by Theorem 2.12, we deduce that $\mathfrak{R}$ has at least one fixed point, which is a solution of problem (1.1)-(1.3). □

4. Applications

Criminal - non criminal model. Here the predator-prey interaction model the dynamics between the criminal minded and non-criminal population living in a particular society is modeled and discussed in [23]. Let $N_p(t)$ denote the non-criminal population density while $C_p(t)$ denotes the criminal population density at time t. The temporal dynamics of C_p and N_p is given by the following nonlinear system of ordinary differential equations

$$\frac{dN_p}{dt} = aN_p - cN_pC_p,$$
$$\frac{dC_p}{dt} = -bC_p + cN_pC_p,$$

with $N_p > 0,\ C_p > 0$ be the initial state. Here the fractional criminal - non criminal model is given by

$$\mathfrak{D}^{\alpha_1,\beta_1} N_p = aN_p - cN_pC_p = \mathfrak{g}_1(N_p(t),C_p(t)),$$
$$\mathfrak{D}^{\alpha_2,\beta_2} C_p = -bC_p + cN_pC_p = \mathfrak{g}_2(N_p(t),C_p(t)),$$

with the initial conditions $\mathfrak{J}^{1-\gamma_1}N_p > 0,\ \mathfrak{J}^{1-\gamma_2}C_p > 0$. According to Theorem 3.1, the system has a unique solution.

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