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AN ALTERNATIVE PARTIAL METRIC APPROACH FOR THE EXISTENCE OF COMMON FIXED POINTS

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Abstract. The main purpose of this work is to present some common fixed point results for multivalued mappings via a new partial metric approach. The proof technique used in this paper provides an alternative way for the existence of common fixed points in partial metric spaces. An example is also provided to support our main results.

Keywords. Partial metric; Pompeiu-Housdorff distance; Multivalued mapping; Common fixed point.

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1. Introduction and preliminaries

Over the last 40 years, multivalued fixed point theory has been developed in many ways. Markin set up the foundation and Nadler [1] obtained fundamental fixed point theorems for multivalued mappings using the notion of the Pompeiu-Hausdorff metric. Nadler fixed point theorem is an extension of the well known Banach Contraction Principle (BCP) to multivalued case. One of the interesting generalization of BCP was obtained by Meir and Keeler [2]. In 2001, Lim [3] characterized the Meir-Keeler contraction by introducing the notion of an L-function. In 2006, Proinov [4] obtained a fixed point theorem for single valued mappings using the L-function. Later, Dhompongsa and Yingtaweesittikul [5] established a multivalued version of the Proinov fixed point theorem on a class of hyperconvex metric space. Recently, Popescu and Stan [6] obtained a new generalization of Dhompongsa and Yingtaweesittikul [5] result on metric spaces which also generalizes Nadlers [1] theorem. To study the denotational semantics of data flow networks, Matthews [7] introduced the notion of the partial metric space. It is well known that such spaces play an important role in constructing models in computation theory. The author of [7] also obtained the partial metric version of BCP. Later on, many authors (see [8, 9, 10, 11] and references

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therein) worked on partial metric spaces and studied their topological properties and obtained various fixed point theorems. In 2012, Ayadi, Abbas and Vetro [12] proved a multivalued fixed point theorem in partial metric spaces. Ahmad, Azam and Arshad [13] obtained a common fixed point theorem for a pair of multivalued mappings in partial metric spaces. Haghi, Rezapour and Shahzad [14] pointed out that some generalizations of fixed point theorems to partial metric spaces can be deduced from the corresponding results in metric spaces.

Motivated by the work of Popescu and Stan [6], in this paper, we established the common fixed point theorems for a pair of multivalued mappings via a new partial metric approach. The results obtained in this article complement, generalize and extend some results in the literature.

Throughout this paper, $\mathbb{N}$ denotes the set of all natural numbers, $\mathbb{R}$ is the set of all real numbers and $\mathbb{R}^+$ is the set of all nonnegative real numbers.

Definition 1.1. [7] A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(p_1) $x = y \Longleftrightarrow p(x,x) = p(x,y) = p(y,y)$,
(p_2) $p(x,x) \leq p(x,y)$,
(p_3) $p(x,y) = p(y,x)$,
(p_4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

The pair (X, p) is called a partial metric space.

One of the main features of this generalization of metric spaces is the "non-zero self-distance". For examples of partial metric spaces, we refer readers to [7, 9].

Remark 1.2. (i) Every partial metric p on X induces a T_0 topology τ_p on X and the family $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$ of open p-balls serve as a basis of this topology, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$, for all $x \in X$ and $\varepsilon > 0$.

(ii) A sequence $\{x_n\}$ in a partial metric space (X, p) converges to a point $x \in X$, with respect to τ_p, if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.

(iii) If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}^+$ given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y),$$

defines a metric on X. Furthermore, a sequence $\{x_n\}$ converges in (X, p^s) to a point $x \in X$ if and only if

$$\lim_{n,m\to\infty} p(x_n,x_m) = \lim_{n\to\infty} p(x_n,x) = p(x,x).$$

Definition 1.3. [7, 9] Let (X, p) be a partial metric space.

(i) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.

(ii) (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x \in X$, that is $\lim_{n\to\infty} p(x_n,x) = p(x,x)$. In this case, we say that the partial metric p is complete.

Definition 1.4. [7, 9] Let (X, p) be a partial metric space.

(i) A sequence $\{x_n\}$ in X is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in metric space (X, p^s).

(ii) A partial metric space (X,p) is complete if and only if the metric space (X,p^s) is complete.

Definition 1.5. A set C is bounded in (X,p) if there exists $x_0 \in X$ and $M > 0$ such that for all $c \in C$, $c \in B_p(x_0,M)$, that is $p(x_0,c) < p(c,c)+M$.

Definition 1.6. [12] Let (X,p) be a partial metric space and let $\mathscr{CB}^p(X)$ be a set of all nonempty, bounded and closed subsets of X, the partial Pompeiu-Hausdorff metric is a function $\mathscr{H}_p : \mathscr{CB}^p(X) \times \mathscr{CB}^p(X) \to [0,\infty)$ such that for all $C,D \in \mathscr{CB}^p(X)$

$$\mathscr{H}_p(C,D) = \max\{\delta_p(C,D), \delta_p(D,C)\},$$

where $\delta_p(C,D) = \sup\{p(c,D), c \in C\}$ and $p(c,D) = \inf\{p(c,d), d \in D\}$.

From the above definition, we have

$$p(c,D) = \inf_{d\in D} p(c,d) \leq \delta_p(C,D) \leq \mathscr{H}_p(C,D),$$

for $C,D \in \mathscr{CB}^p(X)$ and $c \in C$.

Lemma 1.7. [8] Let (X,p) be a partial metric space and C a nonempty subset of X. Then $c \in \overline{C} \Leftrightarrow p(c,C) = p(c,c)$, where $\overline{C}$ is the closure of C with respect to the topology τ_p of (X,p).

Proposition 1.8. [12] Let (X,p) be a partial metric space. The following hold for $C,D,U \in \mathscr{CB}^p(X)$, δ_p and $\mathscr{H}_p$:

(i) $\delta_p(C,C) = \sup\{p(c,c), c \in C\}$,
(ii) $\delta_p(C,C) \leq \delta_p(C,D)$,
(iii) $\delta_p(C,D) = 0 \implies C \subseteq D$,
(iv) $\delta_p(C,D) \leq \delta_p(C,U) + \delta_p(U,D) - \inf_{u\in U} p(u,u)$,
(v) $\mathscr{H}_p(C,C) \leq \mathscr{H}_p(C,D)$,
(vi) $\mathscr{H}_p(C,D) = \mathscr{H}_p(D,C)$,
(vii) $\mathscr{H}_p(C,D) \leq \mathscr{H}_p(C,U) + \mathscr{H}_p(U,D) - \inf_{u\in U} p(u,u)$,
(viii) $\mathscr{H}_p(C,D) = 0 \implies C = D$. Converse not true.

Let $\varphi : [0,\infty) \to [0,\infty)$ satisfy the following conditions:

(i) for any $\alpha > 0$ there exists $\beta > 0$ such that $\alpha < t < \beta$ implies $\varphi(t) \leq \alpha$,
(ii) $\varphi(s+t) \leq \varphi(s) + \varphi(t)$ for all $s,t \in [0,\infty)$,
(iii) $\varphi(t) = 0$ if and only if $t = 0$.

We denote by Φ^*, the collection of all φ.

Lemma 1.9. [6] Let $\varphi \in \Phi^*$ such that $\varphi(x) \leq x$. Then

(i) $\varphi(t) < t$ for every $t > 0$,
(ii) for every sequence $\{l_n\}$ such that $l_n \to l$ as $n \to \infty$, $l_n \geq l$, we have

$$\limsup_{n\to\infty} \varphi(l_n) \leq \varphi(l).$$

Example 1.10. Let $\varphi : [0,\infty) \to [0,\infty)$ defined by $\varphi(t) = \ln(1+t)$. Then $\varphi \in \Phi^*$. Indeed,

(i) for $\alpha > 0$, we consider $\beta = \exp(\alpha) - 1$, then $\beta > \alpha$. For $t \in (\alpha,\beta)$, we have $\varphi(t) = \ln(1+t) < \ln\exp(\alpha) = \alpha$.

(ii) For every $s,t \in [0,\infty)$ we have

$$\begin{aligned}\varphi(s)+\varphi(t) &= \ln(1+s)+\ln(1+t)\\ &= \ln((1+s)(1+t))\\ &= \ln(1+s+t+st) > \ln(1+s+t) = \varphi(s+t).\end{aligned}$$

(iii) Let $\varphi(t) = 0 \Leftrightarrow \ln(1+t) = 0 \Leftrightarrow 1+t = 1 \Leftrightarrow t = 0.$

Thus $\varphi(t) = \ln(1+t) \in \Phi^*$. Also $\varphi(t) = \ln(1+t) < t$.

Theorem 1.11. [6] Let (X,d) be a complete metric space and $T : X \to CB(X)$ be such that there exists $\varphi \in \Phi^*$ satisfying $\varphi(x) \le x$ and

$$\mathscr{H}(Tx,Ty) \le \varphi(d(x,y)) \tag{1.1}$$

for all $x,y \in X$. Then T has a fixed point.

2. Main Results

Theorem 2.1. *Let (X,p) be a complete partial metric space and $S,T : X \to \mathscr{CB}^p(X)$ be multivalued mappings such that there exists $\varphi \in \Phi^*$ satisfying $\varphi(t) \le t$ and*

$$\mathscr{H}_p(Tx,Sy) \le \varphi(M_p(x,y)), \tag{2.1}$$

where

$$M_p(x,y) = \max\{p(x,y), p(Tx,x), p(Sy,y), \frac{1}{2}[p(Tx,y)+p(Sy,x)]\},$$

for all $x,y \in X$. Then there exists $z \in X$ such that $z \in Tz \cap Sz$.

Proof. Let $x_0 \in X$ be arbitrary. Fix an element $x_1 \in Tx_0$. If $p(x_0,x_1) = 0$, then $x_0 = x_1$ and $\mathscr{H}_p(Tx_0,Sx_1) \le \varphi(p(x_0,x_1)) = 0$. Thus $Tx_0 = Sx_1$, which implies that $x_1 = x_0 \in Tx_0 = Sx_1 = Sx_0$. Thus x_0 is a fixed point of S and T. So we assume that $p(x_0,x_1) > 0$. We can now choose $x_2 \in Sx_1$ such that

$$p(x_1,x_2) \le \mathscr{H}_p(Tx_0,Sx_1). \tag{2.2}$$

If $p(x_1,x_2) = 0$, then nothing to prove. Assume $p(x_1,x_2) > 0$. Choose $x_3 \in Tx_2$ such that

$$p(x_2,x_3) \le \mathscr{H}_p(Sx_1,Tx_2). \tag{2.3}$$

Continuing this process, we get a sequence $\{x_n\}$ of points in X, where

$$x_{n+1} \in A_n = \begin{cases} Tx_n & \text{if } n = 2j, \\ & \\ Sx_n & \text{if } n = 2j+1, \end{cases} \quad j \in \mathbb{N}\cup\{0\},$$

such that for each $n \ge 0$, $p(x_n,x_{n+1}) > 0$ and

$$p(x_n,x_{n+1}) \le \mathscr{H}_p(A_{n-1},A_n). \tag{2.4}$$

Case I: Suppose $n = 2j+1$. From (2.4), we have

$$p(x_n,x_{n+1}) \le \mathscr{H}_p(Tx_{2j},Sx_{2j+1}) \le \varphi(M_p(x_{2j},x_{2j+1})) \tag{2.5}$$

where,

$$\begin{aligned}
M_p(x_{2j},x_{2j+1}) &= \max\Big\{p(x_{2j},x_{2j+1}),p(Tx_{2j},x_{2j}),p(Sx_{2j+1},x_{2j+1}),\\
&\qquad \frac{1}{2}[p(Tx_{2j},x_{2j+1})+p(Sx_{2j+1},x_{2j})]\Big\}\\
&\le \max\Big\{p(x_{2j},x_{2j+1}),p(x_{2j+1},x_{2j}),p(x_{2j+2},x_{2j+1}),\\
&\qquad \frac{1}{2}[p(x_{2j+1},x_{2j+1})+p(x_{2j+2},x_{2j})]\Big\}\\
&\le \max\Big\{p(x_{2j},x_{2j+1}),p(x_{2j+2},x_{2j+1}),\\
&\qquad \frac{1}{2}[p(x_{2j+2},x_{2j+1})+p(x_{2j+1},x_{2j})]\Big\}\\
&= \max\{p(x_{2j},x_{2j+1}),p(x_{2j+2},x_{2j+1})\}.
\end{aligned}$$

If $\max\{p(x_{2j},x_{2j+1}),p(x_{2j+2},x_{2j+1})\} = p(x_{2j+2},x_{2j+1})$, then we get from (2.5) that

$$p(x_n,x_{n+1}) \le \varphi(M_p(x_{2j},x_{2j+1})) < M_p(x_{2j},x_{2j+1}) \le p(x_{2j+2},x_{2j+1}) = p(x_{n+1},x_n),$$

which is a contradiction. So $\max\{p(x_{2j},x_{2j+1}),p(x_{2j+2},x_{2j+1})\} = p(x_{2j},x_{2j+1})$. Thus from (2.5), we get

$$p(x_n,x_{n+1}) \le \varphi(M_p(x_{2j},x_{2j+1})) < M_p(x_{2j},x_{2j+1}) < p(x_{2j},x_{2j+1}) = p(x_{n-1},x_n).$$

Case II: Suppose $n = 2j+2$. From (2.4) and following case I, we get

$$\begin{aligned}
p(x_n,x_{n+1}) &\le \mathscr{H}_p(Sx_{2j+1},Tx_{2j+2}) \le \varphi(M_p(x_{2j+1},x_{2j+2}))\\
&< M_p(x_{2j+1},x_{2j+2}) \le p(x_{2j+1},x_{2j+2}) = p(x_{n-1},x_n).
\end{aligned}$$

Thus from both the cases it follows that $\{p_n = p(x_n,x_{n+1})\}$ is strictly decreasing sequence and it converges to a real number $p \ge 0$ (say). Suppose $p > 0$. From case I and II above, we have

$$p(x_n,x_{n+1}) \le M_p(x_{n-1},x_n) < p(x_{n-1},x_n),$$

i.e.,

$$p_n \le M_{p_n} < p_n,$$

where $M_{p_n} = M_p(x_{n-1},x_n)$. Letting $n \to \infty$, we get

$$\lim_{n\to\infty} M_{p_n} = p. \tag{2.6}$$

Also, we have

$$p(x_n,x_{n+1}) \le \varphi(M_p(x_{n-1},x_n)) < p(x_{n-1},x_n).$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} \varphi(M_{p_n}) = p. \tag{2.7}$$

From Lemma 1.9, we have

$$\lim_{n\to\infty} \varphi(M_{p_n}) \le \varphi(p) < p,$$

which contradicts (2.7). Thus $p = 0$, and

$$\lim_{n\to\infty} p(x_n,x_{n+1}) = 0. \tag{2.8}$$

Also

$$\lim_{n\to\infty} M_p(x_n,x_{n+1}) = 0. \tag{2.9}$$

Since $p(x_n,x_n) < p(x_n,x_{n+1})$ and $p(x_{n+1},x_{n+1}) < p(x_n,x_{n+1})$, we have

$$\lim_{n\to\infty} p(x_n,x_n) = 0 = \lim_{n\to\infty} p(x_{n+1},x_{n+1}). \tag{2.10}$$

From (2.1) and (2.9), we get

$$\lim_{n\to\infty} \mathscr{H}_p(Tx_n,Sx_{n+1}) = 0. \tag{2.11}$$

In order to prove the sequence $\{x_n\}$ to be Cauchy, suppose contrary, i.e.,

$$\lim_{n,m\to\infty} p(x_n,x_m) = \infty.$$

This implies that

$$\lim_{n,m\to\infty} p^s(x_n,x_m) = \infty. \tag{2.12}$$

Since p^s is a metric on X. Then (2.12) implies that there exists $\varepsilon > 0$ and $\{x_{n(k)}\}$, $\{x_{m(k)}\}$ be two subsequence of $\{x_n\}$ such that $x_{n(k)+1} \in Tx_{n(k)}$ and $x_{m(k)+1} \in Sx_{m(k)}$ and $n(k)$ be the smallest integer with $n(k) > m(k) > k$ such that

$$p^s(x_{n(k)},x_{m(k)}) \geq \varepsilon$$

for all k. Therefore, we can choose $n(k)$ such that

$$p^s(x_{n(k)-1},x_{m(k)}) < \varepsilon$$

for all k. Using the definition of p^s, we have

$$\begin{aligned}
\varepsilon &\leq p^s(x_{n(k)},x_{m(k)})\\
&\leq p^s(x_{n(k)-1},x_{n(k)}) + p^s(x_{n(k)-1},x_{m(k)})\\
&\leq p^s(x_{n(k)-1},x_{n(k)}) + \varepsilon\\
&\leq 2p(x_{n(k)-1},x_{n(k)}) - p(x_{n(k)-1},x_{n(k)-1}) - p(x_{n(k)},x_{n(k)}) + \varepsilon.
\end{aligned}$$

Letting $k\to\infty$, we get

$$\lim_{k\to\infty} p^s(x_{n(k)},x_{m(k)}) = \varepsilon.$$

This implies that

$$\lim_{k\to\infty} p(x_{n(k)},x_{m(k)}) = \frac{\varepsilon}{2}. \tag{2.13}$$

By (p_4) we have

$$p(x_{n(k)-1},x_{m(k)}) \leq p(x_{n(k)-1},x_{n(k)}) + p(x_{n(k)},x_{m(k)}) - p(x_{n(k)},x_{n(k)}).$$

Taking limit as $k\to\infty$, we get

$$\lim_{k\to\infty} p(x_{n(k)-1},x_{m(k)}) = \frac{\varepsilon}{2}. \tag{2.14}$$

By similar argument, we have

$$\begin{aligned}
\lim_{k\to\infty} p(x_{n(k)},x_{m(k)-1}) &= \frac{\varepsilon}{2} = \lim_{k\to\infty} p(x_{n(k)-1},x_{m(k)+1})\\
&= \lim_{k\to\infty} p(x_{n(k)+1},x_{m(k)-1}) = \lim_{k\to\infty} p(x_{n(k)},x_{m(k)+2}).
\end{aligned} \tag{2.15}$$

Now,

$$M_p(x_{n(k)},x_{m(k)}) = \max\Big\{(p(x_{n(k)},x_{m(k)}), p(Tx_{n(k)},x_{n(k)}), p(Sx_{m(k)},x_{m(k)}),$$
$$\frac{1}{2}[p(Tx_{n(k)},x_{m(k)}) + p(Sx_{m(k)},x_{n(k)})]\Big\}.$$

Taking limit as $k \to \infty$, we get

$$\lim_{k\to\infty} M_p(x_{n(k)},x_{m(k)}) = \frac{\varepsilon}{2}. \tag{2.16}$$

Also, using definition of p^s and (2.1), we have

$$\begin{aligned}
\varepsilon &\le p^s(x_{n(k)+1},x_{m(k)+1}) \\
&= 2p(x_{n(k)+1},x_{m(k)+1}) - p(x_{n(k)+1},x_{n(k)+1}) - p(x_{m(k)+1},x_{m(k)+1}) \\
&\le 2\mathscr{H}_p(Tx_{n(k)},Sx_{m(k)}) \\
&\le 2\varphi(M_p(x_{n(k)},x_{m(k)})) \\
&< 2M_p(x_{n(k)},x_{m(k)}).
\end{aligned}$$

Taking limit as $k \to \infty$, one has

$$\lim_{k\to\infty} \varphi(M_p(x_{n(k)},x_{m(k)})) = \frac{\varepsilon}{2}. \tag{2.17}$$

From Lemma 1.9, we have

$$\lim_{k\to\infty} \varphi(M_p(x_{n(k)},x_{m(k)})) \le \varphi(\frac{\varepsilon}{2}) < \frac{\varepsilon}{2},$$

which is a contradiction of (2.17). Thus $\{x_n\}$ is a Cauchy sequence in (X,p). So by Definition 1.4, $\{x_n\}$ is a Cauchy sequence in (X,p^s). Since (X,p) is complete, (X,p^s) is also complete. Therefore the sequence $\{x_n\}$ converges to some $z \in X$ with respect to metric p^s, and thus we have

$$\lim_{n\to\infty} p(x_n,x_n) = \lim_{n\to\infty} p(x_n,z) = p(z,z) = 0. \tag{2.18}$$

Further,

$$\lim_{n\to\infty} M_p(z,x_{2n+1}) = p(z,Tz).$$

If $p(Tz,z) > 0$, then we find from (2.1) and (p_4) that

$$\begin{aligned}
p(Tz,z) &\le p(Tz,x_{2n+2}) + p(x_{2n+2},z) - p(x_{2n+2},x_{2n+2}) \\
&\le \mathscr{H}_p(Tz,Sx_{2n+1}) + p(x_{2n+2},z) \\
&\le \varphi(M_p(z,x_{2n+1})) + p(x_{2n+2},z) \\
&< M_p(z,x_{2n+1}) + p(x_{2n+2},z).
\end{aligned}$$

Taking limit as $n \to \infty$, we get $p(Tz,z) < p(Tz,z)$, a contradiction. Thus

$$p(z,Tz) = 0. \tag{2.19}$$

Similarly, we get

$$p(z,Sz) = 0. \tag{2.20}$$

From (2.18), (2.19) and (2.20), we have

$$p(z,z) = p(z,Tz) = p(z,Sz).$$

Therefore, $z \in \overline{Tz} = Tz$ and $z \in \overline{Sz} = Sz$. □

Theorem 2.2. *Let (X,p) be a complete partial metric space and $S,T : X \to \mathscr{CB}^p(X)$ be multivalued mappings such that there exists $\varphi \in \Phi^*$ satisfying $\varphi(t) \leq t$ and*

$$\mathscr{H}_p(Tx,Sy) \leq \varphi(p(x,y)) \tag{2.21}$$

for all $x,y \in X$. Then there exists $z \in X$ such that $z \in Tz \cap Sz$.

Proof. The proof follows in the same manner as the proof of Theorem 2.1. □

Theorem 2.3. *Let (X,p) be a complete partial metric space and $T : X \to \mathscr{CB}^p(X)$ be multivalued mappings such that there exists $\varphi \in \Phi^*$ satisfying $\varphi(t) \leq t$ and*

$$\mathscr{H}_p(Tx,Ty) \leq \varphi(p(x,y)) \tag{2.22}$$

for all $x,y \in X$. Then there exists $z \in X$ such that $z \in Tz$.

Theorem 2.4. *Under the assumptions of Theorem 2.2, the set $C_{T,S} = \{x \in X : x \in Tx \cap Sx\}$ of all common fixed point of T and S is closed.*

Proof. Consider any sequence $\{u_n\} \subset C_{T,S}$ such that $\lim\limits_{n\to\infty} p(u_n,u) = p(u,u)$. If $u_k = u$ for some $k \in \mathbb{N}$, then $u \in C_{T,S}$ and nothing to prove. So we assume that $u_n \neq u$ for any $n \in \mathbb{N}$. Then $p(u_n,u_n) \neq p(u,u) \neq p(u_n,u)$ and $p(u_n,u) \neq 0$. Also from (2.21), we have

$$\mathscr{H}_p(Tu,Su_{2n+1}) \leq \varphi(p(u,u_{2n+1})) < p(u,u_{2n+1}). \tag{2.23}$$

Since $\{u_n\}$ is a converging sequence in a complete partial metric space X, so it is Cauchy in (X,p) and hence it is Cauchy in metric space (X,p^s). As (X,p) is complete, (X,p^s) is also complete. Thus

$$\lim_{n,m\to\infty} p(u_n,u_m) = \lim_{n\to\infty} p(u_n,u) = p(u,u). \tag{2.24}$$

Substituting $m = 2k+1 = n$ in (2.24), we get

$$\lim_{k\to\infty} p(u_{2k+1},u_{2k+1}) = \lim_{k\to\infty} p(u_{2k+1},u) = p(u,u). \tag{2.25}$$

Now,

$$\begin{aligned} p(u,Tu) &\leq p(u,u_{2n+2}) + p(u_{2n+2},Tu) - p(u_{2n+2},u_{2n+2}) \\ &\leq p(u,u_{2n+2}) + \mathscr{H}_p(Tu,Su_{2n+1}) - p(u_{2n+2},u_{2n+2}) \\ &< p(u,u_{2n+2}) + p(u,u_{2n+1}) - p(u_{2n+2},u_{2n+2}). \end{aligned}$$

Taking limit as $n \to \infty$, we get

$$p(u,Tu) < p(u,u) + p(u,u) - p(u,u) = p(u,u). \tag{2.26}$$

Since $p(u,Tu) = \inf\limits_{v\in Tu} p(u,v) = p(u,v_0)$ (say). By (p_2) we have

$$p(u,u) \leq p(u,v_0) = p(u,Tu). \tag{2.27}$$

From (2.26) and (2.27), we have

$$p(u,Tu) = p(u,u).$$

Similarly, we get $P(u,Su) = p(u,u)$. Thus $u \in Tu \cap Su$ and hence $u \in C_{T,S}$. □

Corollary 2.5. *Let (X,p) be a complete partial metric space and $S,T:X\to X$ be self-mappings such that there exists $\varphi\in\Phi^*$ satisfying $\varphi(t)\le t$ and $p(Tx,Sy)\le\varphi(p(x,y))$ for all $x,y\in X$. Then T and S have a common fixed point.*

If we take $\varphi(t)=kt$, where $k\in[0,1)$, then we get following result from Theorem 2.2.

Corollary 2.6. *Let (X,p) be a complete partial metric space and $S,T:X\to\mathscr{CB}^p(X)$ be multivalued mappings such that there exists $k\in[0,1)$ satisfying*

$$\mathscr{H}_p(Tx,Sy)\le kp(x,y),$$

for all $x,y\in X$. Then there exists $z\in X$ such that $z\in Tz\cap Sz$.

Example 2.7. Let $X=[0,\infty)$ and $p:X\times X\to[0,\infty)$ be defined as $p(x,y)=\max\{x,y\}$. Then (X,p) is complete partial metric space. Define mappings $T,S:X\to\mathscr{CB}^p(X)$ as

$$T(x)=\begin{cases}\{0\}, & \text{if } x=0,\\ \\ [0,\ln(1+x)]\,, & \text{if } x>0\end{cases}$$

and

$$S(x)=\begin{cases}[0,\frac{1}{7}x^2], & \text{if } 0\le x\le 1,\\ \\ \{0\}, & \text{if } x>1.\end{cases}$$

Let us consider the mapping $\varphi(t)=\ln(1+t)$ (see Example 1.10). Here we will show that the contractive condition (2.21), i.e., $\mathscr{H}_p(Tx,Ty)\le\varphi(p(x,y))$ holds for all $x,y\in X$. Indeed, consider the following cases:

(1) If $x=0$ and $y=0$, then (2.21) holds trivially.
(2) If $x=0,\ y\in(0,1]$, then $Tx=\{0\}$, $Sy=[0,\frac{1}{7}y^2]$. In this case $\mathscr{H}_p(Tx,Sy)=\frac{1}{7}y^2$, $p(x,y)=y$ and one can see that
$$\mathscr{H}_p(Tx,Sy)=\frac{1}{7}y^2<\ln(1+y)=\varphi(y)=\varphi(p(x,y)).$$
(3) If $x=0,\ y>1$, then $Tx=\{0\}$, $Sy=\{0\}$. In this case $\mathscr{H}_p(Tx,Sy)=0$, $p(x,y)=y$ and then
$$\mathscr{H}_p(Tx,Sy)=0<\ln(1+y)=\varphi(y)=\varphi(p(x,y)).$$
(4) If $x>0,\ y=0$, then $Tx=[0,\ln(1+x)]$, $Sy=\{0\}$. In this case $\mathscr{H}_p(Tx,Sy)=\ln(1+x)$ and $p(x,y)=x$. Then
$$\mathscr{H}_p(Tx,Sy)=\ln(1+x)=\varphi(x)=\varphi(p(x,y)).$$
(5) If $x>1,\ y\in(0,1]$, then $Tx=[0,\ln(1+x)]$, $Sy=[0,\frac{1}{7}y^2]$. In this case $\mathscr{H}_p(Tx,Sy)=\ln(1+x)$ and $p(x,y)=x$. Then
$$\mathscr{H}_p(Tx,Sy)=\ln(1+x)=\varphi(x)=\varphi(p(x,y)).$$
(6) If $x>1,\ y>1$. Here $Tx=[0,\ln(1+x)]$, $Sy=\{0\}$. In this case $\mathscr{H}_p(Tx,Sy)=\ln(1+x)$ and $p(x,y)=\max\{x,y\}$. Then we need to consider following two cases:
   (a) If $x<y$, then $p(x,y)=y$. Since $x<y$ it implies that $\ln(1+x)<\ln(1+y)$. Thus $\mathscr{H}_p(Tx,Sy)=\ln(1+x)<\ln(1+y)=\varphi(y)=\varphi(p(x,y))$.

(b) If $x \geq y$, then $p(x,y) = x$ and so we have

$$\mathscr{H}_p(Tx, Sy) = \ln(1+x) = \varphi(x) = \varphi(p(x,y)).$$

(7) If $x, y \in (0,1]$, then $Tx = [0, \ln(1+x)]$, $Sy = [0, \frac{1}{7}y^2]$. In this case $\mathscr{H}_p(Tx, Sy) = \max\{\ln(1+x), \frac{1}{7}y^2\}$ and $p(x,y) = \max\{x,y\}$.

(a) Let $x < y$. Then from Table 1, we have $\mathscr{H}_p(Tx, Sy) \leq \varphi(p(x,y))$.

(b) Let $x \geq y$. In this case, we have $\mathscr{H}_p(Tx, Sy) = \ln(1+x) = \varphi(p(x,y))$.

TABLE 1.
Some selected values of x and y to verify case 7(a)

x	y	$\ln(1+x)$	$\frac{1}{7}y^2$	$\mathscr{H}_p(Tx,Sy)$	$p(x,y)$	$\varphi(p(x,y))$
0.1	0.2	0.0953101798	0.00571428571	0.0953101798	0.2	0.18232155679
	0.4		0.02285714285	0.0953101798	0.4	0.33647223662
	0.6		0.05142857142	0.0953101798	0.6	0.47000362924
	0.8		0.09142857142	0.0953101798	0.8	0.5877866649
	1		0.14285714285	0.14285714285	1	0.69314718056
0.2	0.4	0.18232155679	0.02285714285	0.18232155679	0.4	0.33647223662
	0.6		0.05142857142		0.6	0.47000362924
	0.8		0.09142857142		0.8	0.5877866649
	1		0.14285714285		1	0.69314718056
0.4	0.5	0.33647223662	0.03571428571	0.33647223662	0.5	0.4054651081
	0.7		0.07		0.7	0.53062825106
	0.9		0.11571428571		0.9	0.64185388617
	1		0.14285714285		1	0.69314718056
0.7	0.8	0.53062825106	0.09142857142	0.53062825106	0.8	0.5877866649
	1		0.14285714285		1	0.69314718056
0.9	1	0.64185388617	0.14285714285	0.64185388617	1	0.69314718056

Thus, we can say that (2.21) (and hence (2.1)) is satisfied for all $x, y \in X$. Further, all the required conditions of Theorem 2.2 (and Theorem 2.1) are satisfied. Here we have $x = 0$ such that $x \in Tx \cap Sx$.

REFERENCES

[1] S.B. Nadler, Multi-valued contraction mappings, Pac. J. Math. 30 (1969) 475-488.

[2] A. Meir, E. Keeler, A theorem on contraction mappings, J. Math. Anal. Appl. 28 (1969) 326-329.

[3] T.C. Lim, On characterizations of Meir-Keeler contractive maps, Nonlinear Anal. 46 (2001) 113-120.

[4] P.D. Proinov, Fixed point theorems in metric spaces, Nonlinear Anal. 64 (2006) 546-557 .

[5] S. Dhompongsa, H. Yingtaweesittikul, Diametrically contractive multivalued mappings, Fixed Point Theory Appl. 2007 (2007) Article ID 19745.

[6] O. Popescu and G. Stan, A generalization of Nadler's fixed point theorem, Results Math. doi: 10.1007/s00025-017-0694-4.

[7] S. G. Matthews, Partial metric topology, Proc. 8th Summer Conference on General Topology and Applications, Ann. New York Acad. Sci. 728 (1994) 183-197.

[8] I. Altun, F. Sola, H. Simsek, Generalized contractions on partial metric spaces, Topology Appl. 157 (2010), 2778-2785.

[9] M. Bukatin, R. Kopperman, S. Matthews, H. Pajoohesh, Partial metric spaces, Amer. Math. Monthly 116 (2009), 708-718.

[10] Lj. Ciric, B. Samet, H. Aydi, C. Vetro, Common fixed points of generalized contractions on partial metric spaces and an application, Appl. Math. Comput. 218 (2011), 2398-2406.

[11] S. Romaguera, O. Valero, A quantitative computational model for complete partial metric spaces via formal balls, Math. Structures Comput. Sci. 19 (2009) 541-563.

[12] H. Aydi, M. Abbas, C. Vetro, Common fixed points for multi-valued generalized contractions on partial metric spaces, Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat. doi:10.1007/s13398-013-0120-z.

[13] J. Ahmad, A. Azam , M. Arshad, Fixed points of multivalued mappings in partial metric spaces, Fixed Point Theory Appl. 2013 (2013), 316.

[14] R.H. Haghi, S. Rezapour, N. Shahzad, Be careful on partial metric fixed point results, Topol. Appl. 160 (2012) 450-454.