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ON η-GENERALIZED OPERATOR VARIATIONAL-LIKE INEQUALITIES

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Abstract. In this paper, we introduce and study a class of η-generalized operator variational-like inequalities (for short, η-GOVLI) and η-generalized strong operator variational-like inequalities (for short, η-GSOVLI) in Hausdorff topological vector spaces. An equivalence result concerned with these two classes of η-generalized operator variational-like inequalities is proved under suitable conditions. Some new existence results for η-generalized operator variational-like inequalities and η-generalized strong operator variational-like inequalities are proved by using the Fan KKM theorem.

Keywords. Fan KKM theorem; Hemicontinuous mapping; Topological vector space; Variational-like inequality.

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1. INTRODUCTION

The theory of variational inequalities has become a very effective and powerful tool for studying a wide range of problems arising in pure and applied sciences, such as, differential equations, mechanics, contact problems in elasticity, control problems, general equilibrium problems in economics and transportation, and optimization problems etc. The variational-like inequality was first introduced Aubin and Ekeland [1]. In 2002, Domokos and Kolumban [2] gave a interesting interpretation of variational inequalities and vector variational inequalities in Banach space in terms of variational inequalities with operator solutions; see [3], [4] and the references therien.

Inspired and motivated by recent research work going on this field, we introduce and study a new class of η-generalized operator variational-like inequalities and a class of η-generalized strong operator variational-like inequalities in the setting of Hausdorff topological vector spaces. Our results improve and generalize some known results due to Ahmad and Khan [5], Lee, Khan and Salahuddin [6] and Li, Kim and Huang [7].

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The paper is organized as follows. In Section 2, we recall some necessary definitions and results which play an important role in next sections. Some new existence results for solutions of η-generalized operator variational-like inequalities are proved for both compact and noncompact settings and the existence of solutions of η-generalized strong operator variational-like inequalities are proved in Section 3.

Let Y be a Hausdorff topological vector space. We define the ordering relationships on Y with respect to cone P in Y as follows: For $A, B \subseteq Y$,

$$B - A \subseteq P \;\Leftrightarrow\; A \leq B \;\Leftrightarrow\; a \leq b, \forall a \in A, b \in B,$$

$$B - A \nsubseteq P \;\Leftrightarrow\; A \nleq B \;\Leftrightarrow\; a \nleq b, \forall a \in A, b \in B.$$

If $intP \neq \phi$, then the weak ordering in Y is defined as follows:

$$B - A \subseteq intP \;\Leftrightarrow\; A < B \;\Leftrightarrow\; a < b, \forall a \in A, b \in B,$$

$$B - A \nsubseteq intP \;\Leftrightarrow\; A \not< B \;\Leftrightarrow\; a \not< b, \forall a \in A, b \in B.$$

Next, we work under the following settings

Let X and Y be Hausdorff topological vector spaces. Let $L(X,Y)$ be the space of all continuous linear operators from X to Y, K be a nonempty closed and convex subset of $L(X,Y)$, and $T : K \to L(X,Y)$ be a mapping. Let $C : K \to 2^Y$ be a set-valued mapping such that for each $f \in K$, $C(f)$ is a closed and convex pointed cone in Y with a nonempty interior. Suppose that $\eta : K \times K \to X$ is a mapping and $F : K \times K \to 2^Y$ is a set-valued mapping.

Throughout this paper, we assume that Y is an ordered topological vector space. In particular, for $f \in K$, $(Y, C(f))$ is an ordered topological vector space.

Now, we consider the following two types of operator variational-like inequalities:

(1) *η-Generalized operator variational-like inequalities (for short, η-GOVLI)*: for each $h \in K$ and $\lambda \in (0,1]$, find $f \in K$ such that

$$\langle T(\lambda f + (1-\lambda)h), \eta(g,f)\rangle + F(g,f) \nsubseteq -intC(f), \; \forall g \in K, \tag{1.1}$$

(2) *η-Generalized strong operator variational-like inequality (for short, η-GSOVLI)*: for each $h \in K$ and $\lambda \in (0,1]$, find $f \in K$ such that

$$\langle T(\lambda f + (1-\lambda)h),\; \eta(g,f)\rangle + F(g,f) \nsubseteq -C(f) \setminus \{0\}, \; \forall g \in K. \tag{1.2}$$

Some special cases of η-GOVLI

(i) If the mappings $C : K \to 2^Y$ and $F : K \times K \to 2^Y$ are single valued and $K \subseteq X$, then η-GOVLI reduces to find $f \in K$ such that for each $h \in K, \lambda \in (0,1]$,

$$\langle T(\lambda f + (1-\lambda)h), \eta(g,f)\rangle + F(g,f) \;\notin -intC(f), \; \forall g \in K,$$

which is known as η-generalized vector variational-like inequality given by Li, Kim and Huang [7].

(ii) If the mapping $C : K \to 2^Y$ and $F : K \times K \to 2^Y$ are single-valued maps such that $C(f) = C$, $\forall f \in K$, $F(f,g) = 0$, $\forall f, g \in K$ and $K \subseteq X$, and $\eta : K \times K \to K$, then the $\eta-$GOVLI reduces to find $f \in K$ such that for each $h \in K$, $\lambda \in (0,1]$,

$$\langle T(\lambda f + (1-\lambda)h), \eta(g,f)\rangle \;\notin -intC, \; \forall g \in K,$$

which is known as the vector variational like inequality given by Ahmad and Khan [5].

2. Preliminaries

Definition 2.1. Let $T : K \to L(X,Y)$ and $\eta : K \times K \to X$ be two mappings and let $C : K \to 2^Y$ be a set-valued mapping such that $C(f)$ is a closed and convex pointed cone with nonempty interior. Let $C = \bigcap_{f \in K} C(f) \neq \emptyset$. Then T is said to be monotone in C if $\langle T(x) - T(y), \eta(x,y)\rangle \in C,\ \forall x,y \in K$.

Definition 2.2. Let $T : K \to L(X,Y)$ and $\eta : K \times K \to X$ be two mappings. We say that T is η-hemicontinuous if for any given $x,y,z \in K, \lambda \in (0,1]$, $t \mapsto \langle T(\lambda x + (1-\lambda)(y-x)) + (1-\lambda)z,\ \eta(y,x)\rangle$ is continuous at 0^+.

Definition 2.3. A set-valued mapping $A : X \to 2^Y$ is said to be upper-semicontinuous (for short,u.s.c) at $x_0 \in X$ if for any net $\{x_\lambda\}$ in X such that $x_\lambda \to x_0$ and for any net $\{y_\lambda\}$ in Y with $y_\lambda \in A(x_\lambda)$ such that $y_\lambda \to y_0$ in Y, we have $y_0 \in A(x_0)$. A is said to be upper semicontinuous on X if it is upper semicontinuous at each point of X.

To prove the existence results of solutions of η- GOVLI (1.1), we also need the following lemmas.

Lemma 2.4. [8] *Let M be a nonempty closed and convex subset of a Hausdorff topological vector space. Let $G : M \to 2^M$ be a set-valued map such that G is a KKM-map and $G(x)$ is closed $\forall x \in M$ and compact for some $x \in M$. Then $\cap_{x\in M} G(x) \neq \phi$.*

Lemma 2.5. [9] *Let X be a Hausdorff topological vector space. Let $A_1, A_2, \cdots, A_n$ be nonempty compact convex subsets of X. Then $conv(\cup_{i=1}^n A_i)$ is compact.*

Lemma 2.6. [1] *Let X, Y be two topological spaces. If $A : X \to 2^Y$ is upper-semicontinuous with closed values, then A is closed.*

3. Existence Results

In order to prove the existence results for the solutions of η-GOVLI (1.1), we first prove the following lemma.

Lemma 3.1. *Let (Y,P) be an ordered topological vector space with a closed and convex cone P. For any $A, B \subseteq Y$, we have the followings:*

(i) *If $A - B \subseteq intP$ and $A \nsubseteq intP$, then $B \nsubseteq intP$.*
(ii) *If $A - B \nsubseteq P$ and $A \nsubseteq intP$, then $B \nsubseteq intP$.*
(iii) *If $A - B \subseteq -intP$ and $A \nsubseteq -intP$, then $B \nsubseteq -intP$.*
(iv) *If $A - B \nsubseteq -P$ and $A \nsubseteq -intP$, then $B \nsubseteq -intP$.*

Proof. (i) Suppose $A - B \subseteq intP$ and $A \nsubseteq intP$. Then we have to show that $B \nsubseteq intP$. Since $A - B \subseteq intP \implies -B \subseteq intP - A \implies B \subseteq -intP + A \implies B \subseteq -intP + Y \implies B \subseteq Y \setminus intP \implies B \nsubseteq intP$.

(ii) Suppose $A - B \subseteq P$ and $A \nsubseteq -intP$. Then we have to show that $B \nsubseteq intP$. Since $A - B \subseteq P \implies -B \subseteq P - A \implies B \subseteq -P + A \implies B \subseteq -P + Y \setminus intP \implies B \subseteq Y \setminus intP \implies B \nsubseteq intP$.

(iii) Suppose $A - B \subseteq -intP$ and $A \nsubseteq -intP$. Then we have to show that $B \nsubseteq -intP$. Since $A - B \subseteq -intP \implies -B \subseteq -intP - A \implies B \subseteq intP + A \implies B \subseteq intP + Y \setminus \{-intP\} \implies B \subseteq Y \setminus \{-intP\} \implies B \nsubseteq \{-intP\}$.

(iv) Suppose $A-B\subseteq -P$ and $A\nsubseteq -intP$. Then we have to show that $B\nsubseteq -intP$. Since $A-B\subseteq -P \implies -B\subseteq -P-A \implies B\subseteq P+A \implies B\subseteq P+Y\setminus\{-intP\} \implies B\subseteq Y\setminus\{-intP\} \implies B\nsubseteq -intP$. □

Theorem 3.2. *Let X and Y be Hausdorff topological vector spaces. Let $K\subseteq L(X,Y)$ be a nonempty closed convex subset, and let $C:K\to 2^Y$ be a set-valued mapping such that for all $f\in K$, $C(f)$ is a closed and convex pointed cone with nonempty interior, that is, $(Y,C(f))$ is an ordered topological vector space. Let $\eta:K\times K\to X$ and $F:K\times K\to 2^Y$ be affine mappings such that $\eta(f,f)=F(f,f)=0$, for all $f\in K$. Let $T:K\to L(X,Y)$ be an η-hemicontinuous mapping. If $C=\bigcap_{f\in K}C(f)\neq\emptyset$, T is η-monotone in C, and T_h is defined by $T_h(f)=T(\lambda f+(1-\lambda)h)$, for all $f\in K$, then for each $h\in K,\lambda\in(0,1]$, the following statements are equivalent:*

(i) *Find $f_0\in K$, such that $\langle T_h(f_0),\eta(g,f_0)\rangle+F(g,f_0)\nsubseteq -intC(f_0)$, $\forall g\in K$.*
(ii) *Find $f_0\in K$, suchthat $\langle T_h(g),\eta(g,f_0)\rangle+F(g,f_0)\nsubseteq -intC(f_0)$, $\forall g\in K$.*

Proof. Suppose that (i) holds. We can find $f_0\in K$ such that

$$\langle T_h(f_0),\eta(g,f_0)\rangle+F(g,f_0)\nsubseteq -intC(f_0),\ \forall g\in K.$$

Since T is η-monotone, for each $f,g\in K$, we have

$$\langle T(\lambda g+(1-\lambda)h)-T(\lambda f+(1-\lambda)h),\eta(\lambda g+(1-\lambda)h,\lambda f+(1-\lambda)h)\rangle\in C.$$

Also, since η is affine and $\eta(f,f)=0$, it follows that

$$\begin{aligned}&\langle T_h(g)-T_h(f),\eta(g,f)\rangle\\ =\ &\frac{1}{\lambda}\langle T(\lambda g+(1-\lambda)h)-T(\lambda f+(1-\lambda)h),\eta(\lambda g+(1-\lambda)h,\lambda f+(1-\lambda)h)\rangle\\ \in\ &C.\end{aligned}$$

Hence T_h is also η-monotone, that is,

$$\langle T_h(f_0),\eta(g,f_0)\rangle-\langle T_h(g),\eta(g,f_0)\rangle\in -C,\ \forall g\in K.$$

Note that $C=\bigcap_{f\in K}C(f)$. For all $g\in K$, we have

$$\langle T_h(f_0),\eta(g,f_0)\rangle+F(g,f_0)-\langle T_h(g),\eta(g,f_0)\rangle-F(g,f_0)\subseteq -C\subseteq -C(f_0).$$

By Lemma 3.1 (iv), we have

$$\langle T_h(g),\eta(g,f_0)\rangle+F(g,f_0)\nsubseteq -intC(f_0),\forall g\in K.$$

So, f_0 is a solution of (ii).

Conversely, suppose that (ii) holds. Then there exists $f_0\in K$ such that

$$\langle T_h(g),\eta(g,f_0)\rangle+F(g,f_0)\nsubseteq -intC(f_0),\ \forall g\in K.$$

For each $g\in K,t\in(0,1)$, let

$$g_t=tg+(1-t)f_0.$$

Clearly, $g_t\in K$. It follows that

$$\langle T_h(g_t),\eta(g_t,f_0)\rangle+F(g_t,f_0)\nsubseteq -intC(f_0).$$

Since F and η are affine and $\eta(f_0,f_0)=F(f_0,f_0)=0$, we have

$$\langle T(\lambda(tg+(1-t)f_0)+(1-\lambda)h,\ t\eta(g,f_0)\rangle+tF(g,f_0)\nsubseteq -intC(f_0),$$

that is,

$$\langle T(\lambda f_0+t(g-f_0))+(1-\lambda)h),\eta(g,f_0)\rangle+F(g,f_0)\nsubseteq -intC(f_0).$$

Considering the $\eta-$ hemicontinuity of T and letting $t\to 0^+$, we have

$$\langle T_h(f_0),\eta(g,f_0)\rangle+F(g,f_0)\nsubseteq -intC(f_0),\forall g\in K.$$

Thus (i) holds. This completes the proof. □

Remark 3.3. If $C:K\to 2^Y, F:K\times K\to 2^Y$ are single-valued maps and $C(f)=C,\ \forall f\in K$ and $F(f,g)=0$, for all $f,g\in K$, then the above theorem is reduced to Lemma 5 of [5].

Let $K\subseteq L(X,Y)$ be a closed and convex subset and $C:K\to 2^Y$ such that for all $f\in K$, $C(f)$ is a closed and convex pointed cone with a nonempty interior. Throughout this paper, we define a set-valued map $\overline{C}:K\to 2^Y$ as:

$$\overline{C}(f)=Y\setminus\{-intC(f)\},\ \forall f\in K.$$

Theorem 3.4. *Let $K\subseteq L(X,Y)$ be a nonempty closed and convex subset. Let $C:K\to 2^Y$ be a set-valued map such that, for all $f\in K$, $C(f)$ is a closed and convex pointed cone with a nonempty interior, i.e, $(Y,C(f))$ is an ordered topological vector space. Let $\eta:K\times K\to X$ and $F:K\times K\to 2^Y$ be affine mappings such that, for all $f\in K$, $\eta(f,f)=F(f,f)=0$. Let $T:K\to L(X,Y)$ be an η-hemicontinuous mapping. Assume that the following conditions are satisfied:*

(i) *$C=\cap_{f\in K}C(f)\neq\emptyset$ and T is η-monotone in C,*
(ii) *$\overline{C}:K\to 2^Y$ is an upper-semicontinous set-valued mapping.*

Then, for each $h\in K,\ \lambda\in(0,1]$, there exists $f_0\in K$ such that

$$\langle T(\lambda f_0+(1-\lambda)h),\eta(g,f_0)\rangle+F(g,f_0)\nsubseteq -intC(f_0),\forall g\in K.$$

Proof. For each $g\in K$, we denote $T_h(f)=T(\lambda f+(1-\lambda)h)$, and define

$$F_1(g)=\left\{f\in K:\langle T_h(f),\eta(g,f)\rangle+F(g,f)\nsubseteq -intC(f)\right\},$$

$$F_2(g)=\left\{f\in K:\langle T_h(g),\eta(g,f)\rangle+F(g,f)\nsubseteq -intC(f)\right\}.$$

Then $F_1(g)$ and $F_2(g)$ are nonempty due to $g\in F_1(g)$ and $g\in F_2(g)$. The proof of this theorem is divided into the following three steps:

Step I. We first prove that F_1 is a KKM-mapping. Suppose on the contrary that F_1 is not a KKM-mapping. Then there exists $f_1,f_2,\cdots,f_m\in K,\ t_1\geq 0,t_2\geq 0,\cdots,t_m\geq 0$ with $\sum_{i=1}^m t_i=1$ and $w=\sum_{i=1}^m t_if_i$ such that

$$w\notin\bigcup_{i=1}^m F_1(f_i),\ i=1,2,\cdots,m,$$

that is,

$$\langle\ T_h(w),\eta(f_i,w)\ \rangle+F(f_i,w)\subseteq -intC(w),\forall i=1,2,\cdots,m.$$

Since η and F are affine, we have

$$\begin{aligned}\langle T_h(w), \eta(w,w)\rangle + F(w,w) &= \langle T_h(w), \eta(\sum_{i=1}^{m} t_i f_i,\ w)\rangle + F(\sum_{i=1}^{m} t_i f_i, w)\\ &= \sum_{i=1}^{m} t_i(\langle T_h(w), \eta(f_i,w)\rangle + F(f_i,w))\\ &\subseteq -intC(w).\end{aligned}$$

Since $\eta(w,w) = F(w,w) = 0$, we have

$$\{0\} = \langle T_h(w), \eta(w,w)\rangle + F(w,w) \subseteq -intC(w),$$

which is a contraction. So, $F_1 : K \to 2^K$ is a KKM-mapping.

Step II. Further, we prove that

$$\bigcap_{g\in K} F_1(g) = \cap_{g\in K} F_2(g).$$

For this, first let $f \in F_1(g)$. Then we have

$$\langle T_h(f), \eta(g,f)\rangle + F(g,f) \nsubseteq -intC(f).$$

From Theorem 3.2, T_h is η-monotone in $C(h)$, it follows that

$$\langle T_h(g) - T_h(f), \eta(g,f)\rangle \in C.$$

Hence,

$$\langle T_h(f), \eta(g,f)\rangle + F(g,f) - \langle T_h(g), \eta(g,f)\rangle - F(g,f) \subseteq -C \subseteq -C(f).$$

By Lemma 3.1, we have

$$\langle T_h(g), \eta(g,f)\rangle + F(g,f) \nsubseteq -intC(f).$$

So, $f \in F_2(g)$, for each $g \in K$, that is, $F_1(g) \subseteq F_2(g)$. It follows that

$$\bigcap_{g\in K} F_1(g) \subseteq \bigcap_{g\in K} F_2(g).$$

Next, we suppose that $f \in \cap_{g\in K} F_2(g)$. Then

$$\langle T_h(g), \eta(g,f)\rangle + F(g,f) \nsubseteq -intC(f),\ \forall g \in K.$$

It follows from Theorem 3.2 that

$$\langle T_h(f), \eta(g,f)\rangle + F(g,f) \nsubseteq -intC(f),\ \forall g \in K,$$

that is, $f \in \cap_{g\in K} F_1(g)$. Hence

$$\bigcap_{g\in K} F_2(g) \subseteq \bigcap_{g\in K} F_1(g),$$

which implies that

$$\bigcap_{g\in K} F_1(g) = \bigcap_{g\in K} F_2(g).$$

Step III. Finally, we prove that

$$\bigcap_{g\in K} F_2(g) \neq \emptyset.$$

Since F_1 is a KKM mapping, for any finite set $\{g_1, g_2, ..., g_n\} \subseteq K$, we have

$$conv\{g_1, g_2, \cdots, g_n\} \subseteq \cup_{i=1}^n F_1(g_i) \subseteq \cup_{i=1}^n F_2(g_i).$$

This shows that F_2 is also a KKM-mapping. Now we prove that $F_2(g)$ is closed for all $g \in K$. Assume that there exists a net $\{f_\alpha\} \subseteq F_2(g)$ with $f_\alpha \to f \in K$. Then

$$\langle T_h(f), \eta(g, f_\alpha)\rangle + F(g, f_\alpha) \nsubseteq -intC(f_\alpha).$$

Using the definition of $\overline{C}$, we have

$$\langle T_h(g), \eta(g, f_\alpha)\rangle + F(g, f_\alpha) \subseteq \overline{C}(f_\alpha).$$

Since η and F are continuous, we have

$$\langle T_h(g), \eta(g, f_\alpha)\rangle + F(g, f_\alpha) \to \langle T_h(g), \eta(g, f)\rangle + F(g, f).$$

Since $\overline{C}$ is upper-semicontinuous mapping with closed values, by Lemma 2.6, $\overline{C}$ is closed, and so

$$\langle T_h(g), \eta(g, f)\rangle + F(g, f) \subseteq \overline{C}(f),$$

$$\langle T_h(g), \eta(g, f)\rangle + F(g, f) \nsubseteq -intC(f).$$

This proves that $F_2(g)$ is closed.

Considering the compactness of K and closedness of $F_2(g) \subseteq K$, we know that $F_2(g)$ is compact. By Lemma 2.4, we have $\cap_{g\in K} F_2(g) \neq \emptyset$, which implies that $\cap_{g\in K} F_1(g) \neq \emptyset$, that is, for each $h \in K$ and $\lambda \in (0, 1]$, there exists $f_0 \in K$ such that

$$\langle T(\lambda f_0 + (1-\lambda)h, \eta(g, f_0)\rangle + F(g, f_0) \nsubseteq -intC(f_0), \forall g \in K.$$

Thus η-GOVLI is solvable. This completes the proof. □

Remark 3.5. If $C : K \to 2^Y, F : K \times K \to 2^Y$ are single-valued maps and for all $f, g \in K, C(f) = C$ and $F(f, g) = 0$, then it easy to see that Theorem 3.4 is a generalization of Theorem 6 in [5].

Under some suitable conditions, we prove a new existence result of solution of η-GOVLI without the compactness of K.

Theorem 3.6. *Let $K \subseteq L(X,Y)$ be a nonempty closed and convex set and let $C : K \to 2^Y$ be a set-valued map such that, for all $f \in K$, $C(f)$ is a closed and convex pointed cone with a nonempty interior, i.e, $(Y, C(f))$ is an ordered topological vector space. Let $\eta : K \times K \to X$ and $F : K \times K \to 2^Y$ be affine mappings such that, for all $f \in K$, $\eta(f, f) = F(f, f) = 0$. Let $T : K \to L(X,Y)$ be an η-hemicontinuous mapping. Assume that the following conditions are satisfied:*

(i) *$C = \cap_{f\in K} C(f) \neq \emptyset$ and T is η-monotone in C,*
(ii) *$\overline{C} : K \to 2^Y$ is an upper-semicontinous set-valued mapping,*
(iii) *there exists a nonempty compact and convex subset D of K and for each $h \in K, \lambda \in (0, 1], f \in K \setminus D$, there exists $g_0 \in D$ such that*

$$\langle T(\lambda g_0 + (1-\lambda)h, \eta(g_0, f)\rangle + F(g_0, f) \subseteq -intC(f).$$

Then, for each $h \in K, \lambda \in (0, 1]$, there exists $f_0 \in D$ such that

$$\langle T(\lambda f_0 + (1-\lambda)h, \eta(g, f_0)\rangle + F(g, f_0) \nsubseteq -intC(f_0),\ \forall g \in K. \tag{3.1}$$

Proof. We know that the solution set of problem (ii) in Theorem 3.2 is equivalent to finding the solution of the following variational inequality: find $f \in K$ such that

$$\langle T(\lambda g+(1-\lambda)h), \eta(g,f)\rangle + F(g,f) \nsubseteq -intC(f),\ \forall g \in K.$$

For each $h \in K$ and $\lambda \in (0,1]$, we denote $T_h(f) = T(\lambda f+(1-\lambda)h)$. Let $S : K \to 2^D$ be defined by

$$S(g) = \left\{ f \in D : \langle T_h(g), \eta(g,f)\rangle + F(g,f) \nsubseteq -intC(f) \right\},\ \forall g \in K.$$

Clearly, for each $g \in K$,

$$S(g) = \left\{ f \in K : \langle T_h(g), \eta(g,f)\rangle + F(g,f) \nsubseteq -intC(f) \right\} \cap D.$$

Also, it is easy to see that $S(g)$ is a closed subset of D. Since D is compact, by using the proof of Theorem 3.4, $S(G)$ is compact.

Now we prove that family $\{S(g)\}_{g\in K}$ has the finite intersection property, that is, for any finite set $\{g_1, g_2, \cdots, g_n\} \subseteq K$, one has

$$\bigcap_{i=1}^{n} S(g_i) \neq \emptyset.$$

For this, let $G_n = \cup_{i=1}^{n}\{g_i\}$. Since Y is a Hausdorff topological vector space, $L(X,Y)$ is also Hausdorff and so for each $g_i \in \{g_1, g_2, \cdots, g_n\}$, $\{g_i\}$ is compact and convex. Let $B = conv(D \cup G_n)$. Clearly, $B = conv(D \cup \{g_1, g_2, \cdots, g_n\}) = conv(D \cup \{g_1\}, \{g_2\}, \cdots, \{g_n\})$ and so by Lemma 2.5, B is a compact and convex subset of K. Let $F_1, F_2 : B \to 2^B$ be defined as follows:

$$F_1(g) = \left\{ f \in B : \langle T_h(f), \eta(g,f)\rangle + F(g,f) \nsubseteq -intC(f) \right\}, \forall g \in B,$$

$$F_2(g) = \left\{ f \in B : \langle T_h(g), \eta(g,f)\rangle + F(g,f) \nsubseteq -intC(f) \right\}, \forall g \in B.$$

Then, by using the proof of Theorem 3.4, we obtain

$$\bigcap_{g\in B} F_1(g) = \bigcap_{g\in B} F_2(g) \neq \emptyset.$$

So, there exists $g_0 \in \cap_{g\in B} F_2(g)$.

Next we prove that $g_0 \in D$. For, this suppose on the contrary, that $g_0 \notin D$. Then $g_0 \in K \setminus D$, therefore by given assumption(iii), there exists $l \in D$ such that

$$\langle T(\lambda l+(1-\lambda)h, \eta(l,g_0)\rangle + F(l,g_0) \subseteq -intC(g_0),$$

that is,

$$\langle T_h(l), \eta(l,g_0)\rangle + F(l,g_0) \subseteq -intC(g_0)$$

This implies $g_0 \notin F_2(l)$, which is a contradiction to the fact that $g_0 \in \cap_{g\in B} F_2(g)$. Thus $g_0 \in D$. Since

$$\{g_1, g_2, \cdots, g_n\} \subseteq B \textit{ and } S(g_i) = F_2(g_i) \cap D,$$

for each $g_i \in \{g_1, g_2, \cdots, g_n\}$, we have $g_0 \in S(g_i)$, and so $g_0 \in \bigcap_{i=1}^{n} S(g_i)$. This implies $\bigcap_{i=1}^{n} S(g_i) \neq \emptyset$. This proves that the family $\{S(g)\}_{g\in K}$ has the finite intersection property. Therefore, by using the compactness of $S(g)$ for all $g \in K$, we have $\bigcap_{g\in K} S(g) \neq \emptyset$. That is, there exists $f_0 \in D$ such that $f_0 \in \bigcap_{g\in K} S(g)$. Thus

$$\langle T(\lambda f_0+(1-\lambda)h, \eta(g,f_0)\rangle + F(g,f_0) \nsubseteq -intC(f_0), \forall g \in K.$$

Hence the solution set of η-GOVLI is nonempty. This completes the proof. □

Now we prove the following result for the solution of η-GSOVLI (1.2) under some suitable conditions by using the Fan KKM theorem.

Theorem 3.7. *Let X and Y be two Hausdorff topological vector spaces. Let $K \subseteq L(X,Y)$ be nonempty closed and convex set and let $C: K \to 2^Y$ be a set-valued map such that, for all $f \in K$, $C(f)$ is a closed and convex pointed cone with a nonempty interior, i.e, $(Y, C(f))$ is an ordered topological vector space. Assume that, for each $g \in K$, $f \to \eta(f,g)$ and $f \to F(f,g)$ are affine, $\eta(f,g)+\eta(g,f) = 0$ and $F(f,g)+F(g,f) = 0$, for all $f \in K$. Let $T: K \to L(X,Y)$ be a mapping such that*

(i) *for each $h, g \in K$, $\lambda \in (0,1]$, the set*

$$\{f \in K : \langle T(\lambda f + (1-\lambda)h), \eta(g,f)\rangle + F(g,f) \subseteq -C(f)\setminus\{0\}\}$$

is open in K,

(ii) *there exists a nonempty compact and convex subset D of K and for each $h \in K$, $\lambda \in (0,1]$, $f \in K \setminus D$ there exists $g_0 \in D$ such that*

$$\langle T(\lambda f + (1-\lambda)h), \eta(g_0,f)\rangle + F(g_0,f) \subseteq -C(f)\setminus\{0\}.$$

Then for each $h \in K$, $\lambda \in (0,1]$, there exists $f_0 \in D$ such that

$$\langle T(\lambda f_0 + (1-\lambda h), \eta(g,f_0)\rangle + F(g,f_0) \nsubseteq -C(f_0)\setminus\{0\}, \forall g \in K.$$

Proof. For each $h \in K$ and $\lambda \in (0,1]$, we denote $T_h(f) = T(\lambda f + (1-\lambda)h$. Let $S: K \to 2^D$ be defined as follows

$$S(g) = \left\{f \in D : \langle T_h(f), \eta(g,f)\rangle + F(g,f) \nsubseteq -C(f)\setminus\{0\}\right\}, \ \forall g \in K.$$

Clearly,

$$S(g) = \left\{f \in K : \langle T_h(f), \eta(g,f)\rangle + F(g,f) \nsubseteq -C(f)\setminus\{0\}\right\} \cap D, \ \forall g \in K.$$

From condition (i), $S(g)$ is closed subset of D. Since D is compact, $S(g)$ is a compact.

Now we prove that the family $\{S(g)\}_{g\in K}$ has the finite intersection property, that is, for any finite set $\{g_1, g_2, \ldots.., g_n\} \subseteq K$, we have $\bigcap_{i=1}^n S(g_i) \neq \emptyset$. For this, let $G_n = \bigcup_{i=1}^n \{g_i\}$. Since Y is a Hausdorff topological vector space, L(X,Y) is also Hausdorff and so for each $g_i \in \{g_1, g_2, \ldots, g_n\}$, $\{g_i\}$ is compact and convex. Let $B = conv(D \cup G_n)$. Clearly,

$$B = conv(D \cup \{g_1, g_2, \ldots, g_n\}) = conv(D \cup \{g_1\}, \{g_2\}, \ldots, \{g_n\}).$$

Using Lemma 2.5, B is a compact and convex subset of K. Let $F: B \to 2^B$ be defined as follows:

$$F(g) = \left\{f \in B : \langle T_h(f), \eta(g,f) + F(g,f) \nsubseteq -C(f)\setminus\{0\}\right\}, \ \forall g \in B.$$

Then, we claim that F is a KKM mapping. Suppose on the contrary, that is, F is not a KKM mapping. Then there exists $l_1, l_2, \ldots., l_m \in K, t_1 \geq 0, t_2 \geq 0, \ldots., t_m \geq 0$ with $\sum_{i=1}^m t_i = 1$ such that $conv\{l_1, l_2, \ldots., l_m\} \nsubseteq \bigcup_{i=1}^n F(l_i)$. This means that there exists $w = \sum_{i=1}^m t_i l_i$ such that $w \notin \bigcup_{i=1}^m F(l_i)$. Therefore,

$$\langle T_h(w), \eta(l_i,w)\rangle + F(l_i,w) \subseteq -C(w)\setminus\{0\}.$$

Since η and F are affine,

$$\begin{aligned}\langle T_h(w),\eta(w,w)\rangle+F(w,w) &= \langle T_h(w),\eta(\sum_{i=1}^{m}t_il_i,w)\rangle+F(\sum_{i=1}^{m}t_il_i,w)\\ &= \sum_{i=1}^{m}t_i\langle T_h(w),\eta(l_i,w)\rangle+F(l_i,w)\\ &\subseteq -C(w)\setminus\{0\}.\end{aligned}$$

Since $\eta(w,w)=F(w,w)=0$, we have

$$0=\langle T_h(w),\eta(w,w)\rangle+F(w,w)\subseteq -C(w)\setminus\{0\},$$

which is a contradiction. Therefore, $F:B\to 2^B$ is a KKM mapping. Since $F(g)$ is a closed subset of B, it follows from Theorem 3.4 that $F(g)$ is compact. By Lemma 2.4, we have

$$\bigcap_{g\in B}F(g)\neq\emptyset.$$

Thus there exists $g_0\in\bigcap_{g\in B}F(g)$.

Next we prove that $g_0\in D$. Suppose that $g_0\in B\setminus D$. Then the condition (ii) implies that there exists $l\in D$ such that

$$\langle T(\lambda g_0+(1-\lambda)h),\eta(l,g_0)\rangle+F(l,g_0)\subseteq -C(g_0)\setminus\{0\},$$

This implies $g_0\notin F(l)$, which is a contradiction to the fact that $\bigcap_{g\in B}F(g)\neq\phi$. So we have $g_0\in D$. Since $\{g_1,g_2,....,g_n\}\subseteq B$ and $S(g_i)=F(g_i)\cap D$, for each $g_i\in\{g_1,g_2,....,g_n\}$, it follows $g_0\in\bigcap_{i=1}^{n}S(g_i)$. This proves that the family $\{S(g)\}_{g\in K}$ has the finite intersection property. Using the compactness of $S(g)$, we have

$$\bigcap_{g\in K}S(g)\neq\emptyset.$$

This means that, there exists $f_0\in D$ such that $f_0\in S(g)$ for each $g\in K$, that is, for each $h\in K,\lambda\in(0,1]$, there exists $f_0\in K$ such that

$$\langle T(\lambda f_0+(1-\lambda)h,\eta(g,f_0)\rangle+F(g,f_0)\nsubseteq -C(f_0)\setminus\{0\},\forall g\in K.$$

Thus η-GSOVLI is solvable. This completes the proof. □

Remark 3.8. If $C:K\to 2^Y, F:K\times K\to 2^Y$ are two single-valued maps, K is compact, $C(f)=C$ and $\lambda=1$, then Theorem 3.7 reduces to Theorem 2.1 in [6].

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