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ON SPACES OF INTUITIONISTIC FUZZY ZWEIER LACUNARY IDEAL CONVERGENCE OF SEQUENCES

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Abstract. The concept of I-convergence, introduced and studied by Kostyrko, Šalát and Wilczyński in metric spaces, has wide applications in the field of number theory, trigonometric series, summability theory, probability theory, optimization and approximation theory. In this article, we introduce some new spaces of intuitionistic fuzzy Lacunary ideal convergent sequence via zwier operators. We introduce some new spaces of intuitionistic fuzzy Lacunary ideal convergent sequence via Zweier operators and study some algebraic and topological properties on these spaces.

Keywords. Intuitionistic fuzzy normed space; I_θ-convergence; Topology; Zweier sequence space.

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1. INTRODUCTION

Fuzzy set theory and fuzzy analogues of the classical theories have been considered by many researchers in recent decades. One of the most important contributions on fuzzy sets is the work carried out by Zadeh [1]. Since then different methodologies were involved for the development of these theories. One of the most important developments is the intuitionistic fuzzy normed space provided by [2]. It is known that the definition of I-convergence, is a generalization of statistical convergence [3] which was introduced in 1999; see [4] and the references therein. Vakeel *et al.* [5] base on the definition defined some new spaces of intuitionistic fuzzy normed I-convergence of sequences with the help of the Zweier operator. In 2012, Debhnath [6] introduced and studied the notion of the Lacunary ideal convergence in intuitionistic fuzzy normed linear spaces as a variant of the notion of the ideal convergence.

In this article, we introduce some new spaces of intuitionistic fuzzy Lacunary ideal convergent sequence via zwier operators and study some algebraic and topological properties on these spaces.

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2. PRELIMINARIES

In this section, we present some preliminary definitions and results related to intuitionistic fuzzy normed spaces and that will be used throughout the article.

Definition 2.1. [2] The five-tuple $(X,\mu,\nu,*,\diamond)$ is said to be an intuitionistic fuzzy normed space (for short, IFNS) if X is a vector space, $*$ is a continuous t–norm, $\diamond$ is a continuous t–conorm and μ,ν are fuzzy sets on $X\times(0,\infty)$ satisfying the following conditions, $\forall x,y\in X$ and $s,t>0$:

- **(1):** $\mu(x,t)+\nu(x,t)\leq 1$,
- **(2):** $\mu(x,t)>0$,
- **(3):** $\mu(x,t)=1$ if and only if $x=0$,
- **(4):** $\mu(\alpha x,t)=\mu(x,\frac{t}{|\alpha|})$ for each $\alpha\neq 0$,
- **(5):** $\mu(x,t)*\mu(y,s)\leq\mu(x+y,t+s)$,
- **(6):** $\mu(x,\cdot):(0,\infty)\to[0,1]$ is continuous,
- **(7):** $\lim_{t\to\infty}\mu(x,t)=1$ and $\lim_{t\to 0}\mu(x,t)=0$,
- **(8):** $\nu(x,t)<1$,
- **(9):** $\nu(x,t)=0$ and if and only if $x=0$,
- **(10):** $\nu(\alpha x,t)=\nu(x,\frac{t}{|\alpha|})$ for each $\alpha\neq 0$,
- **(11:** $\nu(x,t)\diamond\nu(y,s)\geq\nu(x+y,t+s)$,
- **(12):** $\nu(x,\cdot):(0,\infty)\to[0,1]$ is continuous,
- **(13):** $\lim_{t\to\infty}\nu(x,t)=0$ and $\lim_{t\to 0}\nu(x,t)=1$.

In this case (μ,ν) is called an intuitionistic fuzzy norm.

Definition 2.2. [2] Let $(X,\mu,\nu,*,\diamond)$ be an intuitionistic fuzzy normed space. A sequence $x=(x_k)$ in X is said to be convergent to $L\in X$ with respect to the intuitionistic fuzzy norm (μ,ν) if, for every $\varepsilon\in(0,1)$ and $t>0$, there exists $k_0\in\mathbb{N}$ such that $\mu(x_k-L,t)>1-\varepsilon$ and $\nu(x_k-L,t)<\varepsilon$ for all $k\geq k_0$. It is denoted by (μ,ν)–$\lim x=L$.

Definition 2.3. [2] Let $(X,\mu,\nu,*,\diamond)$ be an intuitionistic fuzzy normed space. A sequence $x=(x_k)$ in X is said to be a Cauchy sequence with respect to the intuitionistic fuzzy norm (μ,ν) if, for every $\varepsilon\in(0,1)$ and $t>0$, there exists $k_0\in\mathbb{N}$ such that $\mu(x_k-x_m,t)>1-\varepsilon$ and $\nu(x_k-x_m,t)<\varepsilon$ for all $k,m\geq k_0$.

Definition 2.4. [2] Let $(X,\mu,\nu,*,\diamond)$ be an intuitionistic fuzzy normed space. For $t>0$, the open ball $B(x,r,t)$ with center $x\in X$ and radius $r\in(0,1)$ is defined as

$$B(x,r,t)=\{y\in X:\mu(x-y,t)>1-r \text{ and } \nu(x-y,t)<r\}.$$

Definition 2.5. [4] Let X be a non–empty set. Then a family $I\subset P(X)$ of subsets of X is called an ideal in X if and only if

- **(i):** $\emptyset\in I$,
- **(ii):** $A,B\in I$ implies $A\cup B\in I$,
- **(iii):** for each $A\in I$ and $B\subset A$ we have $B\in I$,

where $P(X)$ is the power set of X.

Definition 2.6. [4] Let X be a non–empty set. Then a family of sets $\mathscr{F}\subset P(X)$ is called a filter on X if and only if

(i): $\emptyset \notin \mathscr{F}$,
(ii): $A, B \in \mathscr{F}$ implies that $A \cap B \in \mathscr{F}$,
(iii): for each $A \in \mathscr{F}$ and $B \supset A$ we have $B \in \mathscr{F}$.

An ideal I is said to be non-trivial if $I \neq P(X)$. A non-trivial ideal $I \subset P(X)$ is called an admissible ideal in X if and only if it contains all singletons, i.e., if $I \supset \{\{x\} : x \in X\}$.

Remark 2.7. [4] For each ideal I, there is a filter $\mathscr{F}(I)$ which corresponding to I (filter associate with ideal I), that is,

$$\mathscr{F}(I) = \{K \subseteq X : K^c \in I\}, \text{ where } K^c = X \backslash K. \tag{2.1}$$

Definition 2.8. [7] A Lacunary sequence is an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ is denoted by $J_r = (k_{r-1}, k_r]$ here.

Definition 2.9. [6] Let $I \subset P(\mathbb{N})$, and let $(X, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy normed space. A sequence $x = (x_k)$ in X is said to be Lacunary I–convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\varepsilon \in (0,1)$ and $t > 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \mu(x_k - L, t) \leq 1 - \varepsilon \text{ or } \frac{1}{h_r} \sum_{k \in J_r} \nu(x_k - L, t) \geq \varepsilon \right\} \in I.$$

L is called the Lacunary I-limit of the sequence $x = (x_k)$, and we write $I_\theta^{(\mu,\nu)}$–$\lim x = L$.

Definition 2.10. [6] Let $(X, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy normed space. A sequence $x = (x_k)$ in X is said to be Lacunary I–Cauchy with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\varepsilon \in (0,1)$ and $t > 0$, there exists $m \in \mathbb{N}$ satisfying

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \mu(x_k - x_m, t) \leq 1 - \varepsilon \text{ or } \frac{1}{h_r} \sum_{k \in J_r} \nu(x_k - x_m, t) \geq \varepsilon \right\} \in I.$$

The approach of constructing new sequence spaces by means of the matrix domain of a particular limitation method have been recently employed by Altay, Basar and Mursaleen [8], Basar and Altay [9], Malkowsky [10], Ng and Lee [11], and Wang [12]. In 2007, Sengonul [13] defined the sequence $y = (y_i)$ which is frequently used the Z^p transform of the sequence $x = (x_i)$ i.e, $y_i = px_i + (1-p)x_{i-1}$ where $x - 1 = 0$, $p \mp 1, 1 < p < \infty$ and Z^P denoted the matrix $Z^P = (z_{ik})$ defined by

$$z_{ik} = \begin{cases} p, & \text{if } i = k, \\ 1 - p, & \text{if } i - 1 = k, i, k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Based on Basar and Altay [10], Sengonul [13] introduced the Zweier sequence spaces Z and Z_0. Recently, with the help of the notion I-convergence, Khan and Ebadullah [14] introduced the following Zweier ideal convergence sequence spaces

$$Z^I = \left\{ (x_k) \in \omega : \{k \in \mathbb{N} : |x_k^{'} - L| \geq \varepsilon\} \in I \right\},$$

$$Z_0^I = \left\{ (x_k) \in \omega : \{k \in \mathbb{N} : |x_k^{'}| \geq \varepsilon\} \in I \right\},$$

where $(x_k^{'}) = Z^p(x_k)$.

3. Zweier $I_\theta^{(\mu,\nu)}$-Convergent Sequence Spaces

Following Khan and Ebadullah [14], and using the Zweier operator and I_θ–convergence of sequences in an intuitionistic fuzzy normed spaces, we introduce the following new Zweier sequence spaces and examine some algebraic and topological properties on these spaces. Throughout the article, for the sake of convenience, we denote by $Z^p(x_k) = x^{'}$, $Z^p(y_k) = y^{'}$, $Z^p(z_k) = z^{'}$, where $x = (x_k)$, $y = (y_k)$, $z = (z_k)$ are in ω. Let

$$Z^{I_\theta}_{(\mu,\nu)} = \left\{ x = (x_k) \in \omega : \left\{ k \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \mu(x_k^{'} - L, t) \leq 1 - \varepsilon \text{ or } \frac{1}{h_r} \sum_{k \in J_r} \nu(x_k^{'} - L, t) \geq \varepsilon \right\} \in I \right\} \quad (3.1)$$

and

$$Z^{I_\theta}_{0(\mu,\nu)} = \left\{ x = (x_k) \in \omega : \left\{ k \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \mu(x_k^{'}, t) \leq 1 - \varepsilon \text{ or } \frac{1}{h_r} \sum_{k \in J_r} \nu(x_k^{'}, t) \geq \varepsilon \right\} \in I \right\}. \quad (3.2)$$

Also, we define an open ball with center $x^{'}$ and radius r with respect to t as follows:

$$B(x^{'}, r, t) = \left\{ y = (y_k) \in \omega : \left\{ k \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \mu(x^{'} - y^{'}, t) \leq 1 - \varepsilon \text{ or } \frac{1}{h_r} \sum_{k \in J_r} \nu(x^{'} - y^{'}, t) \geq \varepsilon \right\} \in I \right\}. \quad (3.3)$$

Theorem 3.1. *$Z^{I_\theta}_{(\mu,\nu)}$ and $Z^{I_\theta}_{0(\mu,\nu)}$ are linear spaces.*

Proof. We prove the case for $Z^{I_\theta}_{0(\mu,\nu)}$. The proof for other spaces are not hard to derive. Let $x^{'} = (x_k^{'})$, $y^{'} = (y_k^{'}) \in Z^{I_\theta}_{(\mu,\nu)}$ and let α, β be scalars. For a given $\varepsilon > 0$, we have

$$A_1 = \left\{ k \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \mu\left(x_k^{'} - L_1, \frac{t}{2|\alpha|} \right) \leq 1 - \varepsilon \text{ or } \frac{1}{h_r} \sum_{k \in J_r} \nu\left(x_k^{'} - L_1, \frac{t}{2|\alpha|} \right) \geq \varepsilon \right\} \in I,$$

$$A_2 = \left\{ k \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \mu\left(y_k^{'} - L_2, \frac{t}{2|\beta|} \right) \leq 1 - \varepsilon \text{ or } \frac{1}{h_r} \sum_{k \in J_r} \nu\left(y_k^{'} - L_2, \frac{t}{2|\alpha|} \right) \geq \varepsilon \right\} \in I,$$

$$A_1^c = \left\{ k \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \mu\left(x_k^{'} - L_1, \frac{t}{2|\beta|} \right) > 1 - \varepsilon \text{ or } \frac{1}{h_r} \sum_{k \in J_r} \nu\left(x_k^{'} - L_1, \frac{t}{2|\alpha|} \right) < \varepsilon \right\} \in F(I),$$

$$A_2^c = \left\{ k \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \mu\left(y_k^{'} - L_2, \frac{t}{2|\beta|} \right) > 1 - \varepsilon \text{ or } \frac{1}{h_r} \sum_{k \in J_r} \nu\left(y_k^{'} - L_2, \frac{t}{2|\alpha|} \right) < \varepsilon \right\} \in F(I).$$

Define the set $A_3 = A_1 \cap A_2$ so that $A_3 \in I$. It follows that A_3^c is a non-empty set in $\mathscr{F}(I)$. we next show, for each $(x_k^{'}), (y_k^{'}) \in Z^{I_\theta}(\mu, \nu)$, that

$$A_3^c \subset \left\{ k \in \mathbb{N} : \frac{1}{h_{r,}} \sum_{k \in J_r} \mu\left((\alpha x_k^{'} + \beta y_k^{'}) - (\alpha L_1 + \beta L_2), t) \right) > 1 - \varepsilon, \right.$$
$$\left. \text{or } \frac{1}{h_r} \sum_{(k) \in J_r} \nu\left((\alpha x_k^{'} + \beta y_k^{'}) - (\alpha L_1 + \beta L_2), t \right) < \varepsilon \right\}.$$

Let $m \in A_3^c$. In this case

$$\frac{1}{h_r} \sum_{m \in J_r} \mu(x_m^{'} - L_1, \frac{t}{2|\alpha|}) > 1 - \varepsilon \text{ or } \frac{1}{h_r} \sum_{m \in J_r} \nu(x_m^{'} - L_1, \frac{t}{2|\alpha|}) < \varepsilon$$

and

$$\frac{1}{h_r}\sum_{m\in J_r}\mu(y'_m-L_2,\frac{t}{2|\beta|})>1-\varepsilon \text{ or } \frac{1}{h_r}\sum_{m\in J_r}\nu(y'_m-L_2,\frac{t}{2|\beta|})<\varepsilon.$$

We have

$$\begin{aligned}
&\frac{1}{h_r}\sum_{m\in J_r}\mu\Big((\alpha x'_m+\beta y'_m)-(\alpha L_1+\beta L_2),t\Big)\\
&\quad\geq \frac{1}{h_r}\sum_{m\in J_r}\mu(\alpha x'_m-\alpha L_1,\frac{t}{2})*\frac{1}{h_r}\sum_{m\in J_r}\mu(\beta y'_m-\beta L_2,\frac{t}{2})\\
&\quad= \frac{1}{h_r}\sum_{m\in J_r}\mu(x'_m-L_1,\frac{t}{2|\alpha|})*\frac{1}{h_r}\sum_{m\in J_r}\mu(y'_m-L_2,\frac{t}{2|\beta|})\\
&\quad>(1-\varepsilon)*(1-\varepsilon)=(1-\varepsilon),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{h_r}\sum_{m\in J_r}\nu\Big((\alpha x'_m+\beta y'_m)-(\alpha L_1+\beta L_2),t\Big)\\
&\quad\leq \frac{1}{h_r}\sum_{m\in J_r}\nu(\alpha x'_m-\alpha L_1,\frac{t}{2})\diamond\frac{1}{h_r}\sum_{m\in J_r}\nu(\beta y'_m-\beta L_2,\frac{t}{2})\\
&\quad= \frac{1}{h_r}\sum_{m\in J_r}\nu(x'_m-L_1,\frac{t}{2|\alpha|})\diamond\frac{1}{h_r}\sum_{m\in J_r}\nu(y'_m-L_2,\frac{t}{2|\beta|})\\
&\quad>\varepsilon\diamond\varepsilon=\varepsilon,
\end{aligned}$$

which imply that

$$A_3^c\subset\left\{k\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in J_r}\mu((\alpha x'_k+\beta y'_k)-(\alpha L_1+\beta L_2))>1-\varepsilon \text{ or } \frac{1}{h_r}\sum_{(k\in J_r}\nu((x'_k+\beta y'_k)-(\alpha L_1+\beta L_2))<\varepsilon\right\}.$$

Hence $Z^{I_\theta}_{(\mu,\nu)}$ is a linear space. □

Theorem 3.2. *Every open ball $B(x',t,r)$ is open in $Z^{I_\theta}_{(\mu,\nu)}$.*

Proof. Let $B(x',t,r)$ be an open ball with center x' and radius r with respect to t, that is,

$$B(x',t,r)=\{k\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in J_r}\mu(x'_k-L,t)\leq 1-r \text{ or } \frac{1}{h_r}\sum_{k\in J_r}\nu(x'_k-L,t)\geq r\}\in I.$$

Letting $y'\in B^c(x',t,r)$, one has

$$\frac{1}{h_r}\sum_{k\in J_r}\mu(x'-y',t)>1-r \text{ and } \frac{1}{h_r}\sum_{k\in J_r}\nu(x'-y',t)<r.$$

Since $\mu(x'-y',t)>1-r$, there exists $t_0\in(0,1)$ such that

$$\mu(x'-y',t_0)>1-r \text{ and } \frac{1}{h_r}\sum_{k\in J_r}\frac{1}{h_r}\sum_{k\in J_r}\nu(x'-y',t_0)<r.$$

Putting $r_0=\frac{1}{h_r}\sum_{k\in J_r}\mu(x'-y',t_0)$, we have $r_0>1-r$, and there exists $s\in(0,1)$ such that

$$r_0>1-s>1-r.$$

For $r_0>1-s$, we have

$$r_0*r_1>1-s \text{ and } (1-r_0)\diamond(1-r_2)\leq s,$$

where $r_1, r_2 \in (0,1)$. Put $r_3 = \max\{r_1, r_2\}$. Considering the ball $B^c(y', 1-r_3, t-t_0)$, we prove that

$$B^c(y', 1-r_3, t-t_0) \subset B^c(x', r, t).$$

Letting $z' \in B^c(y', 1-r_3, t-t_0)$, we have

$$\frac{1}{h_r}\sum_{k\in J_r} \mu(y'-z', t-t_0) > r_3 \text{ and } \frac{1}{h_r}\sum_{k\in J_r} \nu(y'-z', t-t_0) < r_3.$$

Therefore

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in J_r} \mu(x'-z', t) &\geq \frac{1}{h_r}\sum_{k\in J_r} \mu(x'-y', t_0) * \frac{1}{h_r}\sum_{k\in J_r} \mu(y'-z', t-t_0)\\ &\geq (r_0 * r_3) \geq (r_0 * r_1) \geq (1-s) > (1-r)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in J_r} \nu(x'-z', t) &\leq \frac{1}{h_r}\sum_{k\in J_r} \nu(x'-y', t_0) \diamond \frac{1}{h_r}\sum_{k\in J_r} \nu(y'-z', t-t_0)\\ &\leq (1-r_0) \diamond (1-r_3) \leq (1-r_0 > (1-r_2) < s < r.\end{aligned}$$

Thus $z' \in B^c(x', t, r)$ and hence $B^c(y', 1-r_3, t-t_0) \subset B^c(x', t, r)$. □

Remark 3.3. $Z^{I_\theta}_{(\mu,\nu)}$ is *IFNS*.

Define $\tau_{(\mu,\nu)} = \{A \subset Z^{I_\theta}_{(\mu,\nu)}$ for each $x \in A, \exists t > 0$ and $r \in (0,1)$ s.t $B^c(x', t, r) \subset A\}$. Then $\tau_{(\mu,\nu)}$ is a topology on $Z^{I_\theta}_{(\mu,\nu)}$.

Theorem 3.4. *The topology $\tau_{(\mu,\nu)}$ on $Z^{I_\theta}_{(\mu,\nu)}$ is first countable.*

Proof. $\{B(x', \frac{1}{n}, \frac{1}{n}) : n = 1,2,3,\ldots\}$ is a local base at x' the topology $\tau_{(\mu,\nu)}$ which is first countable. □

Theorem 3.5. *$Z^{I_\theta}_{(\mu,\nu)}$ and $Z^{I_\theta}_{0(\mu,\nu)}$ are Hausdroff spaces.*

Proof. We prove the case for $Z^{I_\theta}_{(\mu,\nu)}$. Similarly, we can prove the case for $Z^{I_\theta}_{0(\mu,\nu)}$. Let $x', y' \in Z^{I_\theta}_{(\mu,\nu)}$ such that $x' \neq y'$. Then

$$0 < \frac{1}{h_r}\sum_{k\in J_r} \mu(x'-y', t) < 1 \text{ and } \frac{1}{h_r}\sum_{k\in J_r} \nu(x'-y', t) < 1.$$

Put $r_1 = \frac{1}{h_r}\sum_{k\in J_r} \mu(x'-y', t)$, $r_2 = \frac{1}{h_r}\sum_{k\in J_r} \nu(x'-y', t)$ and $r = \max\{r_1, 1-r_2\}$. For each $r_0 \in (r,1)$, there exist r_3 and r_4 such that

$$r_3 * r_4 \geq r_0 \text{ and } (1-r_3) \diamond (1-r_4) \leq (1-r_0).$$

Put $r_5 = \max\{r_3, 1-r_4\}$ and consider the open ball $B(x', 1-r_5, \frac{t}{2})$ and $B(y', 1-r_5, \frac{t}{2})$. Clearly

$$B(x', 1-r_5, \frac{t}{2}) \cap B(y', 1-r_5, \frac{t}{2}) = \phi.$$

If there exists $z' \in B(x', 1-r_5, \frac{t}{2}) \cap B(y', 1-r_5, \frac{t}{2}$, then

$$\begin{aligned}r_1 = \frac{1}{h_r}\sum_{k\in J_r} \mu(x'-y', t) &\geq \frac{1}{h_r}\sum_{k\in J_r} \mu(x'-z', \frac{t}{2}) * \frac{1}{h_r}\sum_{k\in J_r} \mu(z'-y', \frac{t}{2})\\ &\geq r_5 * r_5 \geq r_3 * r_3 \geq r_0 > r_1\end{aligned}$$

and

$$r_2 = \frac{1}{h_r}\sum_{k\in J_r}\nu(x^{'}-y^{'},t) \leq \frac{1}{h_r}\sum_{k\in J_r}\nu(x^{'}-z^{'},\frac{t}{2})\diamond\frac{1}{h_r}\sum_{k\in J_r}\nu(z^{'}-y^{'},\frac{t}{2})$$
$$\leq (1-r_5)\diamond(1-r_5) \leq (1-r_4)\diamond(1-r_4)$$
$$\leq (1-r_0) < r_0 > r.$$

This is a contradiction. Hence $Z^{I_\theta}_{(\mu,\nu)}$ is a Hausdorff space. □

Theorem 3.6. *Let $Z^{I_\theta}_{(\mu,\nu)}$ be an IFNS $\tau_{(\mu,\nu)}$ on $Z^{I_\theta}_{(\mu,\nu)}$. Then a sequence $x_k \in Z^{I_\theta}_{(\mu,\nu)}, x_k \to x$ if and if $\mu(x^{'}_k - x^{'},t) \to 1$ and $\nu(x^{'}_k - x^{'},t) \to 0$ as $k \to \infty$.*

Proof. Fix $t_0 > 0$ and suppose $x_k \to x$. Then for $r \in (0,1)$, there exits $n_0 \in \mathbb{N}$ such that $(x_k) \in B(x^{'},r,t)$ for all $k \geq n_0$

$$B(x^{'},r,t) = \{k \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in J_r}\mu(x^{'}_k - L,t) \leq 1-r \text{ or } \frac{1}{h_r}\sum_{k\in J_r}\nu(x^{'}_k - L,t) \geq r\} \in I$$

such that $B^c(x^{'},r,t) \in \mathscr{F}(I)$. Then

$$1-\frac{1}{h_r}\sum_{k\in J_r}\mu(x^{'}_k - x^{'},t) < r \text{ and } 1-\frac{1}{h_r}\sum_{k\in J_r}\nu(x^{'}_k - x^{'},t) \to 0 \text{ as } k\to\infty.$$

Conversely, if for each $t > 0$,

$$\frac{1}{h_r}\sum_{k\in J_r}\mu(x^{'}_k - x^{'},t) \to 1 \text{ and } \frac{1}{h_r}\sum_{k\in J_r}\nu(x^{'}_k - x^{'},t) \to 0 \text{ as } k\to\infty.$$

Then for $r \in (0,1)$ there exits $n_0 \in \mathbb{N}$ such that

$$1-\frac{1}{h_r}\sum_{k\in J_r}\mu(x^{'}_k - x^{'},t) < r, \quad \forall k \geq n_0.$$

Thus $(x_k) \in B(x^{'},r,t)$ for all $k \geq n_0$ and hence $x_k \to x$. □

Theorem 3.7. *A sequence $x = (x_k) \in Z^{I_\theta}_{(\mu,\nu)}$ is I-convergent if and only if for every $\varepsilon > 0$ and $t > 0$ there exists a number $m = m(x,\varepsilon,t)$ such that*

$$\{m \in \mathbb{N} : \frac{1}{h_r}\sum_{m\in J_r}\mu(x^{'}_m - L,\frac{t}{2}) > 1-\varepsilon \text{ or } \frac{1}{h_r}\sum_{m\in J_r}\nu(x^{'}_m - L,\frac{t}{2}) < \varepsilon\} \in \mathscr{F}(I).$$

Proof. Suppose that $I^{(\mu,\nu)}_\theta$–$\lim x = L$ and let $\varepsilon > 0$ and $t > 0$. For a given $\varepsilon > 0$, choose $s > 0$ such that $(1-\varepsilon)*(1-\varepsilon) > (1-s)$ and $\varepsilon\diamond\varepsilon < s$. For each $Z^{I_\theta}_{(\mu,\nu)}$, we have

$$A^c(x,\varepsilon,t) = \left\{k \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in J_r}\mu(x^{'}_k - L,\frac{t}{2}) \leq 1-\varepsilon \text{ or } \frac{1}{h_r}\sum_{k\in J_r}\nu(x^{'}_k - L,\frac{t}{2}) < \varepsilon\right\} \in \mathscr{F}(I).$$

Conversely, if $N \in A^c(x,\varepsilon,t)$, then

$$\frac{1}{h_r}\sum_{k\in J_r}\mu(x^{'}_N - L,\frac{t}{2}) > 1-\varepsilon \text{ or } \frac{1}{h_r}\sum_{k\in J_r}\nu(x^{'}_N - L,\frac{t}{2}) < \varepsilon.$$

Now we are in a position to show that there exists a number $n = n(x,\varepsilon,t)$ such that

$$\frac{1}{h_r}\sum_{k\in J_r}\mu(x^{'}_k - x^{'}_n,t) > 1-\varepsilon \text{ or } \frac{1}{h_r}\sum_{K\in J_r}\nu(x^{'}_k - x^{'}_n,t) < \varepsilon.$$

For this end, we define, for each $x \in Z_{(\mu,\nu)}^{I_\theta}$

$$B(x^{'},\varepsilon,t) = \{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in J_r}\mu(x_k^{'} - x_n^{'},t) \leq 1-s \text{ or } \frac{1}{h_r}\sum_{k\in J_r}\nu(x_k^{'} - x_n^{'},t) \geq s\} \in I.$$

Now we show that $B(x^{'},\varepsilon,t) \subset A(x,\varepsilon,t)$. Suppose that $B(x^{'},\varepsilon,t) \nsubseteq A(x,\varepsilon,t)$. Then there exists $m \in B(x^{'},\varepsilon,t)$ and $m \notin A(x,\varepsilon,t)$. It follows that

$$\frac{1}{h_r}\sum_{k\in J_r}\mu(x_k^{'} - x_n^{'},t) < 1-s \text{ and } \frac{1}{h_r}\sum_{k\in J_r}\mu(x_n^{'} - L,\frac{t}{2}) > 1-\varepsilon.$$

Hence, one has

$$1-s \geq \frac{1}{h_r}\sum_{k\in J_r}\mu(x_m^{'} - x_n^{'},t) \geq \frac{1}{h_r}\sum_{k\in J_r}\mu(x_m^{'} - L,\frac{t}{2}) * \frac{1}{h_r}\sum_{k\in J_r}\mu(x_n^{'} - L,\frac{t}{2}) \geq (1-\varepsilon)*(1-\varepsilon)1-s,$$

which is a contradiction. On the other hand, one has

$$\frac{1}{h_r}\sum_{k\in J_r}\nu(x_n^{'} - L,\frac{t}{2}) \geq s \text{ and } \frac{1}{h_r}\sum_{k\in J_r}\nu(x_k^{'} - L,\frac{t}{2}) < \varepsilon.$$

In particular,

$$\frac{1}{h_r}\sum_{k\in J_r}\nu(x_n^{'} - L,\frac{t}{2}) < \varepsilon.$$

Therefore,

$$s \leq \frac{1}{h_r}\sum_{k\in J_r}\nu(x_m^{'} - x_n^{'},t) \leq \frac{1}{h_r}\sum_{k\in J_r}\nu(x_m^{'} - L,\frac{t}{2}) \diamond \frac{1}{h_r}\sum_{k\in J_r}\nu(x_n^{'} - L,\frac{t}{2}) \leq \varepsilon \diamond \varepsilon < s,$$

which is not possible. Hence $B(x^{'},\varepsilon,t) \subset A(x,\varepsilon,t), A(x,\varepsilon,t) \in I$. This implies that $B(x^{'},\varepsilon,t) \in I$. This completes the proof. □

4. CONCLUSION

The spaces of intuitionistic fuzzy zweier lacunary ideal convergence of sequences were defined in this article and some algebraic and topological properties of these spaces are investigated. These definitions and results provided in this paper can be viewed as new tools to deal with the convergence problems of sequences in the fuzzy settings which occur in many branches of science and engineering.

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