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ON THE EXISTENCE OF POSITIVE SOLUTIONS FOR NONLINEAR THREE-POINT BOUNDARY VALUE PROBLEMS

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Abstract. In this paper we consider the existence of positive solutions of a nonlinear three-point boundary value problem. By applying fixed point theorems in cones, we prove the existence of at least one positive solution when f is superlinear or sublinear.

Keywords. Banach space; Boundary value problem; Positive solution; Superlinear; Sublinear.

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1. Introduction

Boundary value problems (BVPs) play a major role in many fields of engineering design and manufacturing. Major established industries such as the automobile, aerospace chemical, pharmaceutical, petroleum, electronics and communication, as well as emerging technologies such as nanotechnology and biotechnology. In the past twenty years or so, various BVPs for ordinary differential equations have been extensively studied; see, for example, [1]-[7], [9]-[12], [14]-[20] and the references therein.

In this paper, we are concerned with the existence of positive solutions of the second-order BVP

$$u''(t)+w(t)f(u(t))=0, \qquad 0<t<1, \tag{1.1}$$

$$\beta u(0)-\gamma u'(0)=0, \qquad \alpha u(\eta)=u(1), \tag{1.2}$$

where $0<\eta<1,\ 0<\alpha<1/\eta,\ \beta,\gamma\geq 0,\ \beta+\gamma>0$, and $d=\beta(1-\alpha\eta)+\gamma(1-\alpha)>0$. Here we give some existence results for positive solutions to (1.1) and (1.2). Our main results are obtained via a fixed point theorem in a cone. To the best of our knowledge, there are still no results for the existence of positive solutions to BVP (1.1) and (1.2) by using the fixed point theorem.

Throughout the paper, we suppose that the following conditions are satisfied:

(C1) $w:[0,1]\to[0,\infty)$ is continuous and there exists $x_0\in[\eta,1]$ such that $w(x_0)>0$;

(C2) $f:[0,\infty)\to[0,\infty)$ is continuous.

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2. PRELIMINARIES

We will employ the following fixed point theorem due to Krasnoselskii [13], that can also be found in the book by Guo [8].

Theorem 2.1. *Let E be a Banach space, and let $P \subset E$ be a cone. Suppose Ω_1, Ω_2 are open bounded subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $F : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ be completely continuous. Assume that one of the two conditions*

(a) $\|Fu\| \leq \|u\|, \quad \forall u \in P \cap \partial\Omega_1$, *and* $\|Fu\| \geq \|u\|, \forall u \in P \cap \partial\Omega_2$,
(b) $\|Fu\| \geq \|u\|, \quad \forall u \in P \cap \partial\Omega_1$, *and* $\|Fu\| \leq \|u\|, \forall u \in P \cap \partial\Omega_2$

is satisfied. Then F has a least one fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Lemma 2.2. *Let $d = \beta(1-\alpha\eta) + \gamma(1-\alpha) \neq 0$. Then for $h \in C[0,1]$, the BVP*

$$u''(t) + h(t) = 0, \qquad 0 < t < 1, \tag{2.1}$$

$$\beta u(0) - \gamma u'(0) = 0, \qquad \alpha u(\eta) = u(1), \tag{2.2}$$

has a unique solution

$$u(t) = -\int_0^t (t-s)h(s)ds + \frac{\beta t + \gamma}{d}\left[\int_0^1 (1-s)h(s)ds - \alpha\int_0^\eta (\eta - s)h(s)ds\right]. \tag{2.3}$$

Proof. From (2.1), we have

$$u(t) = -\int_0^t (t-s)h(s)ds + C_1 t + C_2.$$

In particular,

$$\begin{aligned} u(0) &= C_2, \qquad u'(0) = C_1, \\ u(\eta) &= -\int_0^\eta (\eta - s)h(s)ds + C_1\eta + C_2, \qquad u(1) = -\int_0^1 (1-s)h(s)ds + C_1 + C_2. \end{aligned}$$

Combining this with boundary conditions (2.2), we conclude that

$$\begin{aligned} C_1 &= \frac{\beta}{\beta(1-\alpha\eta) + \gamma(1-\alpha)}\left[\int_0^1 (1-s)h(s)ds - \alpha\int_0^\eta (\eta - s)h(s)ds\right], \\ C_2 &= \frac{\gamma}{\beta(1-\alpha\eta) + \gamma(1-\alpha)}\left[\int_0^1 (1-s)h(s)ds - \alpha\int_0^\eta (\eta - s)h(s)ds\right]. \end{aligned}$$

Therefore, BVP (2.1) and (2.2) has a unique solution

$$u(t) = -\int_0^t (t-s)h(s)ds + \frac{\beta t + \gamma}{\beta(1-\alpha\eta) + \gamma(1-\alpha)}\left[\int_0^1 (1-s)h(s)ds - \alpha\int_0^\eta (\eta - s)h(s)ds\right].$$

□

Lemma 2.3. *Let $0 < \alpha < 1/\eta$ and $d > 0$. If $h \in C[0,1]$ and $h \geq 0$, then unique solution u of the BVP (2.1) and (2.2) satisfies*

$$u(t) \geq 0, \qquad t \in [0,1].$$

Proof. From the fact that $u''(t) = -h(t) \leq 0$, we know that the graph of u is concave down on $[0,1]$. It suffices to verify that $u(0) \geq 0, \;\; u(1) \geq 0$.

We shall show that $u(0) \geq 0$, there are two cases to be considered. First we consider the case $0 < \alpha < 1$.

$$\begin{aligned} u(0) &= \frac{\gamma}{d}\left[\int_0^1 (1-s)h(s)ds - \alpha \int_0^\eta (\eta - s)h(s)ds\right] \\ &\geq \frac{\gamma}{d}\left[\int_0^1 (1-s)h(s)ds - \alpha \int_0^1 (1-s)h(s)ds\right] \\ &= \frac{\gamma}{d}(1-\alpha)\int_0^1 (1-s)h(s)ds \geq 0. \end{aligned}$$

Next we consider the case $\alpha \geq 1$.

$$\begin{aligned} u(0) &= \frac{\gamma}{d}\left[\int_0^1 (1-s)h(s)ds - \alpha \int_0^\eta (\eta - s)h(s)ds\right] \\ &= \frac{\gamma}{d}\left[\int_0^\eta (1-\alpha\eta) + s(\alpha - 1)\right]h(s)ds + \frac{\gamma}{d}\int_\eta^1 (1-s)h(s)ds \geq 0. \end{aligned}$$

Furthermore, we observe that

$$\begin{aligned} u(1) &= -\int_0^1 (1-s)h(s)ds + \frac{(\beta+\gamma)}{d}\left[\int_0^1 (1-s)h(s)ds - \alpha \int_0^\eta (\eta - s)h(s)ds\right] \\ &= \frac{\alpha(\beta\eta+\gamma)}{d}\int_0^1 (1-s)h(s)ds - \frac{\alpha(\beta+\gamma)}{d}\int_0^\eta (\eta - s)h(s)ds \\ &\geq \frac{\alpha}{d}\left[(\beta\eta+\gamma)\int_0^\eta (1-s)h(s)ds - (\beta+\gamma)\int_0^\eta (\eta - s)h(s)ds\right] \\ &= \frac{\alpha(1-\eta)}{d}\int_0^\eta (\beta s+\gamma)h(s)ds \geq 0. \end{aligned}$$

□

Lemma 2.4. *Let $\alpha\eta > 1$. If $h \in C[0,1]$ and $h(t) \geq 0$, for $0 < t < 1$, then the BVP (2.1) and (2.2) has no positive solution.*

Proof. Suppose the BVP (2.1) and (2.2) has a positive solution u. If $u(1) > 0$, then $u(\eta) > 0$ and

$$\frac{u(1)}{1} = \frac{\alpha u(\eta)}{1} > \frac{u(\eta)}{\eta}$$

a contradiction of the concavity of u. If $u(1) = 0$ and $u(\tau) > 0$ for some $\tau \in (0,1)$, then $u(\eta) = u(1) = 0$, where $\tau \neq \eta$. If $\tau \in (0,\eta)$, then $u(\tau) > u(\eta) = u(1)$, a contradiction of the concavity of u. If $\tau \in (\eta,1)$, then $u(0) = u(\eta) < u(\tau)$, another violation of the concavity of u. So $u(1) < 0$, that is, there is no positive solution exists. □

Lemma 2.5. *Let $0 < \alpha < 1/\eta$ and $d > 0$. If $h \in C[0,1]$ and $h(t) \geq 0$, then the unique solution u of the BVP (2.1) and (2.2) satisfies*

$$\inf_{\eta \leq t \leq 1} u(t) \geq r\|u\|,$$

where $r = \min\left\{\dfrac{\alpha(1-\eta)}{1-\alpha\eta}, \alpha\eta, \eta\right\}$.

Proof. First we consider the case where $0 < \alpha < 1$. By the second boundary condition, we know that $u(\eta) \geq u(1)$. Pick $t_0 \in (0,1)$ such that $u(t_0) = \|u\|$. If $t_0 \leq \eta < 1$, then $\min_{t\in[\eta,1]} u(t) = u(1)$ and

$$\begin{aligned} u(t_0) &\leq u(1) + \frac{u(1)-u(\eta)}{1-\eta}(0-1) \\ &= \frac{-\eta u(1) + u(\eta)}{1-\eta} \\ &= \frac{(1-\alpha\eta)u(1)}{\alpha(1-\eta)}. \end{aligned}$$

Therefore

$$\min_{t\in[\eta,1]} u(t) \geq \frac{\alpha(1-\eta)}{(1-\alpha\eta)}\|u\|.$$

If $\eta \leq t_0 < 1$, we get $u(1) = \min_{t\in[\eta,1]} u(t)$. From the concavity of u, we know that

$$\frac{u(\eta)}{\eta} \geq \frac{u(t_0)}{t_0}.$$

Using the boundary condition $\alpha u(\eta) = u(1)$, we find that

$$\frac{u(1)}{\alpha\eta} \geq \frac{u(t_0)}{t_0} \geq u(t_0) = \|u\|,$$

which implies that

$$\min_{t\in[\eta,1]} u(t) \geq \alpha\eta\|u\|.$$

Now we consider the case where $1 \leq \alpha < \frac{1}{\eta}$.The boundary condition this time implies $u(\eta) \leq u(1)$. Set $u(t_0) = \|u\|$. By the concavity of u, we have $t_0 \in [\eta, 1]$ and

$$\min_{t\in[\eta,1]} u(t) = u(\eta).$$

Using the concavity of u and Lemma 2.3, it follows that

$$\frac{u(\eta)}{\eta} \geq \frac{u(t_0)}{t_0}.$$

Hence $\min_{t\in[\eta,1]} u(t) \geq \eta\|u\|$. □

3. Main results

In this section, we will state and verify our main results.

We define

$$f_0 = \lim_{u\to 0^+} \frac{f(u)}{u}, \qquad f_\infty = \lim_{u\to\infty} \frac{f(u)}{u}.$$

f is said to be *superlinear,* when $f_0 = 0$ and $f_\infty = \infty$, and f is said to be *sublinear,* when $f_0 = \infty$ and $f_\infty = 0$.

Theorem 3.1. *Suppose that conditions (C1) and (C2) are satisfied. If, either*

(i) $f_0 = 0$ *and* $f_\infty = \infty$ *(i.e.* f *is superlinear), or*

(ii) $f_0 = \infty$ *and* $f_\infty = 0$ *(i.e.*f *is sublinear),*

then the BVP (1.1) *and* (1.2) *has at least one positive solution in* P*.*

Proof. Let the Banach space $E = C[0,1]$ be equipped with the norm

$$\|u\| = \max_{0\le t\le 1} |u(t)|.$$

We define

$$P = \left\{ u \in E:\ u(t) \ge 0,\ \min_{\eta\le t\le 1} u(t) \ge r\|u\| \right\}.$$

Then it is obvious that P is a cone in E. The BVP (1.1) and (1.2) has a solution $u = u(t)$ if and only if u is a fixed point of the operator equation

$$\begin{aligned} Fu(t) &= -\int_0^t (t-s)w(s)f(u(s))ds \\ &+\frac{\beta t+\gamma}{d}\left[\int_0^1 (1-s)w(s)f(u(s))ds - \alpha\int_0^\eta (\eta-s)w(s)f(u(s))ds\right],\ t\in[0,1]. \end{aligned} \tag{3.1}$$

In addition, by Lemma 2.5, $FP \subset P$. It is also easy to check that $F: P \to P$ is completely continuous.

First, we consider the superlinear case: $f_0 = 0$ and $f_\infty = \infty$.

Now since $f_0 = 0$, there exists an $H_1 > 0$ such that $f(u) \le \varepsilon u$, for $0 < u < H_1$, where $\varepsilon > 0$ is such that

$$\frac{\varepsilon(\beta+\gamma)}{d}\int_0^1 (1-s)w(s)ds \le 1. \tag{3.2}$$

If $u \in P$ with $\|u\| = H_1$, then by (3.1) and (3.2), we have

$$\begin{aligned} Fu(t) &= -\int_0^t (t-s)w(s)f(u(s))ds \\ &+\frac{\beta t+\gamma}{d}\left[\int_0^1 (1-s)w(s)f(u(s))ds - \alpha\int_0^\eta (\eta-s)w(s)f(u(s))ds\right] \\ &\le \frac{\beta t+\gamma}{d}\int_0^1 (1-s)w(s)f(u(s))ds \\ &\le \frac{\beta t+\gamma}{d}\int_0^1 (1-s)w(s)\varepsilon u(s)ds \\ &\le \frac{\varepsilon(\beta+\gamma)}{d}\int_0^1 (1-s)w(s)ds\|u\| \\ &\le \frac{\varepsilon(\beta+\gamma)}{d}\int_0^1 (1-s)w(s)dsH_1 \le H_1. \end{aligned} \tag{3.3}$$

Now if we set

$$\Omega_1 = \{u \in E: \|u\| < H_1\},$$

then (3.3) shows that $\|Fu\| \le \|u\|$, for $u \in P \cap \partial\Omega_1$.

On the other hand, since $f_\infty = \infty$, there is $H > 0$ such that $f(u) \ge \rho u$, for $u \ge H$, where $\rho > 0$ is chosen such that

$$\frac{\rho(\beta\eta+\gamma)r}{d}\int_\eta^1 (1-s)w(s)ds \ge 1. \tag{3.4}$$

Set $H_2 = \max\left\{2H_1, H/r\right\}$ and

$$\Omega_2 = \{u \in E: \|u\| < H_2\}.$$

Then $u \in P$ and $\|u\| = H_2$, imply that

$$\min_{\eta \le t \le 1} u(t) \ge r\|u\| \ge H.$$

Hence,

$$\begin{aligned}
Fu(\eta) &= -\int_0^\eta (\eta - s)w(s)f(u(s))ds \\
&\quad + \frac{\beta\eta+\gamma}{d}\left[\int_0^1 (1-s)w(s)f(u(s))ds - \alpha\int_0^\eta (\eta-s)w(s)f(u(s))ds\right] \\
&= -\left[1+\frac{\alpha(\beta\eta+\gamma)}{d}\right]\int_0^\eta (\eta-s)w(s)f(u(s))ds + \frac{\beta\eta+\gamma}{d}\int_0^1 (1-s)w(s)f(u(s))ds \\
&= -\left[1+\frac{\alpha(\beta\eta+\gamma)}{d}\right]\int_0^\eta \eta w(s)f(u(s))ds + \left[1+\frac{\alpha(\beta\eta+\gamma)}{d}\right]\int_0^\eta sw(s)f(u(s))ds \\
&\quad + \frac{(\beta\eta+\gamma)}{d}\int_0^1 w(s)f(u(s))ds - \frac{(\beta\eta+\gamma)}{d}\int_0^1 sw(s)f(u(s))ds \\
&\ge \frac{(\beta\eta+\gamma)}{d}\int_\eta^1 w(s)f(u(s))ds - \frac{(\beta\eta+\gamma)}{d}\int_\eta^1 sw(s)f(u(s))ds \\
&= \frac{(\beta\eta+\gamma)}{d}\int_\eta^1 (1-s)w(s)f(u(s))ds.
\end{aligned}$$

Hence, for $u \in P \cap \partial\Omega_2$, one has

$$\|Fu\| \ge \frac{\rho(\beta\eta+\gamma)r}{d}\int_\eta^1 (1-s)w(s)ds\|u\| \ge \|u\|.$$

So, it follows from the first part of the Fixed Point Theorem that F has a fixed point in $u^\star \in P \cap (\overline{\Omega}_2 \backslash \Omega_1)$, which is a convenient positive solution of the BVP (1.1) and (1.2).

Next, we consider the sublinear case: $f_0 = \infty$ and $f_\infty = 0$.

We first choose $H_3 > 0$ such that $f(u) \ge \lambda u$ for $0 \le u \le H_3$, where $\lambda > 0$ satisfies

$$\lambda r\left(\frac{\beta\eta+\gamma}{d}\right)\int_\eta^1 (1-s)w(s)ds \ge 1. \tag{3.5}$$

For $u \in P$ and $\|u\| = H_3$, by (3.5), one has

$$\begin{aligned}
Fu(\eta) &= -\int_0^\eta (\eta - s)w(s)f(u(s))ds \\
&\quad + \frac{\beta\eta+\gamma}{d}\left[\int_0^1 (1-s)w(s)f(u(s))ds - \alpha\int_0^\eta (\eta-s)w(s)f(u(s))ds\right] \\
&\ge \frac{(\beta\eta+\gamma)}{d}\int_\eta^1 (1-s)w(s)f(u(s))ds \\
&\ge \lambda\frac{(\beta\eta+\gamma)}{d}\int_\eta^1 (1-s)w(s)u(s)ds \\
&\ge \lambda r\frac{(\beta\eta+\gamma)}{d}\|u\|\int_\eta^1 (1-s)w(s)ds \ge \|u\|.
\end{aligned}$$

Thus, we may let $\Omega_3 = \{u \in E : \|u\| < H_3\}$ so that $\|Fu\| \geq \|u\|$, for $u \in P \cap \partial\Omega_3$. Since $f_\infty = 0$, there exists $M > 0$ such that $f(u) \leq \mu u$ for $u \geq M$, where $\mu > 0$ satisfies

$$\frac{\mu(\beta+\gamma)}{d}\int_0^1 (1-s)w(s)ds \leq 1. \tag{3.6}$$

There are the two cases, (i) f is bounded, or (ii) f is unbounded.

(i) Assume that f is bounded, say $f(u) \leq N$ for all $u \in [0,\infty)$ for some constant $N > 0$. Pick

$$H_4 = \max\left\{2H_3, \frac{N(\beta+\gamma)}{d}\int_0^1 (1-s)w(s)ds\right\}.$$

If $u \in P$ with $\|u\| = H_4$, then

$$\begin{aligned} Fu(t) &= -\int_0^t (t-s)w(s)f(u(s))ds \\ &\quad + \frac{\beta t+\gamma}{d}\left[\int_0^1 (1-s)w(s)f(u(s))ds - \alpha\int_0^\eta (\eta-s)w(s)f(u(s))ds\right] \\ &\leq \frac{(\beta t+\gamma)}{d}\int_0^1 (1-s)w(s)f(u(s))ds \\ &\leq \frac{(\beta+\gamma)}{d}\int_0^1 (1-s)w(s)f(u(s))ds \\ &\leq N\frac{(\beta+\gamma)}{d}\int_0^1 (1-s)w(s)ds \leq H_4,\ t \in [0,1]. \end{aligned}$$

Thus, $\|Fu\| \leq \|u\|$.

(ii) If f is unbounded, then we let $H_4 = \max\left\{2H_3, \frac{1}{r}M\right\}$ such that

$$f(u) \leq f(H_4) \text{ for } 0 \leq u \leq H_4.$$

If $u \in P$ with $\|u\| = H_4$, then

$$\begin{aligned} Fu(t) &= -\int_0^t (t-s)w(s)f(u(s))ds \\ &\quad + \frac{\beta t+\gamma}{d}\left[\int_0^1 (1-s)w(s)f(u(s))ds - \alpha\int_0^\eta (\eta-s)w(s)f(u(s))ds\right] \\ &\leq \frac{(\beta t+\gamma)}{d}\int_0^1 (1-s)w(s)f(H_4)ds \\ &\leq \mu H_4\frac{(\beta+\gamma)}{d}\int_0^1 (1-s)w(s)ds \leq H_4 = \|u\|,\ t \in [0,1]. \end{aligned}$$

so that $\|Fu\| \leq \|u\|$. For this case, if we let

$$\Omega_4 = \{u \in E : \|u\| < H_4\},$$

then $\|Fu\| \leq \|u\|$, for $u \in P \cap \partial\Omega_4$. By the second part of the Fixed Point Theorem, we know that the BVP (1.1) and (1.2) has at least one positive solution.

This completes the proof.

□

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