Commun. Optim. Theory 2017 (2017), Article ID 9 https://doi.org/10.23952/cot.2017.9

MATHRES

Communications in Optimization Theory

Available online at http://cot.mathres.org

# SMOOTH STABLE INVARIANT MANIFOLDS ON TIME SCALES

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**Abstract.** This paper focuses on the problems of invariant manifolds for nonuniformly hyperbolic systems on time scales. We establish the existence of smooth stable invariant manifolds for a nonlinear dynamical system on time scales in Banach spaces assuming that the corresponding linearized system admits a nonuniform exponential dichotomy.

**Keywords.** Time scales; Nonuniform exponential dichotomies; Stable invariant manifold; Nonuniformly hyperbolic system.

**2010 Mathematics Subject Classification.** 34C45.

## 1. INTRODUCTION

Dynamical systems or dynamic equations on time scales originate from [1, 2] and allow a simultaneous treatment of continuous dynamical systems, discrete dynamical systems and dynamical systems on the general time scales. The concept of uniform or nonuniform exponential dichotomies on time scales introduced in [3, 4, 5, 6] is a very important method and tool to explore the dynamic behavior of nonautonomous dynamical systems such as the existence and roughness [4, 5, 7], the Hartman-Grobman theorems [6, 8, 9], periodic solutions [4, 11], (pseudo) almost-periodic solutions [10, 12, 13, 14, 15, 16] and impulsive dynamic systems [17].

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Received August 3, 2016; Accepted 6 December, 2016.

As one of the most important and useful properties, the classical theory of invariant manifolds provides a geometric structure to describe and understand the qualitative behavior of nonlinear dynamical systems and has been widely recognized both in mathematics and in applications [18]. In particular, with the help of nonuniform exponential dichotomies, Zhang, Fan and Chang [6] investigated the existence of Lipschitz stable invariant manifolds for nonuniformly hyperbolic systems on measure chains. However, there are few results to consider smooth stable invariant manifolds on time scales, which is quite different from the existing results and is much more challenging.

The content of this paper is as follows. In Section 2, we introduce some basic preliminary results on time scales in order to make this paper self-contained. Section 3 focuses on establishing smooth stable invariant manifolds for nonuniformly hyperbolic systems on time scales with the help of nonuniform exponential dichotomies.

## 2. PRELIMINARIES

We first introduce some basic terminologies and results of the calculus on time scales. We refer the readers to [1, 2] for more details.

Let $(X,\|\cdot\|)$ be a Banach space. A time scale $\mathbb{T}$ is defined as a nonempty closed subset of the real numbers. Define the forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ and the graininess function $\mu(t)=\sigma(t)-t$ for any $t\in\mathbb{T}$. We assume that $\mathbb{T}$ is unbounded below and above. Let $\mathbb{T}_\tau^+:=\{t\in\mathbb{T}:\tau\le t\}$ and $\mathbb{T}_\tau^-:=\{t\in\mathbb{T}:t\le\tau\}$ for any $\tau\in\mathbb{T}$. $\mathrm{C_{rd}}(\mathbb{T},X)$ denotes the set of rd-continuous functions $g:\mathbb{T}\to X$. $\mathscr{R}^+(\mathbb{T},\mathbb{R}):=\{g\in\mathrm{C_{rd}}(\mathbb{T},\mathbb{R}):1+\mu(t)g(t)>0,\ \text{for } t\in\mathbb{T}\}$ is the space of positively regressive functions. Define $(\varphi\oplus\psi)(t):=\varphi(t)+\psi(t)+\mu(t)\varphi(t)\psi(t),\ominus\varphi:=-\dfrac{\varphi(t)}{1+\mu(t)\varphi(t)},(\omega\odot\varphi)(t):=\lim\limits_{h\searrow\mu(t)}\dfrac{(1+h\varphi(t))^\omega-1}{h}$ for a given $\omega\in\mathbb{R}^+$ and for any $t\in\mathbb{T},\varphi,\psi\in\mathscr{R}^+(\mathbb{T},\mathbb{R})$. If $\varphi\in\mathscr{R}^+(\mathbb{T},\mathbb{R})$, then we define *the exponential function* by

$$e_\varphi(t,s)=\exp\left\{\int_s^t\zeta_{\mu(\tau)}(\varphi(\tau))\Delta\tau\right\}\text{ with }\zeta_h(z)=\begin{cases}z & \text{if}\quad h=0,\\ \mathrm{Log}(1+hz)/h & \text{if}\quad h\neq0,\end{cases}$$

for $s,t\in\mathbb{T}$, where Log is the principal logarithm. Let $\kappa=\min\{t\in\mathbb{T},0\le t\}$. For any $\varphi\in\mathrm{C_{rd}}(\mathbb{T}_\kappa^+,\mathbb{R})$, we introduce the abbreviation $[\varphi]^*:=\sup_{t\in\mathbb{T}_\kappa^+}(\varphi(t)),[\varphi]_*:=\inf_{t\in\mathbb{T}_\kappa^+}(\varphi(t))$ and the notations $0 \lhd \varphi \Leftrightarrow 0 < [\varphi]_*$. It is clear to see that

$$\lim_{t\to\infty} e_{\ominus\varphi}(t,\tau)=0,\ \lim_{\tau\to-\infty} e_{\ominus\varphi}(t,\tau)=0,\ e_{\varphi}(t,\kappa)\geq 1 \text{ for } \kappa\leq t,$$

for $0 < [\varphi]_*$.

Let $\mathscr{B}(X)$ be the space of bounded linear operators defined on $X$. We consider the following systems

$$x^{\Delta}=A(t)x, \tag{2.1}$$

$$x^{\Delta}=A(t)x+f(t,x), \tag{2.2}$$

where $A(t)\in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}_{\kappa}^{+},\mathscr{B}(X))$, $f:\mathbb{T}_{\kappa}^{+}\times X\to X$. Let $T(t,s)$ be the evolution operator satisfying $T(t,s)x(s)=x(t)$ for $s\leq t, t,s\in\mathbb{T}$ and any solution $x(t)$ of equation (2.1).

**Definition 2.1** ([6]). (2.1) is said to have a nonuniform exponential dichotomy on a time scale $\mathbb{T}$ if there are a projection $P(t):X\to X$ for $\kappa\leq t$, a constant $K>1$ and growth rates $0\lhd a, 0\lhd b, 0\unlhd\rho$ such that, for any $\kappa\leq s\leq t$, $P(t)T(t,s)=T(t,s)P(s)$, $T_Q(t,s):=T(t,s)|Q(s)X: Q(s)X\to Q(t)X$ is invertible, where $Q(t)=\mathrm{id}-P(t)$, and

$$\begin{aligned} \|T(t,s)P(s)\| &\leq Ke_{\ominus a}(t,s)e_{\rho}(s,\kappa), \\ \|T_Q(t,s)^{-1}Q(t)\| &\leq Ke_{\ominus b}(t,s)e_{\rho}(t,\kappa). \end{aligned} \tag{2.3}$$

We then define the *stable* and *unstable subspaces* for each $\kappa\leq t$ by

$$E(t)=P(t)(X) \quad \text{and} \quad F(t)=Q(t)(X).$$

We denote by $\partial$ the partial derivative with respect to the second variable of any given function of two variables and assume that:

(1) $f(t,0)=0$ for any $\kappa\leq t$;

(2) there exists $c>0$ such that

$$\left\|\frac{\partial f}{\partial x}(t,x)\right\|\leq ce_{\ominus(3\odot\rho)}(t,\kappa) \tag{2.4}$$

and

$$\left\|\frac{\partial f}{\partial x}(t,x_1)-\frac{\partial f}{\partial x}(t,x_2)\right\|\leq ce_{\ominus(3\odot\rho)}(t,\kappa)\|x_1-x_2\| \tag{2.5}$$

for any $\kappa \le t, x_1, x_2 \in X$.

We also denote by $\mathscr{X}$ the space of rd-continuous functions $\Phi : \mathbb{T}_\kappa^+ \times E(s) \to X$ of class $C^1$ in $\xi \in E(s)$ such that for each $\kappa \le s$:

(1) $\Phi(s, E(s)) \subset F(s)$ and $\Phi(s, 0) = 0$;

(2)

$$\|(\partial\Phi/\partial\xi)(s,\xi)\| \le 1, \tag{2.6}$$
$$\|(\partial\Phi/\partial\xi)(s,\xi_1) - (\partial\Phi/\partial\xi)(s,\xi_2)\| \le \|\xi_1 - \xi_2\|$$

for every $\xi_1, \xi_2 \in E(s)$.

It is easy to show that $\mathscr{X}$ is a Banach space with the norm

$$|\Phi|' = \sup\{\|\Phi(s,\xi)\|/\|\xi\| : \kappa \le s \text{ and } \xi \in E(s) \setminus \{0\}\}.$$

Given $\Phi \in \mathscr{X}$, we consider the graph

$$\mathscr{W} = \left\{(s, \xi, \Phi(s,\xi)) : (s,\xi) \in \mathbb{T}_\kappa^+ \times E(s)\right\}. \tag{2.7}$$

Moreover, for each $(s, u(s), v(s)) \in \mathbb{T}_\kappa^+ \times E(s) \times F(s)$ we consider

$$\Psi_\gamma(s, u(s), v(s)) = (t, u(t), v(t)), \quad t - s = \gamma \ge 0 \tag{2.8}$$

generated by equation (2.2), where

$$u(t) = U(t,s)u(s) + \int_s^t U(t, \sigma(\tau)) f(\tau, u(\tau), v(\tau)) \Delta\tau, \tag{2.9}$$

$$v(t) = V(t,s)v(s) + \int_s^t V(t, \sigma(\tau)) f(\tau, u(\tau), v(\tau)) \Delta\tau, \tag{2.10}$$

where $U(t,s) = T(t,s)P(s)$ and $V(t,s) = T_Q(t,s)Q(s)$.

## 3. Existence of smooth stable manifolds

In this section, we establish the existence of smooth stable invariant manifolds for nonuniformly hyperbolic systems on time scales. Let $\mathscr{X}^*$ be the space of rd-continuous functions $z : \mathbb{T}_\kappa^+ \times E(s) \to X$ of class $C^1$ in $\xi \in E(s)$ such that for $\kappa \le s$:

(1) $z(t,\xi) \in E(t)$, $z(t,0) = 0$ for every $s \le t$ and $z(s,\xi) = \xi$;

(2)

$$\|(\partial z/\partial \xi)(t,\xi)\| \leq 2Ke_{\ominus a}(t,s)e_{\rho}(s,\kappa) \tag{3.1}$$

and

$$\|(\partial z/\partial \xi)(t,\xi_1)-(\partial z/\partial \xi)(t,\xi_2)\| \leq 2Ke_{\ominus a}(t,s)e_{2\odot\rho}(s,\kappa)\|\xi_1-\xi_2\| \tag{3.2}$$

for every $s \leq t$ and $\xi_1, \xi_2 \in E(s)$.

It is not difficult to show that $\mathscr{X}^*$ is a Banach space with the norm

$$\|z\|^* = \sup\left\{\frac{\|z(t,\xi)\|}{\|\xi\|e_{\ominus a}(t,s)e_{\rho}(s,\kappa)} : s \leq t, \xi \in E(s)\setminus\{0\}\right\} \tag{3.3}$$

and $\|z\|^* \leq K$ for each $z \in \mathscr{X}^*$.

The following is our main result. It establishes the existence of a smooth stable invariant manifold for equation (2.2).

**Theorem 3.1.** *Assume that equation* (2.1) *admits a nonuniform exponential dichotomy and* $[a\mu]^* < \infty$, $[\rho\mu]^* < \infty$. *If* $[a \oplus b \ominus \rho]_* > 0$ *and* $c$ *is sufficiently small, then there exists a unique function* $\Phi \in \mathscr{X}$ *such that the set* $\mathscr{W}$ *is forward invariant under* $\Psi_\gamma$, *in the sense that*

$$\Psi_\gamma(s,\xi,\Phi(s,\xi)) \in \mathscr{W} \quad \textit{for every} \quad (s,\xi) \in \mathbb{T}^+_\kappa \times E(s),\ \gamma \geq 0. \tag{3.4}$$

*Furthermore, the graph* $\mathscr{W}$ *is of class* $C^1$ *for* $\xi \in E(s)$, *and there exist constants* $d_1, d_2 > 0$ *such that*

$$\|\Psi_\gamma(s,\xi_1,\Phi(s,\xi_1)) - \Psi_\gamma(s,\xi_2,\Phi(s,\xi_2))\| \leq d_1 e_{\ominus a}(t,s)e_{\rho}(s,\kappa)\|\xi_1-\xi_2\| \tag{3.5}$$

*and*

$$\|\frac{\partial \Psi_\gamma}{\partial \xi}(s,\xi_1,\Phi(s,\xi_1)) - \frac{\partial \Psi_\gamma}{\partial \xi}(s,\xi_2,\Phi(s,\xi_2))\| \leq d_2 e_{\ominus a}(t,s)e_{2\odot\rho}(s,\kappa)\|\xi_1-\xi_2\| \tag{3.6}$$

*for* $t-s=\gamma \geq 0$ *and* $(s,\xi_1),(s,\xi_2) \in \mathbb{T}^+_\kappa \times E(s)$.

The proof of Theorem 3.1 will be obtained in several steps.

**Lemma 3.1.** *For $s \le t$, we have*

$$\|z(t,\xi_1)-z(t,\xi_2)\| \le 2Ke_{\ominus a}(t,s)e_\rho(s,\kappa)\|\xi_1-\xi_2\|, \tag{3.7}$$

$$\|(\partial\Phi/\partial\xi)(t,z(t,\xi))\| \le 2Ke_{\ominus a}(t,s)e_\rho(s,\kappa), \tag{3.8}$$

$$\|(\partial f/\partial\xi)(t,z(t,\xi),\Phi(t,z(t,\xi)))\| \le 4Kce_{\ominus(3\odot\rho)}(t,\kappa)e_{\ominus a}(t,s)e_\rho(s,\kappa), \tag{3.9}$$

$$\|\Phi(t,z(t,\xi_1))-\Phi(t,z(t,\xi_2))\| \le 2Ke_{\ominus a}(t,s)e_\rho(s,\kappa)\|\xi_1-\xi_2\|, \tag{3.10}$$

$$\|(\partial\Phi/\partial\xi)(t,z(t,\xi_1))-(\partial\Phi/\partial\xi)(t,z(t,\xi_2))\| \le 6K^2e_{\ominus a}(t,s)e_{2\odot\rho}(s,\kappa)\|\xi_1-\xi_2\|, \tag{3.11}$$

$$\begin{aligned}&\|(\partial f/\partial\xi)(t,z(t,\xi_1),\Phi(t,z(t,\xi_1)))-(\partial f/\partial\xi)(t,z(t,\xi_2),\Phi(t,z(t,\xi_2)))\|\\ &\quad\le 24K^2ce_{\ominus(3\odot\rho)}(t,\kappa)e_{\ominus a}(t,s)e_{2\odot\rho}(s,\kappa)\|\xi_1-\xi_2\|\end{aligned} \tag{3.12}$$

*for every $\Phi\in\mathscr{X}, z\in\mathscr{X}^*, \xi_1,\xi_2\in E(s)$.*

**Proof.** It follows from (3.1) that

$$\begin{aligned}\|z(t,\xi_1)-z(t,\xi_2)\| &\le \sup_{\theta\in[0,1]}\left\|\frac{\partial z}{\partial\xi}(t,\xi_1+\theta(\xi_2-\xi_1))\right\|\cdot\|\xi_1-\xi_2\|\\ &\le 2Ke_{\ominus a}(t,s)e_\rho(s,\kappa)\|\xi_1-\xi_2\|.\end{aligned}$$

By (2.6) and (3.1), we have

$$\left\|\frac{\partial\Phi}{\partial\xi}(t,z(t,\xi))\right\| \le \left\|\frac{\partial\Phi}{\partial z}(t,z)\right\|\cdot\left\|\frac{\partial z}{\partial\xi}(t,\xi)\right\| \le 2Ke_{\ominus a}(t,s)e_\rho(s,\kappa).$$

By (2.4), (3.1) and (3.8), we have

$$\begin{aligned}\left\|\frac{\partial f}{\partial\xi}(t,z(t,\xi),\Phi(t,z(t,\xi)))\right\| &\le \left\|\frac{\partial f}{\partial z}(t,z,\Phi)\right\|\cdot\left\|\frac{\partial z}{\partial\xi}(t,\xi)\right\|\\ &\quad+\left\|\frac{\partial f}{\partial\Phi}(t,z,\Phi)\right\|\cdot\left\|\frac{\partial\Phi}{\partial\xi}(t,z)\right\|\\ &\le 4Kce_{\ominus(3\odot\rho)}(t,\kappa)e_{\ominus a}(t,s)e_\rho(s,\kappa).\end{aligned}$$

Writing $z^i = z(t,\xi_i)$, $i = 1,2$. It follows from (2.6) and (3.7) that

$$\begin{aligned}
&\|\Phi(t,z(t,\xi_1)) - \Phi(t,z(t,\xi_2))\| \\
&\leq \sup_{\theta\in[0,1]} \left\| \frac{\partial \Phi}{\partial z}(t, z^1 + \theta(z^2 - z^1)) \right\| \cdot \|z(t,\xi_1) - z(t,\xi_2)\| \\
&\leq 2Ke_{\ominus a}(t,s)e_\rho(s,\kappa)\|\xi_1 - \xi_2\|.
\end{aligned}$$

By (2.6), (3.1), (3.2) and (3.7), we have

$$\begin{aligned}
A_1(t) :&= \left\| \frac{\partial \Phi}{\partial z}(t,z^1)\frac{\partial z}{\partial \xi}(t,\xi_1) - \frac{\partial \Phi}{\partial z}(t,z^1)\frac{\partial z}{\partial \xi}(t,\xi_2) \right\| \\
&\leq 2Ke_{\ominus a}(t,s)e_{2\odot\rho}(s,\kappa)\|\xi_1 - \xi_2\|
\end{aligned}$$

and

$$\begin{aligned}
A_2(t) :&= \left\| \frac{\partial \Phi}{\partial z}(t,z^1)\frac{\partial z}{\partial \xi}(t,\xi_2) - \frac{\partial \Phi}{\partial z}(t,z^2)\frac{\partial z}{\partial \xi}(t,\xi_2) \right\| \\
&\leq 2Ke_{\ominus a}(t,s)e_\rho(s,\kappa)\|z^1 - z^2\| \\
&\leq 4K^2e_{\ominus a}(t,s)e_{2\odot\rho}(s,\kappa)\|\xi_1 - \xi_2\|.
\end{aligned}$$

Therefore, one has

$$\begin{aligned}
&\left\| \frac{\partial \Phi}{\partial \xi}(t,z(t,\xi_1)) - \frac{\partial \Phi}{\partial \xi}(t,z(t,\xi_2)) \right\| \\
&= \left\| \frac{\partial \Phi}{\partial z}(t,z^1)\frac{\partial z}{\partial \xi}(t,\xi_1) - \frac{\partial \Phi}{\partial z}(t,z^2)\frac{\partial z}{\partial \xi}(t,\xi_2) \right\| \\
&\leq A_1(t) + A_2(t) \leq 6K^2e_{\ominus a}(t,s)e_{2\odot\rho}(s,\kappa)\|\xi_1 - \xi_2\|
\end{aligned}$$

since $K > 1$. Writing $\Phi^i = \Phi(t,z(t,\xi_i)), i = 1,2$. By (2.4), (2.5), (3.1), (3.2), (3.7) and (3.10) we have

$$\begin{aligned}
B_1(t) :&= \left\| \frac{\partial f}{\partial z}(t,z^1,\Phi^1)\frac{\partial z}{\partial \xi}(t,\xi_1) - \frac{\partial f}{\partial z}(t,z^1,\Phi^1)\frac{\partial z}{\partial \xi}(t,\xi_2) \right\| \\
&\leq 2Kce_{\ominus(3\odot\rho)}(t,\kappa)e_{\ominus a}(t,s)e_{2\odot\rho}(s,\kappa)\|\xi_1 - \xi_2\|
\end{aligned}$$

and

$$\begin{aligned}B_2(t) &:= \left\| \frac{\partial f}{\partial z}(t,z^1,\Phi^1)\frac{\partial z}{\partial \xi}(t,\xi_2) - \frac{\partial f}{\partial z}(t,z^2,\Phi^2)\frac{\partial z}{\partial \xi}(t,\xi_2) \right\| \\ &\le 2Kce_{\ominus(3\odot\rho)}(t,\kappa)e_{\ominus a}(t,s)e_{\rho}(s,\kappa)(\|z^1-z^2\|+\|\Phi^1-\Phi^2\|) \\ &\le 8K^2ce_{\ominus(3\odot\rho)}(t,\kappa)e_{\ominus a}(t,s)e_{2\odot\rho}(s,\kappa)\|\xi_1-\xi_2\|.\end{aligned}$$

On the other hand, it follows from (3.8) and (3.11) that

$$\begin{aligned}B_3(t) &:= \left\| \frac{\partial f}{\partial \Phi}(t,z^1,\Phi^1)\frac{\partial \Phi}{\partial \xi}(t,z^1) - \frac{\partial f}{\partial \Phi}(t,z^1,\Phi^1)\frac{\partial \Phi}{\partial \xi}(t,z^2) \right\| \\ &\le 6K^2ce_{\ominus(3\odot\rho)}(t,\kappa)e_{\ominus a}(t,s)e_{2\odot\rho}(s,\kappa)\|\xi_1-\xi_2\|\end{aligned}$$

and

$$\begin{aligned}B_4(t) &:= \left\| \frac{\partial f}{\partial \Phi}(t,z^1,\Phi^1)\frac{\partial \Phi}{\partial \xi}(t,z^2) - \frac{\partial f}{\partial \Phi}(t,z^2,\Phi^2)\frac{\partial \Phi}{\partial \xi}(t,z^2) \right\| \\ &\le 2Kce_{\ominus(3\odot\rho)}(t,\kappa)e_{\ominus a}(t,s)e_{\rho}(s,\kappa)(\|z^1-z^2\|+\|\Phi^1-\Phi^2\|) \\ &\le 8K^2ce_{\ominus(3\odot\rho)}(t,\kappa)e_{\ominus a}(t,s)e_{2\odot\rho}(s,\kappa)\|\xi_1-\xi_2\|.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\left\| \frac{\partial f}{\partial \xi}(t,z(t,\xi_1),\Phi(t,z(t,\xi_1))) - \frac{\partial f}{\partial \xi}(t,z(t,\xi_2),\Phi(t,z(t,\xi_2))) \right\| \\ &\le B_1(t)+B_2(t)+B_3(t)+B_4(t) \\ &\le 24K^2ce_{\ominus(3\odot\rho)}(t,\kappa)e_{\ominus a}(t,s)e_{2\odot\rho}(s,\kappa)\|\xi_1-\xi_2\|.\end{aligned}$$

**Lemma 3.2.** *Given $c>0$ sufficiently small and $(s,\xi,\Phi)\in\mathbb{T}_\kappa^+\times E(s)\times\mathscr{X}$, there exists a unique function $z\in\mathscr{X}^*$ such that* (2.9) *holds for every $s\le t$.*

**Proof.** Given $(s,\xi)\in\mathbb{T}_\kappa^+\times E(s)$ and $\Phi\in\mathscr{X}$, we define an operator $L$ in $\mathscr{X}^*$ by

$$(Lz)(t,\xi) = U(t,s)\xi + \int_s^t U(t,\sigma(\tau))f(\tau,z(\tau,\xi),\Phi(\tau,z(\tau,\xi)))\Delta\tau.$$

Obviously, $(Lz)(t,\xi)\in E(t)$, $(Lz)(t,0)=0$ and $(Lz)(s,\xi)=\xi$. It follows from (2.4), (2.6) and (3.3) that

$$\begin{aligned}\|f(\tau,z(\tau,\xi),\Phi(\tau,z(\tau,\xi)))\| &\le ce_{\ominus(3\odot\rho)}(\tau,\kappa)\|(z(\tau,\xi),\Phi(\tau,z(\tau,\xi)))\|\\ &\le 2ce_{\ominus(3\odot\rho)}(\tau,\kappa)\|z(\tau,\xi)\|\\ &\le 4Kce_{\ominus(3\odot\rho)}(\tau,\kappa)e_{\ominus a}(\tau,s)e_{\rho}(s,\kappa)\|\xi\|.\end{aligned}\tag{3.13}$$

By (2.3), we have

$$\begin{aligned}&\int_s^t \|U(t,\sigma(\tau))\|\|f(\tau,z(\tau,\xi),\Phi(\tau,z(\tau,\xi)))\|\\ &\le 4K^2c\|\xi\|\int_s^t e_{\ominus a}(t,\sigma(\tau))e_{\rho}(\sigma(\tau),\kappa)e_{\ominus(3\odot\rho)}(\tau,\kappa)e_{\ominus a}(\tau,s)e_{\rho}(s,\kappa)\Delta\tau\\ &\le 4K^2ce_{\ominus a}(t,s)e_{\rho}(s,\kappa)\|\xi\|\int_s^t(1+a\mu(\tau))(1+\rho\mu(\tau))e_{\ominus(2\odot\rho)}(\tau,\kappa)\Delta\tau\\ &\le 4K^2c\lambda' e_{\ominus a}(t,s)e_{\rho}(s,\kappa)\|\xi\|,\end{aligned}$$

where

$$\lambda'=\frac{(1+[a\mu]^*)(1+[\rho\mu]^*)(1+[(2\odot\rho)\mu]^*)}{[2\odot\rho]_*}.$$

Then

$$\begin{aligned}\|(Lz)(t,\xi)\| &\le \|U(t,s)\xi\|+\int_s^t\|U(t,\sigma(\tau))\|\|f(\tau,z(\tau,\xi),\Phi(\tau,z(\tau,\xi)))\|\Delta\tau\\ &\le Ke_{\ominus a}(t,s)e_{\rho}(s,\kappa)\|\xi\|+4K^2c\lambda' e_{\ominus a}(t,s)e_{\rho}(s,\kappa)\|\xi\|.\end{aligned}$$

This implies that $\|Lz\|^*\le K+4K^2c\lambda'<\infty$. By (2.3) and (3.9), one has

$$\begin{aligned}\left\|\frac{\partial(Lz)}{\partial\xi}(t,\xi)\right\| &\le Ke_{\ominus a}(t,s)e_{\rho}(s,\kappa)\\ &\quad+4K^2c\int_s^t e_{\ominus a}(t,\sigma(\tau))e_{\rho}(\sigma(\tau),\kappa)e_{\ominus(3\odot\rho)}(\tau,\kappa)e_{\ominus a}(\tau,s)e_{\rho}(s,\kappa)\Delta\tau\\ &\le Ke_{\ominus a}(t,s)e_{\rho}(s,\kappa)+4K^2c\lambda' e_{\ominus a}(t,s)e_{\rho}(s,\kappa)\le 2Ke_{\ominus a}(t,s)e_{\rho}(s,\kappa),\end{aligned}$$

since $c$ is sufficiently small so that $4K^2c\lambda' \le K$. For any $\xi_1, \xi_2 \in E(s)$, we have

$$\begin{aligned}\left\|\frac{\partial(Lz)}{\partial\xi}(t,\xi_1)-\frac{\partial(Lz)}{\partial\xi}(t,\xi_2)\right\| &\le 24K^2c\|\xi_1-\xi_2\| \\ &\times\int_s^t e_{\ominus a}(t,\sigma(\tau))e_\rho(\sigma(\tau),s)e_{\ominus(3\odot\rho)}(\tau,\kappa)e_{\ominus a}(\tau,s)e_{(2\odot\rho)}(s,\kappa)\Delta\tau \\ &\le 24K^2c\lambda' e_{\ominus a}(t,s)e_{(2\odot\rho)}(s,\kappa)\|\xi_1-\xi_2\| \le 2Ke_{\ominus a}(t,s)e_{(2\odot\rho)}(s,\kappa)\|\xi_1-\xi_2\|,\end{aligned}$$

since $c$ is sufficiently small. Therefore, $L(\mathscr{X}^*) \subset \mathscr{X}^*$. On the other hand, for each $z_1, z_2 \in \mathscr{X}^*$, we conclude that

$$\begin{aligned}&\|f(\tau,z_1(\tau,\xi),\Phi(\tau,z_1(\tau,\xi)))-f(\tau,z_2(\tau,\xi),\Phi(\tau,z_2(\tau,\xi)))\| \\ &\le ce_{\ominus(3\odot\rho)}(\tau,\kappa)\|(z_1(\tau,\xi),\Phi(\tau,z_1(\tau,\xi)))-(z_2(\tau,\xi),\Phi(\tau,z_2(\tau,\xi)))\| \\ &\le 2ce_{\ominus(3\odot\rho)}(\tau,\kappa)\|z_1(\tau,\xi)-z_2(\tau,\xi)\| \\ &\le 2ce_{\ominus(3\odot\rho)}(\tau,\kappa)e_{\ominus a}(\tau,s)e_\rho(s,\kappa)\|\xi\|\|z_1-z_2\|^*.\end{aligned}$$

It follows from (2.3) that

$$\begin{aligned}\|(Lz_1)(t)-(Lz_2)(t)\| &\le 2Kc\|\xi\|\|z_1-z_2\|^* \\ &\times\int_s^t e_{\ominus a}(t,\sigma(\tau))e_\rho(\sigma(\tau),\kappa)e_{\ominus(3\odot\rho)}(\tau,\kappa)e_{\ominus a}(\tau,s)e_\rho(s,\kappa)\Delta\tau \\ &\le 2Kc\lambda' e_{\ominus a}(t,s)e_\rho(s,\kappa)\|\xi\|\|z_1-z_2\|^*.\end{aligned}$$

This implies that $\|Lz_1-Lz_2\|^* \le 2Kc\lambda'\|z_1-z_2\|^*$ and $L$ is a contraction if $c$ is sufficiently small. Therefore, there exists a unique function $z \in \mathscr{X}^*$ such that $Lz = z$.

Let $z = z^\Phi(t,\xi)$ be the unique function given by Lemma 3.2, that is,

$$z(t,\xi) = U(t,s)\xi + \int_s^t U(t,\sigma(\tau))f(\tau,z(\tau,\xi),\Phi(\tau,z(\tau,\xi)))\Delta\tau \tag{3.14}$$

for each $s \le t$.

**Lemma 3.3.** *Let $\Phi \in \mathscr{X}$ and $z$ be the unique function given by Lemma 3.2. Then the following properties hold:*

*(1) if*

$$\Phi(t,z(t,\xi)) = V(t,s)\Phi(s,\xi) + \int_s^t V(t,\sigma(\tau))f(\tau,z(\tau,\xi),\Phi(\tau,z(\tau,\xi)))\Delta\tau \tag{3.15}$$

*for every* $(s,\xi)\in\mathbb{T}_\kappa^+\times E(s)$ *and* $s\le t$*, then*

$$\Phi(s,\xi)=-\int_s^\infty V(\sigma(\tau),s)^{-1}f(\tau,z(\tau,\xi),\Phi(\tau,z(\tau,\xi)))\Delta\tau. \tag{3.16}$$

*(2) if identity* (3.16) *holds for every* $(s,\xi)\in\mathbb{T}_\kappa^+\times E(s)$*, then* (3.15) *holds for every* $(s,\xi)\in\mathbb{T}_\kappa^+\times E(s)$.

**Proof.** It follows from (2.3) and (3.13) that

$$\begin{aligned}
&\int_s^\infty \|V(\sigma(\tau),s)^{-1}\|\|f(\tau,z(\tau,\xi),\Phi(\tau,z(\tau,\xi)))\|\Delta\tau\\
&\le 4K^2c\|\xi\|\int_s^t e_{\ominus b}(\sigma(\tau),s)e_\rho(\sigma(\tau),\kappa)e_{\ominus(3\odot\rho)}(\tau,\kappa)e_{\ominus a}(\tau,s)e_\rho(s,\kappa)\Delta\tau\\
&\le 4K^2c\|\xi\|e_{a\oplus b\oplus\rho}(s,\kappa)\int_s^t\frac{1+\rho\mu(\tau)}{1+b\mu(\tau)}e_{\ominus(a\oplus b\oplus(2\odot\rho))}(\tau,\kappa)\Delta\tau\\
&\le 4K^2c\|\xi\|\left(\frac{1+[\rho\mu]^*}{1+[b\mu]_*}\right)\left(\frac{1}{[a\oplus b\oplus(2\odot\rho)]_*}+[\mu]^*\right)<\infty.
\end{aligned}$$

This implies that (3.16) is well-defined. If (3.15) holds for every $(s,\xi)\in\mathbb{T}_\kappa^+\times E(s)$ and $s\le t$, then identity (3.15) can be written in the form

$$\Phi(s,\xi)=V(t,s)^{-1}\Phi(t,z(t,\xi))-\int_s^t V(\sigma(\tau),s)^{-1}f(\tau,z(\tau,\xi),\Phi(\tau,z(\tau,\xi)))\Delta\tau. \tag{3.17}$$

By (2.3), (2.6), and (3.1), we have

$$\begin{aligned}
\|V(t,s)^{-1}\Phi(t,z(t,\xi))\|&\le Ke_{\ominus b}(t,s)e_\rho(t,\kappa)\|z(t,\xi)\|\\
&\le 2K^2\|\xi\|e_{\ominus b}(t,s)e_\rho(t,\kappa)e_{\ominus a}(t,s)e_\rho(s,\kappa)\\
&\le 2K^2\|\xi\|e_{\ominus(a\oplus b\ominus\rho)}(t,\kappa)e_{a\oplus b\oplus\rho}(s,\kappa).
\end{aligned}$$

Therefore, (3.16) holds when letting $t\to\infty$ since $[a\oplus b\ominus\rho]_*>0$.

We now assume that (3.16) holds for every $(s,\xi)\in\mathbb{T}_\kappa^+\times E(s)$. It follows from (3.16) that

$$\begin{aligned}
V(t,s)\Phi(s,\xi)&=-\int_s^\infty V(t,\sigma(\tau))f(\tau,z(\tau,\xi),\Phi(\tau,z(\tau,\xi)))\Delta\tau\\
&=-\int_s^t V(t,\sigma(\tau))f(\tau,z(\tau,\xi),\Phi(\tau,z(\tau,\xi)))\Delta\tau\\
&\quad-\int_t^\infty V(t,\sigma(\tau))f(\tau,z(\tau,\xi),\Phi(\tau,z(\tau,\xi)))\Delta\tau\\
&=-\int_s^t V(t,\sigma(\tau))f(\tau,z(\tau,\xi),\Phi(\tau,z(\tau,\xi)))\Delta\tau+\Phi(t,z(t,\xi)).
\end{aligned}$$

This completes the proof of the lemma.

**Lemma 3.4.** *If $c$ is sufficiently small, then there exists $K_1 > 0$ such that*

$$\|z^{\Phi_1}(t,\xi) - z^{\Phi_2}(t,\xi)\| \le K_1 e_{\gamma\ominus a}(t,s) e_\rho(s,\kappa)\|\xi\| \cdot |\Phi_1 - \Phi_2|' \tag{3.18}$$

*for every $\Phi_1, \Phi_2 \in \mathscr{X}$, $(s,\xi) \in \mathbb{T}^+_\kappa \times E(s)$ and $s \le t$.*

**Proof.** Write $z_i = z^{\Phi_i}$ for $i = 1,2$. We first note that

$$\begin{aligned}
&\|\Phi_1(\tau, z_1(\tau,\xi)) - \Phi_2(\tau, z_2(\tau,\xi))\| \\
&\le \|\Phi_1(\tau, z_1(\tau,\xi)) - \Phi_2(\tau, z_1(\tau,\xi))\| + \|\Phi_2(\tau, z_1(\tau,\xi)) - \Phi_2(\tau, z_2(\tau,\xi))\| \\
&\le \|z_1(\tau,\xi)\| \cdot |\Phi_1 - \Phi_2|' + \|z_1(\tau,\xi) - z_2(\tau,\xi)\|.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\|f(\tau, z_1(\tau,\xi), \Phi_1(\tau, z_1(\tau,\xi))) - f(\tau, z_1(\tau,\xi), \Phi_2(\tau, z_2(\tau,\xi)))\| \\
&\le c e_{\ominus(3\odot\rho)}(\tau,\kappa)(\|z_1(\tau,\xi)\| \cdot |\Phi_1 - \Phi_2|' + 2\|z_1(\tau,\xi) - z_2(\tau,\xi)\|) \\
&\le 2Kc e_{\ominus(3\odot\rho)}(\tau,\kappa) e_{\ominus a}(\tau,s) e_\rho(s,\kappa)\|\xi\| \cdot |\Phi_1 - \Phi_2|' \\
&\quad + 2c e_{\ominus(3\odot\rho)}(\tau,\kappa)\|z_1(\tau,\xi) - z_2(\tau,\xi)\|.
\end{aligned} \tag{3.19}$$

It follows from (2.3) that

$$\begin{aligned}
e_{\ominus a}(s,t)\|z_1(t,\xi) - z_2(t,\xi)\| \le{}& 2K^2 c\lambda' e_\rho(s,\kappa)\|\xi\| \cdot |\Phi_1 - \Phi_2|' \\
&+ \gamma \int_s^t e_{\ominus a}(s,\tau)\|z_1(\tau,\xi) - z_2(\tau,\xi)\|\Delta\tau,
\end{aligned}$$

where

$$\gamma = 2Kc(1 + [a\mu]^*)(1 + [\rho\mu]^*).$$

By using Gronwall's inequality (see Section 6 in [2]), we have

$$\|z_1(t,\xi) - z_2(t,\xi)\| \le 2K^2 c\lambda' e_{\gamma\ominus a}(t,s) e_\rho(s,\kappa)\|\xi\| \cdot |\Phi_1 - \Phi_2|'.$$

**Lemma 3.5.** *If $c$ is sufficiently small, then there exists a unique function $\Phi \in \mathscr{X}$ such that* (3.16) *holds for every $(s,\xi) \in \mathbb{T}^+_\kappa \times E(s)$.*

**Proof.** For each $\Phi \in \mathscr{X}$ and $(s,\xi) \in \mathbb{T}_\kappa^+ \times E(s)$, we define an operator $J$ by

$$(J\Phi)(s,\xi) = -\int_s^\infty V(\sigma(\tau),s)^{-1} f(\tau, z^\Phi(\tau,\xi), \Phi(\tau, z^\Phi(\tau,\xi)))\Delta\tau,$$

where $z^\Phi$ is the unique function given by Lemma 3.2. It is easy to show that $J\Phi$ is of class $C^1$ in $\xi \in E(s)$, $J\Phi(s,E(s)) \subset F(s)$ and $J\Phi(s,0) = 0$. By (2.3) and (3.9), we have

$$\begin{aligned}\left\|\frac{\partial(J\Phi)}{\partial\xi}(s,\xi)\right\| &\le \int_s^\infty \|V(\sigma(\tau),s)^{-1}\| \left\|\frac{\partial f}{\partial\xi}\right\|\Delta\tau \\ &\le 4K^2c\left(\frac{1+[\rho\mu]^*}{1+[b\mu]_*}\right)\left(\frac{1}{[a\oplus b\oplus(2\odot\rho)]_*} + [\mu]^*\right),\end{aligned}$$

which implies that $\|(\partial(J\Phi)/\partial\xi)(s,\xi)\| \le 1$ since $c$ is sufficiently small. It follows from (2.3) and (3.12) that

$$\begin{aligned}&\left\|\frac{\partial(J\Phi)}{\partial\xi}(s,\xi_1) - \frac{\partial(J\Phi)}{\partial\xi}(s,\xi_2)\right\| \\ &\le \int_s^\infty \|V(\sigma(\tau),s)^{-1}\| \left\|\frac{\partial f}{\partial\xi}(\tau,\xi_1) - \frac{\partial f}{\partial\xi}(\tau,\xi_2)\right\| \\ &\le 24K^3c\left(\frac{1+[\rho\mu]^*}{1+[b\mu]_*}\right)\left(\frac{1}{[a\oplus b\oplus(2\odot\rho)]_*} + [\mu]^*\right)\|\xi_1-\xi_2\|.\end{aligned}$$

Therefore, $J(\mathscr{X}) \subset \mathscr{X}$.

Now we show that $J$ is a contraction. Let $(s,\xi) \in \mathbb{T}_\kappa^+ \times E(s)$, for each $\Phi_1, \Phi_2 \in \mathscr{X}$ and $z_i = z_\xi^{\Phi_i}$ for $i = 1,2$. By (3.18) and (3.19), we have

$$\begin{aligned}C(\tau) :&= \|f(\tau, z_1(\tau,\xi), \Phi_1(\tau, z_1(\tau,\xi))) - f(\tau, z_2(\tau,\xi), \Phi_2(\tau, z_2(\tau,\xi)))\| \\ &\le 2(K+K_1)c e_{\ominus(3\odot\rho)}(\tau,\kappa)(e_{\ominus a}(\tau,s) \\ &\quad + e_{\gamma\ominus a}(\tau,s))e_\rho(s,\kappa)\|\xi\| \cdot |\Phi_1 - \Phi_2|'.\end{aligned}$$

Then

$$\begin{aligned}&\|(J\Phi_1)(s,\xi) - (J\Phi_2)(s,\xi)\| \\ &\le \int_s^\infty \|V(\sigma(\tau),s)^{-1}\| C(\tau)\Delta\tau \\ &\le 2K(K+K_1)c\left(\frac{1+[\rho\mu]^*}{1+[b\mu]_*}\right)\|\xi\| \cdot |\Phi_1-\Phi_2|' \\ &\quad \times \int_s^\infty e_{\ominus b}(\tau,s)e_\rho(\tau,\kappa)e_{\ominus(3\odot\rho)}(\tau,\kappa)(e_{\ominus a}(\tau,s) + e_{\gamma\ominus a}(\tau,s))e_\rho(s,\kappa)\Delta\tau.\end{aligned}$$

If $c$ is sufficiently small, then

$$\|(J\Phi_1)(s,\xi)-(J\Phi_2)(s,\xi)\|\leq\eta\|\xi\||\Phi_1-\Phi_2|',$$

where

$$\eta=2K(K+K_1)c\left(\frac{1+[\rho\mu]^*}{1+[b\mu]_*}\right)\left(\frac{1}{[\gamma\ominus(a\oplus b\oplus(2\odot\rho))]_*}+[\mu]^*\right)<1,$$

which means that $J$ is a contraction and has a unique fixed point $\Phi$ in $\mathscr{X}$.

We are now at the right position to establish Theorem 3.1.

**Proof of Theorem 3.1.** It follows from Lemma 3.2 that, for each $(s,\xi)\in\mathbb{T}_\kappa^+\times E(s)$ and $\Phi\in\mathscr{X}$, there exists a unique function $z=z_\xi^\Phi\in\mathscr{X}^*$. By Lemmas 3.3 and 3.5, for each $\kappa\leq s$ and $\xi\in E(s)$ there exists a unique function $\Phi$ such that (3.15) holds. Therefore, (3.4) holds and $\mathscr{W}$ is forward invariant under $\Psi_\gamma$. Meanwhile, since the function $\Phi$ is of class $C^1$ for $\xi\in E(s)$, the graph $\mathscr{W}$ is of class $C^1$ for $\xi\in E(s)$. For each $(s,\xi_1),(s,\xi_2)\in\mathbb{T}_\kappa^+\times E(s)$ and $\gamma=t-s\geq 0$, by (3.7) and (3.10) we have

$$\begin{aligned}
&\|\Psi_\gamma(s,\xi_1,\Phi(s,\xi_1))-\Psi_\gamma(s,\xi_2,\Phi(s,\xi_2))\|\\
&=\|(t,z(t,\xi_1),\Phi(t,z(t,\xi_1)))-(t,z(t,\xi_2),\Phi(t,z(t,\xi_2)))\|\\
&\leq 4Ke_{\ominus a}(t,s)e_\rho(s,\kappa)\|\xi_1-\xi_2\|,
\end{aligned}$$

and it follows from (3.2) and (3.12) that

$$\begin{aligned}
&\|\frac{\partial\Psi_\gamma}{\partial\xi}(s,\xi_1,\Phi(s,\xi_1))-\frac{\partial\Psi_\gamma}{\partial\xi}(s,\xi_2,\Phi(s,\xi_2))\|\\
&\leq\|(\partial z)(\partial\xi)(t,\xi_1)-(\partial z)(\partial\xi)(t,\xi_2)\|\\
&\quad+\|(\partial\Phi)(\partial\xi)(t,z(t,\xi_1))-(\partial\Phi)(\partial\xi)(t,z(t,\xi_2))\|\\
&\leq(2K+6K^2)e_{\ominus a}(t,s)e_{2\odot\rho}(s,\kappa)\|\xi_1-\xi_2\|.
\end{aligned}$$

This completes the proof of the theorem. □

**Acknowledgements**

This research is supported by the Fund of Heilongjiang Education Committee (No. 12541127).

## REFERENCES


[1] S. Hilger, Analysis on measure chains — a unified approach to continuous and discrete calculus, Results Math. 18 (1990), 18-56.

[2] M. Bohner, A. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhäuser, Boston, 2001.

[3] M. Bohner, D.A. Lutz, Asymptotic behavior of dynamic equations on time scales, J. Difference Equ. Appl. 7 (2001), 21-50.

[4] J.M. Zhang, M. Fan, H.P. Zhu, Existence and roughness of exponential dichotomies of linear dynamic equations on time scales, Comput. Math. Appl. 59 (2010), 2658-2675.

[5] J.M. Zhang, M. Fan, H.P. Zhu, Necessary and sufficient criteria for the existence of exponential dichotomy on time scales, Comput. Math. Appl. 60 (2010), 2387-2398.

[6] J.M. Zhang, M. Fan, X.Y. Chang, Nonlinear perturbations of nonuniform exponential dichotomy on measure chains, Nonlinear Anal. 75 (2012), 670-683.

[7] J.M. Zhang, Y.J. Song, Z.T. Zhao, General exponential dichotomies on time scales and parameter dependence of roughness, Adv. Differ. Equ. 2013 (2013), Article ID 339.

[8] Y.H. Xia, J. Cao, M. Han, A new analytical method for the linearization of dynamic equations on measure chains, J. Differ. Equations 235 (2007), 527-543.

[9] Y.H. Xia, J.B. Li, P.J.Y. Wong, On the topological classification of dynamic equations on time scales. Nonlinear Anal. RWA 14 (2013), 2231-2248.

[10] J. Gao, Q.R. Wang, L.W. Zhang, Existence and stability of almost-periodic solutions for cellular neural networks with time-varying delays in leakage terms on time scales, Appl. Math. Comput. 237 (2014), 639-649.

[11] L. Yang, Y.K. Li, Existence and exponential stability of periodic solution for stochastic Hopfield neural networks on time scales, Neurocomputing, 167 (2015), 543-550.

[12] Y.K. Li, C. Wang, Pseudo almost periodic functions and pseudo almost periodic solutions to dynamic equations on time scales. Adv. Differ. Equ. 77 (2012), 1-24.

[13] Y.K. Li, L. Yang, Almost periodic solutions for neutral-type BAM neural networks with delays on time scales, J. Appl. Math. 2013 (2013), Article ID 942309.

[14] Z.J. Yao, Uniqueness and global exponential stability of almost periodic solution for Hematopoiesis model on time scales, J. Nonlinear Sci. Appl. 8 (2015), 142-152.

[15] Y.Z. Liao, L.J. Xu, Almost periodic solution for a delayed Lotka-Volterra system on time scales, Adv. Differ. Equ. 2014 (2014), Article ID 96.

[16] S.H. Hong, Y.Z. Peng, Almost periodicity of set-valued functions and set dynamic equations on time scales, Inform. Sciences 330 (2016), 157-174.

[17] C. Wang, R.P. Agarwal, Exponential dichotomies of impulsive dynamic systems with applications on time scales, Math. Method. Appl. Sci. 38 (2015), 3879-3900.

[18] L. Barreira, C. Valls, Stability of Nonautonomous Differential Equations. Lecture Notes in Mathematics, vol. 1926. Springer, Berlin (2008).