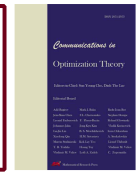
Commun. Optim. Theory 2017 (2017), Article ID 7 https://doi.org/10.23952/cot.2017.7

Communications in Optimization Theory
Available online at http://cot.mathres.org



FIXED POINT THEOREMS VIA C-CLASS FUNCTIONS IN SYMMETRIC SPACES

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Abstract. This paper is devoted to prove the existence of fixed points for self maps satisfying some C-class type contractive conditions in symmetric spaces. Without assuming continuity, we prove coincidence and fixed point theorems. Moreover, as an application, we provide common fixed point theorems via a family of C-class functions in a generating space of a symmetric family under a contractive condition of the Lebesgue integral type.

Keywords. Symmetric space; (E.A)-property; Weak commutativity; Compatible mappings, Coincidence point.

2010 Mathematics Subject Classification. 47H10, 54H25.

1. Introduction-Preliminaries

The authors of [1] introduced the notion of $(E.A)$-property which generalizes the concept of non-compatible mappings in metric spaces. They proved some common fixed-point theorems concerning non-compatible mappings under strict contractive conditions. In [7], The authors studied commutative maps as a tool for generalizing maps. Since then, a large number of generalizations of Theorem 1 of [7] which utilized the commuting map concept appeared; see [6, 12, 13] and the references therin. In [18, 19] the authors proved various common fixed-point

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Received May 19, 2016; Accepted December 2, 2016.

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theorems for strict contractive non-compatible mappings in metric spaces. Clearly, commuting mappings are weakly commuting and weakly commuting pairs are compatible. Examples in [8] and [21] shows that neither converse is true. Some common fixed point theorems in [1, 18, 10] proved for strict contractive mappings in metric spaces are extended to symmetric (semi-metric) spaces under tight conditions.

In this paper, we present a few theorems that establish the existence of common periodic points for a pair of maps via the concept of C-class functions in a symmetric space when the maps have a unique common fixed point. Moreover, we prove common fixed point theorems via family of C-class functions in a generating space of symmetric family under a contractive condition of Lebesgue integral type.

Definition 1.1. A symmetric on a set X is a function $d : X \times X \to [0,\infty)$ such that for all $x,y \in X$,

(i) $d(x,y) = 0$ iff $x = y$,

(ii) $d(x,y) = d(y,x)$.

If d is symmetric on a set X, then for $x \in X$ and $\varepsilon > 0$, we write $\mathscr{B}(x,\varepsilon)$=$\{y \in X : d(x,y) < \varepsilon\}$. A topology $\mathfrak{F}$ (d) on X is given by $\mathscr{U} \in \mathfrak{F}(d)$ if and only if for each $x \in X$, $\mathscr{B}(x,\varepsilon) \subset \mathscr{U}$ for some $\varepsilon > 0$. A set $S \subset X$ is a neighborhood of $b \in X$ iff there exists $\mathscr{U} \in \mathfrak{F}(d)$ such that $b \in \mathscr{U} \subset S$. A symmetric d is a semi-metric if for each $x \in X$ and for each $\varepsilon > 0, \mathscr{B}(x,\varepsilon)$ is a neighborhood of x in the topology $\mathfrak{F}(d)$.

Definition 1.2. A semi-metric space is a topological space whose topology $\mathfrak{F}(d)$ on X is induced by semi-metric d. In what follows symmetric space as well as semi-metric space will be denoted by (X,d). The distinction between a symmetric and a semi-metric is evident as one can easily construct a symmetric d such that $\mathscr{B}(x,\varepsilon)$ need not be a neighborhood of x in $\mathfrak{F}(d)$. We can find generalized symmetric space in [14, 20].

For a symmetric d on X the following two axioms were given by Wilson [22]:

$$W_3 : \text{For a sequence } \{x_n\} \text{ in } X \text{ and } x,y \in X,$$

$$\lim_{n\to\infty} d(x_n,x) = 0 \text{ and } \lim_{n\to\infty} d(x_n,y) = 0 \text{ imply } x = y.$$

$$W_4 : \text{For a sequence } \{x_n\},\{y_n\} \text{ in } X \text{ and } x \in X,$$

$$\lim_{n\to\infty} d(x_n,x) = 0 \text{ and } \lim_{n\to\infty} d(y_n,x_n) = 0 \text{ imply } \lim_{n\to\infty} d(y_n,x) = 0.$$

Definition 1.3. [17] A pair of self-mappings (f,g) on a symmetric (semi-metric) space, (X,d) said to be R-weakly commuting if there exists some real number $R>0$ such that $d(fgx,gfx)\leq Rd(fx,gx)$ for all $x\in X$, where as the pair (f,g) is said to be point wise R-weakly commuting if given $x\in X$ there exists $R>0$ such that $d(fgx,gfx)\leq Rd(fx,gx)$.

Here it may be noted that on the points of coincidence R-weak commutativity is equivalent to commutativity and remains a necessary minimal condition for the existence of common fixed points of contractive type mappings.

Definition 1.4. [17] A pair of self-mappings (f,g) on a symmetric (semi-metric) space, (X,d) said to be compatible if $\lim_{n\to\infty} d(fgx_n,gfx_n)=0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} f(x_n)=\lim_{n\to\infty} g(x_n)=t\in X$. Here it may be noted that R-weakly commuting mappings need not be compatible.

Definition 1.5. [17] A pair of self-mappings (f,g) on a symmetric (semi-metric) space, (X,d) said to be weakly compatible(or coincidentally commuting) if $fx=gx$ implies $fgx=gfx$.

Definition 1.6. [17] A pair of self-mappings (f,g) on a symmetric (semi-metric) space, (X,d) said to enjoy $E.A$-property if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} f(x_n)=\lim_{n\to\infty} g(x_n)=t$, for some $t\in X$.

Clearly non compatible pairs satisfy property $(E.A)$. The concept of C-class functions was introduced by Ansari in [2] that is pivotal result in fixed point theory; see [3] , [4] and [5].

Definition 1.7. [2] A mapping $f:[0,\infty)^2\to\mathbb{R}$ is called C-class function if it is continuous and satisfies following axioms:

(1) $f(s,t)\leq s$,

(2) $f(s,t)=s$ implies that either $s=0$ or $t=0$; for all $s,t\in[0,\infty)$.

For some f we have that $f(0,0)=0$. We denote C-class functions as $\mathscr{C}$.

Example 1.8. [2] The following functions $F:[0,\infty)^2\to\mathbb{R}$ are elements of $\mathscr{C}$, for all $s,t\in[0,\infty)$:

(1) $f(s,t)=s-t$, $f(s,t)=s\Rightarrow t=0$;

(2) $f(s,t)=ms$, $0<m<1$, $f(s,t)=s\Rightarrow s=0$;

(3) $f(s,t)=\frac{s}{(1+t)^r}$; $r\in(0,\infty)$, $f(s,t)=s\Rightarrow s=0$ or $t=0$;

(4) $f(s,t)=\log(t+a^s)/(1+t)$, $a>1$, $f(s,t)=s\Rightarrow s=0$ or $t=0$;

(5) $F(s,t) = \ln(1+a^s)/2$, $a > e$, $F(s,1) = s \Rightarrow s = 0$;

(6) $F(s,t) = (s+l)^{(1/(1+t)^r)} - l$, $l > 1, r \in (0,\infty)$, $F(s,t) = s \Rightarrow t = 0$;

(7) $F(s,t) = s\log_{t+a} a$, $a > 1$, $F(s,t) = s \Rightarrow s = 0$ or $t = 0$;

(8) $f(s,t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$, $f(s,t) = s \Rightarrow t = 0$;

(9) $F(s,t) = s\beta(s)$, $\beta : [0,\infty) \to [0,1)$,and is continuous, $F(s,t) = s \Rightarrow s = 0$;

(10) $F(s,t) = s - \frac{t}{k+t}, F(s,t) = s \Rightarrow t = 0$;

(11) $F(s,t) = s - \varphi(s), F(s,t) = s \Rightarrow s = 0$,here $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$;

(12) $F(s,t) = sh(s,t), F(s,t) = s \Rightarrow s = 0$,here $h : [0,\infty) \times [0,\infty) \to [0,\infty)$is a continuous function such that $h(t,s) < 1$ for all $t,s > 0$;

(13) $f(s,t) = s - (\frac{2+t}{1+t})t$, $f(s,t) = s \Rightarrow t = 0$.

(14) $f(s,t) = \sqrt[n]{\ln(1+s^n)}$, $f(s,t) = s \Rightarrow s = 0$.

(15) $f(s,t) = \phi(s), f(s,t) = s \Rightarrow s = 0$,here $\phi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\phi(0) = 0$, and $\phi(t) < t$ for $t > 0$,

(16) $f(s,t) = \frac{s}{(1+s)^r}$; $r \in (0,\infty)$, $f(s,t) = s \Rightarrow s = 0$.

Definition 1.9. [9] A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is non-decreasing and continuous,

(ii) $\psi(t) = 0$ if and only if $t = 0$.

In this paper, we denote Ψ set altering distance functions.

Definition 1.10. [2] An ultra altering distance function is a continuous, nondecreasing mapping $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t) > 0$, $t > 0$ and

An ultra altering distance function is a continuous, nondecreasing mapping $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t) > 0$, $t > 0$.

We denote Φ_u, the set of ultra altering distance functions. We can find some convergence axioms in [20].

$C_1 : \lim_{n\to\infty} d(x_n,y_n) = 0 = \lim_{n\to\infty} d(x_n,x) \Rightarrow \lim_{n\to\infty} d(y_n,x) = 0$.

$C_2 : \lim_{n\to\infty} d(x_n,x) = 0, \lim_{n\to\infty} d(y_n,x) = 0 \Rightarrow \lim_{n\to\infty} d(x_n,y_n) = 0$.

2. Some theorems related to coincidence points and fixed points

Theorem 2.1. [16] *Let (X,d) be a symmetric (semi-metric) space with W_3 or a Hausdorff semi-metric space. Let (f,g) be a pair of self maps of X that has the $(E.A)$-property and*

(1) $d(gx,gy) < max\{d(fx,gx),d(fy,gy),d(fx,fy)\}$,

(2) $f(X)$ is a closed subset of a X.

Then f and g have a point of coincidence.

The following variant of Theorem 2.1 also holds.

Theorem 2.2. [16] *Theorem 2.1 remains true if d-closedness ($\mathfrak{F}(d)$-closedness) of $f(X)$ is replaced by d-closedness ($\mathfrak{F}(d)$-closedness) of $g(X)$ along with $g(X) \subset f(X)$ retaining the rest of the hypotheses.*

Theorems 2.1 and 2.2 ensure common fixed point instead of point of coincidence if contractive condition (1) of theorem 2.1 is replaced by a slightly weaker condition.

Theorem 2.3. [16] *In the setting of Theorems 2.1 and 2.2, f and g have a unique common fixed point provided f and g are weakly compatible and satisfy the contraction condition (1) of Theorem 2.1 for all $x \neq y \in X$, $d(gx,gy) < max\{d(fx,gx),d(fy,gy),d(fx,fy)\}$.*

Proof. In view of Theorems 2.1 and 2.2, f and g have a point of coincidence 'a'. $i.e.$, $f(a) = g(a)$. Now due to weak compatibility one can write $fg(a) = ff(a) = gg(a) = gf(a)$. If $gg(a) = g(a)$ then (1) of Theorem2.3 implies

$$d(ga,gga) < max\{d(fa,ga),d(fga,gga),d(fa,fga)\} = d(ga,gga),$$

which is a contradiction. Hence $ga = gga = gfa = fga = ffa$, which shows that ga is a common fixed point of f and g. Uniqueness of the common fixed point follows easily.

Corollary 2.4. [16] *Let (X,d) be a symmetric (semi-metric) space that enjoys W_3 (the Hausdorff separation axiom). Let g be a self map of X such that for all $x \neq y \in X$, $d(gx,gy) < max\{d(x,gx),d(y,gy),d(x,y)\}$. Then g has a unique fixed point.*

Proof. If we take $f = I$ the identity mapping in Theorem 2.3, and follow a similar proof as that in Theorem 2.3, we establish this Corollary 2.4.

In [10], Sumati Kumari presented a few results that establish the existence of common periodic points for a pair of maps on a symmetric (metric) space when the maps have a unique common fixed point. These results are supported by suitable examples.

Theorem 2.5. [16] *Let be a self map of a symmetric space X,d satisfying*

$$d(fx,fy) < max\{d(x,y),d(x,fx),d(y,fy),d(x,fy),d(y,fx)\}$$

for each $x,y \in X(x \neq y)$ for which the right hand side of above inequality is not zero. Then $u \in X$ is a periodic point of f if and only if u is the unique fixed point of f.

By using Corollary 2.4 and Theorem 2.5, we have the following result.

Theorem 2.6. [16] *Let g be a self map on a symmetric space (X,d) satisfying $d(gx,gy) < max\{d(x,y),d(x,gx),d(y,gy)\}$ for each $x,y \in X(x \neq y)$ for which the right hand side of above inequality is not zero. Then $u \in X$ is a periodic point of g if and only if u is the unique fixed point of g.*

To illustrate the above theorem, we have the following example.

Example 2.7. [16] *Let $X = [0,1)$ and $d(x,y) = |x-y|^2$. The inequality can be easily checked. Then next theorem involves a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfies the following conditions:*

(1) ϕ is non-decreasing on $\mathbb{R}^+$,

(2) $0 < \phi(t) < t$ for each $t \in (0,\infty)$.

Theorem 2.8. *Let A,B,S and T be self-mappings of a symmetric (semi-metric) space (X,d) that enjoy W_3 (the Hausdorff's T_2 separation axiom) and $F \in \mathscr{C}$, $\psi \in \Psi$, $\phi \in \Phi_u$. Suppose that*

(1) $A(X) \subset T(X)$, $B(X) \subset S(X)$,

(2) The pair (B,T) enjoys the property $(E.A)$ (or alternatively the pair (A,S) enjoys the property $(E.A)$),

(3) $d(Ax,By) \leq F(\psi(m(x,y)),\phi(m(x,y)))$, where $m(x,y) = max\{d(Sx,Ax),d(Ty,By),d(Sx,Ty)\}$,

(4) $S(X)$ is d-closed($\mathfrak{F}(d)$-closed) subset of X (or alternatively, $T(X)$ is d-closed($\mathfrak{F}(d)$-closed) subset of X.)

Then pairs (A,S) has a point of coincidence u and the pair (B,T) has a point of coincidence w.

Proof. Since the pair (B,T) enjoys the property $(E.A)$), there exists a sequence $\{X_n\} \subset X$ such that $\lim_{n\to\infty} B(x_n) = \lim_{n\to\infty} T(x_n) = t \in X$.. Since $B(X) \subset S(X)$, for each x_n there exists y_n such that $Bx_n = Sy_n$. Thus in all $Bx_n \to t, Sy_n \to t$ and $Tx_n \to t$. Now we assert that $Ay_n \to t$. Suppose, to obtain a contradiction in each case.

Case (i) $Ay_m = Bx_m$ for only finitely many m.

Let $Ay_m \neq Bx_m$ for all $m \geq n_0$, where n_0 is a positive integer. Then for $m \geq n_0$,

$$\begin{aligned}
\psi(d(Ay_m,Bx_m)) \leq F(&\psi(max\{d(Sy_m,Ay_m),d(Tx_m,Bx_m),d(Sy_m,Tx_m)\}),\\
&\phi(max\{d(Sy_m,Ay_m),d(Tx_m,Bx_m),d(Sy_m,Tx_m)\}))\\
&= F(\psi(max\{d(Bx_m,Ay_m),d(Tx_m,Bx_m),d(Bx_m,Tx_m)\}),\\
&\phi(max\{d(Bx_m,Ay_m),d(Tx_m,Bx_m),d(Bx_m,Tx_m)\}))\\
&= F(\psi(max\{d(Bx_m,Ay_m),d(Tx_m,Bx_m)\}),\\
&\phi(max\{d(Bx_m,Ay_m),d(Tx_m,Bx_m)\}))\\
&\leq \psi(max\{d(Bx_m,Ay_m),d(Tx_m,Bx_m)\}).
\end{aligned}$$

Hence $\max\{d(Bx_m,Ay_m),d(Bx_m,Tx_m)\} = d(Bx_m,Ay_m)$. It follows that

$$\psi(d(Ay_m,Bx_m)) \leq F(\psi(d(Bx_m,Ay_m)),\phi(d(Bx_m,Ay_m))).$$

So $\psi(d(Bx_m,Ay_m)) = 0$, or $\phi(d(Bx_m,Ay_m)) = 0$. Therefore, we have $d(Bx_m,Ay_m) = 0$, which is a contradiction. Hence $max\{d(Bx_m,Ay_m),d(Bx_m,Tx_m)\} = d(Bx_m,Tx_m)$. This implies that

$$\psi(d(Ay_m,Bx_m)) \leq F(\psi(d(Bx_m,Ay_m)),\phi(d(Bx_m,Ay_m))) \leq \psi(d(Bx_m,Ay_m)).$$

Hence, we have

$$d(Ay_m,Bx_m) \leq d(Bx_m,Ay_m).$$

Letting $m \to \infty$ and using C_2, we get

$$\begin{aligned}
\limsup_{m\to\infty} d(Ay_m,Bx_m) &\leq \lim_{m\to\infty} d(Bx_m,Tx_m)\\
&= 0 \text{ by } (C_2).
\end{aligned}$$

From the fact that $\lim Bx_m = \lim Tx_m = t$, we have $\lim supd(Ay_m,Bx_m) = 0$, which implies $\lim d(Ay_m,Bx_m) = 0$. This implies $\lim Ay_m = \lim Bx_m = t$ by C_1. Therefore $\lim Ay_m = t$.

Case (ii) Suppose $Ay_m = Bx_m$ for infinitely many values of m. Let the sequence $\mathscr{K} = \{k_1 < k_2 < k_3 < ...\}$ enjoying the property that $Ay_{k_i} = Bx_{k_i}$ for $i = 1,2...$ and $\mathscr{J} = \{j_1 < j_2 < j_3 < ...\}$ have the property that $Ay_{j_i} \neq Bx_{j_i}$for $j = 1,2...$ and let $\mathscr{K} \cup \mathscr{J} = \mathbb{N}$. Then the sequence $Ay_{k_i} \to t$ since the sequence $Bx_{k_i} \to t$. If $\mathscr{J}$ is a finite set then we may suppose $\mathscr{J} = \emptyset$ and conclude $Ay_m \to t$. Otherwise we can conclude that $Ay_{j_i} \to t$ as in case (i). Since $\mathscr{K} \cup \mathscr{J} = \mathbb{N}$ and since $Ay_{k_i} \to t$ and also $Ay_{j_i} \to t$, It is clear that $Ay_m \to t$.

Suppose that $S(X)$ is a d-closed subset of X then $Sy_n \to t$ and one can find a point $u \in X$ such that $Su = t$. Now we suppose that $Au \neq Su$. Then

$$\begin{aligned}\psi(d(Au,Bx_n) \leq & F(\psi(max\{d(Su,Au),d(Tx_n,Bx_n),d(Su,Tx_n)\}),\\ & \phi(\max\{d(Su,Au),d(Tx_n,Bx_n),d(Su,Tx_n)\}))),\end{aligned}$$

which on letting $n \to \infty$ yields

$$\psi(d(Au,Su)) \leq F(\psi(d(Su,Au)),\phi(d(Su,Au))).$$

So $\psi(d(Su,Au)) = 0$, or $\phi(d(Su,Au)) = 0$. Therefore $d(Su,Au) = 0$, which is a contradiction. Hence $Au = Su$. Also $A(X) \subset T(X)$, there exists $w \in X$ such that $Au = Tw$. We assert that $Tw = Bw$. If not, then using inequality (3) of Theorem 2.7, one gets

$$\begin{aligned}\psi(d(Au,Bw)) &\leq F(\psi(max\{d(Su,Au),d(Tw,Bw),d(Su,Tw)\}),\\ &\quad \phi(max\{d(Su,Au),d(Tw,Bw),d(Su,Tw)\}))\\ &= F(\psi(d(Tw,Bw)),\phi(d(Tw,Bw))\\ &= F(\psi(d(Au,Bw)),\phi(d(Au,Bw)).\end{aligned}$$

It follows that $\psi(d(Au,Bw)) = 0$, or $\phi(d(Au,Bw)) = 0$. Therefore $d(Au,Bw) = 0$, which is a contradiction. Hence $Au = Su = Bw = Tw$. This shows that the pairs (A,S) and (B,T) have a point of coincidence $u\&w$. The proof is similar if we consider the case when pair (A,S) enjoys property $(E.A)$, and $T(X)$ is d-closed subset of X. Hence it is omitted. This completes the proof of the theorem.

Theorem 2.9. *In the setting of Theorem 2.7 , A,B,S and T have a unique common fixed point provided one adds the weak compatibility of the pair (A,S) (or weak compatibility of the pair*

(B,T) and satisfying the contractive condition (3) of Theorem 2.7 for $x \neq y \in X$,

$$\psi(d(Ax,By)) \leq F(\psi(m(x,y)),\phi(m(x,y))),$$

where $m(x,y) = max\{d(Sx,Ay),d(Ty,By),d(Sx,Ty)\}$,

Proof. In view of Theorem 2.7, one concludes that $Au = Su = Bw = Tw$. Now the weak compatibility of (A,S) implies that $ASu = SAu$ and $AAu = ASu = SAu = SSu$. Suppose that $Au \neq AAu$ then using (3) 0f Theorem 2.7, one gets

$$\begin{aligned} d(Au,AAu) &= d(AAu,Bw) \\ &\leq F(\psi(max\{d(SAu,AAu),d(Tw,Bw),d(SAu,Tw)\}), \\ &\quad ,\phi(max\{d(SAu,AAu),d(Tw,Bw),d(SAu,Tw)\})) \\ &= F(\psi(d(Au,AAu)),\varphi(d(Au,AAu))) \end{aligned}$$

It follows that $\psi(d(Au,AAu)) = 0$, or $\phi(d(Au,AAu)) = 0$. Therefore $d(Au,AAu) = 0$, which is a contradiction. Thus $Au = AAu = SAu$. Then Au is the common fixed point of A and S. Also Au is a common fixed point of the pair (B,T). Uniqueness of the common fixed point follows easily. The proof is similar in the other case. This completes the proof.

Corollary 2.10. *Let f be self map of a symmetric (semi-metric) space that enjoys W_3 (the Hausdorffness of $\mathfrak{F}(d)$) and satisfying $d(fx,fy) \leq F(\psi(m(x,y)),\phi(m(x,y)))$, where $m(x,y) = max\{d(x,fx),d(y,fy),d(x,y).\}$ Then f has a unique fixed point.*

Proof. Take $A = B = f$ and $S = T = I$ an identity mapping in Theorem 2.8, and follow the similar proof as that in Theorem 2.8, we find the desired conclusion immediately.

3. Common fixed point theorems via a family of C-class functions

Definition 3.1. [11] Let X be a non-empty set and $\{d_\alpha : \alpha \in (0,1]\}$ a family of mapping d_α of $X \times X$ into $\mathbb{R}^+$. Then (X,d_α) is called a generating space of symmetric family if it satisfied the following conditions for any $x,y \in X$.

(i) $d_\alpha(x,y) = 0$ if and only if $x = y$ $\forall\alpha \in (0,1]$;

(ii) $d_\alpha(x,y) = d_\alpha(y,x)$ $\forall\alpha \in (0,1]$.

Definition 3.2. A pair (A,χ) of self mappings of a G_s-family (X,d_α) is said to satisfy the common limit range of χ property, abbreviated as (CLR_χ)-property, if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} \chi x_n = \eta$, where $\eta \in \chi(X)$.

Definition 3.3. Two pairs (A,χ) and (B,ϑ) of self mappings of a G_s-family (X,d_α) is said to satisfy the common limit range of χ andϑ property, abbreviated as $(CLR_{\chi\vartheta})$-property, if there exist two sequence $\{x_n\}$ and $\{y_n\}$ in X such that,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} \chi x_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} \vartheta y_n = \eta, \text{ where } \eta \in \chi(X) \cap \vartheta(X).$$

Also, let Φ denote the set of all increasing functions $\Upsilon : [0,+\infty) \to [0,+\infty)$ that satisfy the below conditions

(1) Υ is lower semi-continuous on $[0,+\infty)$;
(2) $\Upsilon(0) \geq 0$;
(3) $\Upsilon(\lambda) > 0$ for each $\lambda > 0$.

Theorem 3.4. *Let A,B,χ and ϑ be self mappings of a G_s-family (X,d_α). Suppose that the following criteria hold.*

(1) the pair (A,χ) satisfies the (CLR_χ)-property (or the pair (B,ϑ) satisfies the (CLR_ϑ)-property).
(2) $A(X) \subset \chi(X)$ (or $B(X) \subset \vartheta(X)$).
(3) $\vartheta(X)$ (or $\chi(X)$) is a closed subset of X.
(4) $\{By_n\}$ converges for every sequence $\{y_n\}$ in X whenever ϑy_n converges (or $\{Ax_n\}$ converges for every sequence $\{x_n\}$ in X whenever χx_n converges).
(5) there exists $\Upsilon \in \Phi$ such that

$$\int_0^{d_\alpha(Ax,By)} \phi(t)dt \leq F(M(x,y), \Upsilon(M(x,y))), \forall x,y \in X, \tag{3.1}$$

where $M(x,y) = \int_0^{max\{d_\alpha(By,\chi x),\frac{k}{2}[d_\alpha(\chi x,\vartheta y)+d_\alpha(By,\vartheta y)],\frac{k}{2}[d_\alpha(By,\chi x)+d_\alpha(Ax,\vartheta y)]\}} \phi(t)dt$
and $\phi : [0,\infty) \to [0,\infty)$ *is a Lebesgue-integrable mapping which is summable such that*

$$\int_0^{\varepsilon} \phi(t)dt > 0, \tag{3.2}$$

for all $\varepsilon > 0$ *and* $1 \leq k < 2$.

Then the pairs (A,χ) *and* (B,ϑ) *satisfies* $(CLR_{\chi\vartheta})$*-property and have a coincidence point. Moreover,* A,B,χ *and* ϑ *have a unique common fixed point if both the pairs are weakly compatible.*

Proof. From given hypothesis, the pair (A,χ) satisfies the (CLR_χ)-property. Thus there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} \chi x_n = \eta, \text{ where } \eta \in \chi(X).$$

From the statement of the theorem, we have $A(X) \subset \chi(X)$ ($\vartheta(X)$ is closed subset of X,) so for each $\{x_n\} \subset X$, there exists a sequence $\{y_n\} \subset X$ such that $Ax_n = \vartheta y_n$.
Hence $\lim_{n\to\infty} \vartheta y_n = \lim_{n\to\infty} Ax_n = \eta$. Since $\vartheta(X)$ is closed, $\eta \in \chi(X)$. Therefore $\eta \in \chi(X) \cap \vartheta(X)$. Thus we have $Ax_n \to \eta,\ \chi x_n \to \eta$ and $\vartheta y_n \to \eta$ as $n \to \infty$. Again from the statement of the theorem, the sequences By_n converges.

Now we will prove that the pairs (A,χ) and (B,ϑ) satisfies $(CLR_{\chi\vartheta})$-property. And in all we need to show that $By_n \to \eta$ as $n \to \infty$. If we take x_n instead of x and y_n instead of y, we get

$$\int_0^{d_\alpha(Ax_n,By_n)} \phi(t)dt \leq F(M(x_n,y_n),\Upsilon(M(x_n,y_n))) \tag{3.3}$$

where $M(x_n,y_n) = \int_0^{max\{d_\alpha(By_n,\chi x_n),\frac{k}{2}[d_\alpha(\chi x_n,\vartheta y_n)+d_\alpha(By_n,\vartheta y_n)],\frac{k}{2}[d_\alpha(By_n,\chi x_n)+d_\alpha(Ax_n,\vartheta y_n)]\}} \phi(t)dt$. Let us assume that $By_n \to \xi(\neq \eta)$ for $t > 0$ as $n \to \infty$. By taking limit as $n \to \infty$ in (3.3), we get

$$\int_0^{d_\alpha(\eta,\xi)} \phi(t)dt \leq F(\lim_{n\to\infty} M(x_n,y_n),\Upsilon(\lim_{n\to\infty} M(x_n,y_n))), \tag{3.4}$$

where

$$\begin{aligned}\lim_{n\to\infty} M(x_n,y_n) &= \int_0^{max\{d_\alpha(\xi,\eta),\frac{k}{2}[d_\alpha(\eta,\eta)+d_\alpha(\xi,\eta)],\frac{k}{2}[d_\alpha(\xi,\eta)+d_\alpha(\eta,\eta)]\}} \phi(t)dt \\ &= \int_0^{max\{d_\alpha(\xi,\eta),\frac{k}{2}d_\alpha(\xi,\eta)\}} \phi(t)dt \\ &= \int_0^{d_\alpha(\xi,\eta)} \phi(t)dt\end{aligned} \tag{3.5}$$

since $1 \leq k < 2$. Hence from (3.5), we get

$$\int_0^{d_\alpha(\xi,\eta)} \phi(t)dt \leq F\Big(\int_0^{d_\alpha(\xi,\eta)} \phi(t)dt, \Upsilon\Big(\int_0^{d_\alpha(\xi,\eta)} \phi(t)dt\Big)\Big),$$

which implies

$$\int_0^{d_\alpha(\xi,\eta)} \phi(t)dt = 0 \quad \text{or} \quad \Upsilon\Big(\int_0^{d_\alpha(\xi,\eta)} \phi(t)dt\Big) = 0.$$

From the definition of ϕ, Υ, $d_\alpha(\xi,\eta) = 0$ or equivalently $\xi = \eta$, which contradicts to $\xi \neq \eta$. Hence (A,χ) and (B,ϑ) share the $(CLR_{\chi\vartheta})$-property. Since the pairs (A,χ) and (B,ϑ) satisfies the (CLR_χ)-property, then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} \chi x_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} \vartheta y_n = \eta, \text{ where } \eta \in \chi(X) \cap \vartheta(X).$$

As $\eta \in \chi(X)$, there exists a point $\ell \in X$ such that $\chi\ell = \eta$. Now we prove that $A\ell = \eta$. In order to prove this, let $A\ell \neq \eta$. Note that

$$\int_0^{d_\alpha(A\ell,By_n)} \phi(t)dt \leq F(M(\ell,y_n), \Upsilon(M(\ell,y_n))), \tag{3.6}$$

where $M(\ell,y_n) = \int_0^{max\{d_\alpha(By_n,\chi\ell),\frac{k}{2}[d_\alpha(\chi\ell,\vartheta y_n)+d_\alpha(By_n,\vartheta y_n)],\frac{k}{2}[d_\alpha(By_n,\chi\ell)+d_\alpha(A\ell,\vartheta y_n)]\}} \phi(t)dt$. By letting $n \to \infty$ in (3.6), we get

$$\int_0^{d_\alpha(A\ell,\eta)} \phi(t)dt \leq F(\lim_{n\to\infty} M(\ell,y_n), \Upsilon(\lim_{n\to\infty} M(\ell,y_n))), \tag{3.7}$$

where,

$$\begin{aligned}\lim_{n\to\infty} M(\ell, y_n) &= \int_0^{max\{d_\alpha(\eta,S\ell),\frac{k}{2}[d_\alpha(\chi\ell,\eta)+d_\alpha(\eta,\eta)],\frac{k}{2}[d_\alpha(\eta,\chi\ell)+d_\alpha(A\ell,\eta)]\}} \phi(t)dt \\ &= \int_0^{\frac{k}{2}d_\alpha(A\ell,\eta)} \phi(t)dt\end{aligned} \tag{3.8}$$

From (3.7), we find that

$$\begin{aligned}\int_0^{d_\alpha(A\ell,\eta)} \phi(t)dt &\leq F\Big(\int_0^{\frac{k}{2}d_\alpha(A\ell,\eta)} \phi(t)dt, \Upsilon\Big(\int_0^{\frac{k}{2}d_\alpha(A\ell,\eta)} \phi(t)dt\Big)\Big) \\ &\leq F\Big(\int_0^{d_\alpha(A\ell,\eta)} \phi(t)dt, \Upsilon\Big(\int_0^{\frac{k}{2}d_\alpha(A\ell,\eta)} \phi(t)dt\Big)\Big),\end{aligned} \tag{3.9}$$

which yields $A\ell = \eta$. Therefore $\chi\ell = A\ell = \eta$. This implies that ℓ is a coincidence point of the pair (A,χ). As $\eta \in \vartheta(X)$, there exists a point $\eta_1 \in X$ such that $\vartheta\eta_1 = \eta$. On the other hand, we have

$$\int_0^{d_\alpha(Ax_n,B\eta_1)} \phi(t)dt \leq F(M(x_n,\eta_1), \Upsilon(M(x_n,\eta_1))), \tag{3.10}$$

where,

$$\begin{aligned}M(x_n,\eta_1) &= \int_0^{max\{d_\alpha(B\eta_1,\chi x_n),\frac{k}{2}[d_\alpha(\chi x_n,\vartheta\eta_1)+d_\alpha(B\eta_1,\vartheta\eta_1)],\frac{k}{2}[d_\alpha(B\eta_1,\chi x_n)+d_\alpha(Ax_n,\vartheta\eta_1)]\}} \phi(t)dt \\ &= \int_0^{max\{d_\alpha(B\eta_1,\eta),\frac{k}{2}[d_\alpha(\eta,\eta)+d_\alpha(B\eta_1,\eta)],\frac{k}{2}[d_\alpha(B\eta_1,\eta)+d_\alpha(\eta,\eta)]\}} \phi(t)dt \\ &= \int_0^{d_\alpha(B\eta_1,\eta)} \phi(t)dt\end{aligned} \tag{3.11}$$

Equation (3.10) yields

$$\int_0^{d_\alpha(\eta,B\eta_1)} \phi(t)dt \leq F\Big(\int_0^{d_\alpha(B\eta_1,\eta)} \phi(t)dt, \Upsilon\Big(\int_0^{d_\alpha(B\eta_1,\eta)} \phi(t)dt\Big)\Big).$$

Thus $\int_0^{d_\alpha(B\eta_1,\eta)} \phi(t)dt = 0$, or $\Upsilon(\int_0^{d_\alpha(B\eta_1,\eta)} \phi(t)dt) = 0$. From the property of ϕ, Υ, $d_\alpha(B\eta_1,\eta) = 0$, which yields $B\eta_1 = \eta$. Thus $B\eta_1 = \vartheta\eta_1 = \eta$, which shows that η_1 is a coincidence point of the pair (B,ϑ). Since the pair (A,χ) and (B,ϑ) are weakly compatible, $A\ell = S\ell$ and $B\eta_1 = \vartheta\eta_1$. Therefore $A\eta = A\chi\ell = \chi A\ell = \chi\eta$ and $B\eta = B\vartheta\eta_1 = \vartheta B\eta_1 = \vartheta\eta$. Note that

$$\int_0^{d_\alpha(A\eta,B\eta_1)} \phi(t)dt \leq F(M(\eta,\eta_1),\Upsilon(M(\eta,\eta_1))), \tag{3.12}$$

where,

$$\begin{aligned} M(\eta,\eta_1) &= \int_0^{max\{d_\alpha(B\eta_1,\chi\eta),\frac{k}{2}[d_\alpha(\chi\eta,\vartheta\eta_1)+d_\alpha(B\eta_1,\vartheta\eta_1)],\frac{k}{2}[d_\alpha(B\eta_1,\chi\eta)+d_\alpha(A\eta,\vartheta\eta_1)]\}} \phi(t)dt \\ &= \int_0^{max\{d_\alpha(\eta,A\eta),\frac{k}{2}[d_\alpha(A\eta,\eta)+d_\alpha(\eta,\eta)],\frac{k}{2}[d_\alpha(\eta,A\eta)+d_\alpha(A\eta,\eta)]\}} \phi(t)dt \\ &= \int_0^{d_\alpha(\eta,A\eta)} \phi(t)dt \end{aligned} \tag{3.13}$$

From (3.12), we get

$$\int_0^{d_\alpha(A\eta,\eta)} \phi(t)dt \leq F(\int_0^{d_\alpha(\eta,A\eta)} \phi(t)dt, \Upsilon(\int_0^{d_\alpha(\eta,A\eta)} \phi(t)dt)).$$

It follows that $\int_0^{d_\alpha(\eta,A\eta)} \phi(t)dt = 0$, or $\Upsilon(\int_0^{d_\alpha(\eta,A\eta)} \phi(t)dt) = 0$. Therefore $A\eta = \eta$. Thus $A\eta = \chi\eta = \eta$ and therefore η is a common fixed point of the pair (A,χ). If we take $x = \ell$ and $y = \eta$ in (3.1), we get

$$\int_0^{d_\alpha(A\ell,B\eta)} \phi(t)dt \leq F(M(\ell,\eta),\Upsilon(M(\ell,\eta))), \tag{3.14}$$

where,

$$\begin{aligned} M(\ell,\eta) &= \int_0^{\max\{d_\alpha(B\eta,\chi\ell),\frac{k}{2}[d_\alpha(\chi\ell,\vartheta\eta)+d_\alpha(B\eta,\vartheta\eta)],\frac{k}{2}[d_\alpha(B\eta,\chi\ell)+d_\alpha(A\ell,\chi\eta)]\}} \phi(t)dt \\ &= \int_0^{\max\{d_\alpha(B\eta,\eta),\frac{k}{2}[d_\alpha(\eta,\chi\eta)+d_\alpha(\chi\eta,\chi\eta)],\frac{k}{2}[d_\alpha(B\eta,\eta)+d_\alpha(\eta,B\eta)]\}} \phi(t)dt \qquad (3.15) \\ &= \int_0^{d_\alpha(B\eta,\eta)} \phi(t)dt \end{aligned}$$

From (3.14), we get

$$\int_0^{d_\alpha(\eta,B\eta)} \phi(t)dt \le F(\int_0^{d_\alpha(\eta,B\eta)} \phi(t)dt, \Upsilon(\int_0^{d_\alpha(\eta,B\eta)} \phi(t)dt)).$$

Thus $\int_0^{d_\alpha(\eta,B\eta)} \phi(t)dt = 0,, \Upsilon(\int_0^{d_\alpha(\eta,B\eta)} \phi(t)dt) = 0$. Therefore $\eta = B\eta$. Which implies $B\eta = \chi\eta = \eta$. Therefore η is a common fixed point of A, B, χ and ϑ. In order to prove uniqueness, suppose z be another common fixed point of A, B, χ and ϑ. *i.e.*, $Az = Bz = \vartheta z = \chi z = z$. Putting $x = z, y = \eta$ in (3.1), we have

$$\int_0^{d_\alpha(Az,B\eta)} \phi(t)dt \le F(M(z,\eta), \Upsilon(M(z,\eta))), \qquad (3.16)$$

where,

$$\begin{aligned} M(z,\eta) &= \int_0^{\max\{d_\alpha(B\eta,\chi z),\frac{k}{2}[d_\alpha(\chi z,\vartheta\eta)+d_\alpha(B\eta,\vartheta\eta)],\frac{k}{2}[d_\alpha(B\eta,\chi z)+d_\alpha(Az,\vartheta\eta)]\}} \phi(t)dt \\ &= \int_0^{\max\{d_\alpha(\eta,z),\frac{k}{2}[d_\alpha(z,\eta)+d_\alpha(\eta,\eta)],\frac{k}{2}[d_\alpha(\eta,z)+d_\alpha(z,\eta)]\}} \phi(t)dt \\ &= \int_0^{d_\alpha(\eta,z)} \phi(t)dt. \end{aligned}$$

From (3.16), we get

$$\int_0^{d_\alpha(z,\eta)} \phi(t)dt \le F(\int_0^{d_\alpha(z,\eta)} \phi(t)dt, \Upsilon(\int_0^{d_\alpha(z,\eta)} \phi(t)dt)).$$

Thus $\int\limits_{0}^{d_\alpha(z,\eta)} \phi(t)dt = 0$, or$\Upsilon(\int\limits_{0}^{d_\alpha(z,\eta)} \phi(t)dt) = 0$. Therefore $z = \eta$. Hence A, B, χ and ϑ have a unique common fixed point.

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