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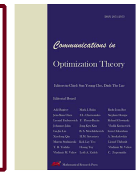

EVEN TUPLED COINCIDENCE AND COMMON FIXED POINT RESULTS FOR WEAKLY CONTRACTIVE MAPPINGS IN COMPLETE METRIC SPACES VIA NEW FUNCTIONS

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Abstract. In this paper, we prove results on even tupled coincidence and common fixed points in ordered complete metric spaces for a pair of weakly contractive compatible mappings under some new control functions. Moreover, we also illustrate our main result with an example in arbitrary even order case.

Keywords. Partially ordered set; Control function; Compatible mapping; Mixed g-monotone property; n-tupled coincidence point.

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1. Introduction

Branciari [7] established a fixed point result for an integral-type inequality, which is a generalization of Banach contraction principle. Vijayaraju *et al.* [27] obtained a general principle, which made it possible to prove many fixed point theorems for pairs of integral type maps. Kada *et al.* [14] defined the concept of w-distance in a metric space and studied some fixed point

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theorems. Afterwards, Razani *et al.* [25] proved a fixed point theorem which is a new version of the main theorem in [7], by considering the concept of the w-distance, as follows:

Theorem 1.1. ([25]) *Let p be a w-distance on a complete metric space (X,d). Let ϕ be nondecreasing, continuous and $\phi(\varepsilon) > 0$ for each $\varepsilon > 0$ and ψ be nondecreasing, right continuous and $\psi(t) < t$ for all $t > 0$. Suppose T is a (ϕ, ψ, p)-contractive map on X. Then T has a unique fixed point in X. Moreover, $\lim_{n\to\infty} T^n x$ is a fixed point of T for each $x \in X$.*

The investigation of fixed points in ordered metric spaces is a relatively new development which appears to have its origin in the paper of Ran and Reurings [24] which was well complimented by Nieto and López [22]. The concept of multi-dimensional fixed point was introduced by Guo and Lakshmikantham [10]. In [6], Bhaskar and Lakshmikantham proved some coupled fixed point theorems for a mapping $F : X^2 \to X$ in ordered complete metric space. In this continuation, Lakshmikantham and Ćirić [20] generalized these results for non-linear ϕ-contraction mapping by introducing two ideas namely: coupled coincidence point and mixed g-monotone property. In an attempt to extend the definition from X^2 to X^3, Berinde and Borcut [5] introduced the concept of tripled fixed point and utilize the same to prove some tripled fixed point theorems. After that, Karapınar [15] introduced the quadrupled fixed point to prove some quadrupled fixed point theorems for nonlinear contraction mappings satisfying mixed g-monotone property; see [16, 17] and the references therein.

Recently, Samet and Vetro [26] extended the idea of coupled as well as quadrupled fixed point to higher dimensions by introducing the notion of fixed point of n-order (or n-tupled fixed point, where $n \in \mathbb{N}$ and $n \geq 3$) and presented some n-tupled fixed point results in complete metric spaces, using a new concept of f-invariant set. Here it can be pointed out that the notion of tripled fixed point due to Berinde and Borcut [5] is different from the one defined by Samet and Vetro [26] for $n = 3$ in the case of ordered metric spaces in order to keep the mixed monotone property working. Recently, Imdad *et al.* [11] extended the idea of mixed g-monotone property to the mapping $F : X^n \to X$ (where n is even natural number) and proved an even-tupled coincidence point theorem for nonlinear contraction mappings satisfying mixed g-monotone property.

2. Preliminaries

Definition 2.1. Let X be a non-empty set. A relation '$\preccurlyeq$' on X is said to be a partial order if the following properties are satisfied:

(i) reflexive: $x \preccurlyeq x$ for all $x \in X$,

(ii) anti-symmetric: $x \preccurlyeq y$ and $y \preccurlyeq x$ implies $x = y$,

(iii) transitive: $x \preccurlyeq y$ and $y \preccurlyeq z$ implies $x \preccurlyeq z$ for all $x, y, z \in X$.

A non-empty set X together with a partial order '$\preccurlyeq$' is said to be an ordered set and we denote it by $(X, \preccurlyeq)$.

Definition 2.2. Let $(X, \preccurlyeq)$ be an ordered set. Any two elements x and y are said to be comparable elements in X if either $x \preccurlyeq y$ or $y \preccurlyeq x$.

Definition 2.3. ([23]) A triplet $(X, d, \preccurlyeq)$ is called an ordered metric space if (X, d) is a metric space and $(X, \preccurlyeq)$ is an ordered set. Moreover, if d is a complete metric on X, then we say that $(X, d, \preccurlyeq)$ is an ordered complete metric space.

Throughout the paper, n stands for a general even natural number. Let us denote by X^n the product space $X \times X \times \ldots \times X$ of n identical copies of X.

Definition 2.4. ([11]) Let $(X, \preccurlyeq)$ be an ordered set and $F : X^n \to X$ and $g : X \to X$ two mappings. Then F is said to have the mixed g-monotone property if F is g-nondecreasing in its odd position arguments and g-nonincreasing in its even position arguments, that is, for $x^1, x^2, x^3, \ldots, x^n \in X$, if

for all $x_1^1, x_2^1 \in X$, $gx_1^1 \preccurlyeq gx_2^1 \Rightarrow F(x_1^1, x^2, x^3, \ldots, x^n) \preccurlyeq F(x_2^1, x^2, x^3, \ldots, x^n)$,

for all $x_1^2, x_2^2 \in X$, $gx_1^2 \preccurlyeq gx_2^2 \Rightarrow F(x^1, x_2^2, x^3, \ldots, x^n) \preccurlyeq F(x^1, x_1^2, x^3, \ldots, x^n)$,

for all $x_1^3, x_2^3 \in X$, $gx_1^3 \preccurlyeq gx_2^3 \Rightarrow F(x^1, x^2, x_1^3, \ldots, x^n) \preccurlyeq F(x^1, x^2, x_2^3, \ldots, x^n)$,

$$\vdots$$

for all $x_1^n, x_2^n \in X$, $gx_1^n \preccurlyeq gx_2^n \Rightarrow F(x^1, x^2, x^3, \ldots, x_2^n) \preccurlyeq F(x^1, x^2, x^3, \ldots, x_1^n)$.

For $g = I$ (identity mapping), Definition 2.4 reduces to mixed monotone property (for details see [11]).

Definition 2.5. ([26]) An element $(x^1,x^2,...,x^n) \in X^n$ is called an n-tupled fixed point of the mapping $F : X^n \to X$ if

$$\begin{cases} F(x^1,x^2,x^3,...,x^n) = x^1, \\ F(x^2,x^3,...,x^n,x^1) = x^2, \\ F(x^3,...,x^n,x^1,x^2) = x^3, \\ \vdots \\ F(x^n,x^1,x^2,...,x^{n-1}) = x^n. \end{cases}$$

Example 2.6. *Let (R,d) be a partially ordered metric space under natural setting and $F : R^n \to R$ a mapping defined by $F(x^1,x^2,...,x^n) = sin(x^1,x^2,...,x^n)$, for any $x^1,x^2,...,x^n \in R$. Then $(0,0,...,0)$ is an n-tupled fixed point of F.*

Definition 2.7. ([11]) An element $(x^1,x^2,...,x^n) \in X^n$ is called an n-tupled coincidence point of mappings $F : X^n \to X$ and $g : X \to X$ if

$$\begin{cases} F(x^1,x^2,x^3,...,x^n) = g(x^1), \\ F(x^2,x^3,...,x^n,x^1) = g(x^2), \\ F(x^3,...,x^n,x^1,x^2) = g(x^3), \\ \vdots \\ F(x^n,x^1,x^2,...,x^{n-1}) = g(x^n). \end{cases}$$

Example 2.8. *Let (R,d) be a partially ordered metric space under natural setting and $F : R^n \to R$ be a mapping defined by $F(x^1,x^2,...,x^n) = \frac{x^1+x^2+...+x^n}{n}$, for any $x^1,x^2,...,x^n \in R$ while $g : R \to R$ is defined as $g(x) = \frac{x}{2}$. Then $(0,0,...,0)$ is an n-tupled coincidence point of F and g.*

Remark 2.9. For $n = 2$, Definitions 2.5 and 2.6 yield the definitions of coupled fixed point and coupled coincidence point respectively while on the other hand, for $n = 4$ these definitions yield the definitions of quadrupled fixed point and quadrupled coincidence point respectively.

Definition 2.10. An element $(x^1,x^2,...,x^n)\in X^n$ is called an n-tupled common fixed point of $F:X^n\to X$ and $g:X\to X$ if

$$\begin{cases} F(x^1,x^2,x^3,...,x^n)=g(x^1)=x^1,\\ F(x^2,x^3,...,x^n,x^1)=g(x^2)=x^2,\\ F(x^3,...,x^n,x^1,x^2)=g(x^3)=x^3,\\ \vdots\\ F(x^n,x^1,x^2,...,x^{n-1})=g(x^n)=x^n. \end{cases}$$

Definition 2.11. Let $F:X^n\to X$ and $g:X\to X$ be two mappings. Then F and g are said to be compatible if

$$\begin{cases} \lim_{m\to\infty} d(g(F(x_m^1,x_m^2,x_m^3,...,x_m^n)),F(gx_m^1,gx_m^2,gx_m^3,...,gx_m^n))=0,\\ \lim_{m\to\infty} d(g(F(x_m^2,x_m^3,...,x_m^n,x_m^1)),F(gx_m^2,gx_m^3,...,gx_m^n,x_m^1))=0,\\ \vdots\\ \lim_{m\to\infty} d(g(F(x_m^n,x_m^1,x_m^2,...,x_m^{n-1})),F(gx_m^n,gx_m^1,gx_m^2,...,gx_m^{n-1}))=0, \end{cases}$$

where $\{x_m^1\},\{x_m^2\},...,\{x_m^n\}$ are sequences in X such that

$$\begin{cases} \lim_{m\to\infty} F(x_m^1,x_m^2,x_m^3,...,x_m^n)=\lim_{m\to\infty} g(x_m^1)=x^1,\\ \lim_{m\to\infty} F(x_m^2,x_m^3,...,x_m^n,x_m^1)=\lim_{m\to\infty} g(x_m^2)=x^2,\\ \vdots\\ \lim_{m\to\infty} F(x_m^n,x_m^1,x_m^2,...,x_m^{n-1})=\lim_{m\to\infty} g(x_m^n)=x^n, \end{cases}$$

for some $x^1,x^2,...,x^n\in X$ are satisfied.

Definition 2.12. ([18]) A function $\psi:[0,\infty)\to[0,\infty)$ is called an altering distance function if the following properties are satisfied;

(a) ψ is monotonically increasing and continuous;

(b) $\psi(t)=0$ if and only if $t=0$.

Definition 2.13. A function $\phi:[0,\infty)\to[0,\infty)$ is called an ultra-altering distance function if the following properties are satisfied;

(a) ϕ is continuous;

(b) $\phi(0) \geq 0$, and $\phi(\varepsilon) > 0$ for each $\varepsilon > 0$.

Now we state the main result of Choudhury *et al.* [9].

Theorem 2.14. *Let $(X,d,\preccurlyeq)$ be a complete ordered metric space. Let $\varphi : [0,\infty) \to [0,\infty)$ be a continuous function with $\varphi(t) = 0$ if and only if $t = 0$ while ψ an altering distance function. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that F has the mixed g-monotone property on X and*

$$\psi(d(F(x,y),F(u,v))) \leq \psi(max\{d(gx,gu),d(gy,gv)\}) - \varphi(max\{d(gx,gu),d(gy,gv)\})$$

for all $x,y,u,v \in X$ for which $gu \preccurlyeq gx$ and $gy \preccurlyeq gv$. Suppose that $F(X \times X) \subseteq g(X)$, g is continuous and F and g are compatible. Also, suppose that

(a) F is continuous or

(b) X has the following properties:

(i) if a nondecreasing sequence $\{x_n\} \to x$, then $g(x_n) \preccurlyeq g(x)$ for all $n \geq 0$;

(ii) if a nonincreasing sequence $\{y_n\} \to y$, then $g(y) \preccurlyeq g(y_n)$ for all $n \geq 0$.

If there exist $x_0, y_0 \in X$ such that $g(x_0) \preccurlyeq F(x_0,y_0)$ and $F(y_0,x_0) \preccurlyeq g(y_0)$, then there exist $x,y \in X$ such that $g(x) = F(x,y)$ and $g(y) = F(y,x)$, i.e., F and g have a coupled coincidence point in X.

Ansari [4] introduced the concept of C-class functions which cover a large class of contractive conditions.

Definition 2.15. A continuous function $f : [0,\infty)^2 \to \mathbb{R}$ is called a C-function if for any $s,t \in [0,\infty)$, the following conditions hold:

(1) $f(s,t) \leq s$;

(2) $f(s,t) = s$ implies that either $s = 0$ or $t = 0$.

An extra condition on f is that $f(0,0) = 0$ could be imposed in some cases if required. The letter $\mathscr{C}$ denotes the class of all C-functions. The following example shows that the class C is nonempty:

Example 2.16. *Define $f : [0,\infty)^2 \to \mathscr{R}$ by*

(1) $f(s,t) = s - t$,

(2) $f(s,t) = \frac{s}{(1+t)^r}$ *for some* $r \in (0,\infty)$,

(3) $f(s,t) = \log(t + a^s)/(1+t)$, *for some* $a > 1$,

(4) $f(s,t) = \ln(1 + a^s)/2$, *for* $a > e$. *Indeed* $f(s,1) = s$ *implies that* $s = 0$,

(5) $f(s,t) = (s+l)^{(1/(1+t)^r)} - l$, $l > 1$, *for* $r \in (0,\infty)$,

(6) $f(s,t) = s\log_{t+a} a$, *for* $a > 1$,

(7) $f(s,t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$,

(8) $f(s,t) = s\beta(s)$, *where* $\beta : [0,\infty) \to [0,1)$ *and semi-continuous*,

(9) $f(s,t) = s - \frac{t}{k+t}$,

(10) $f(s,t) = s - \varphi(s)$, *where* $\varphi : [0,\infty) \to [0,\infty)$ *is a continuous function such that* $\varphi(t) = 0$ *if and only if* $t = 0$,

(11) $f(s,t) = sh(s,t)$, *where* $h : [0,\infty) \times [0,\infty) \to [0,\infty)$ *is a continuous function such that* $h(s,t) < 1$ *for all* $t,s > 0$,

(12) $f(s,t) = s - (\frac{2+t}{1+t})t$,

(13) $f(s,t) = \sqrt[n]{\ln(1+s^n)}$,

(14) $f(s,t) = \phi(s)$, *where* $\phi : [0,\infty) \to [0,\infty)$ *is an upper semi-continuous function such that* $\phi(0) = 0$ *and* $\phi(t) < t$ *for* $t > 0$,

(15) $f(s,t) = \frac{s}{(1+s)^r}$; $r \in (0,\infty)$, *for all* $s,t \in [0,\infty)$.

Then f *is an element of* C.

3. Main results

Now, we are in a position to prove our main results.

Theorem 3.1. *Let* $(X,d,\preccurlyeq)$ *be a complete ordered metric space. Let* φ *be an ultra-altering distance function,* ψ *an altering distance function and* f *a C-class function. Let* $F : X^n \to X$ *and* $g : X \to X$ *be two mappings such that* F *has the mixed g-monotone property on* X *and*

$$\psi(d(F(x^1,x^2,...,x^n),F(y^1,y^2,...,y^n))) \leq f(\psi(\max\{d(gx^1,gy^1),...,d(gx^n,gy^n)\}), \varphi(\max\{d(gx^1,gy^1),...,d(gx^n,gy^n)\})) \quad (1)$$

for all $x^1, x^2, ..., x^n, y^1, y^2, ..., y^n \in X$ *for which* $gy^1 \preccurlyeq gx^1, gx^2 \preccurlyeq gy^2, gy^3 \preccurlyeq gx^3, ..., gx^n \preccurlyeq gy^n$. *Suppose that* $F(X^n) \subseteq g(X)$, *g is continuous and F and g are compatible. Also, suppose that*

(a) F is continuous or

(b) X has the following properties:

(i) if nondecreasing sequence $\{x_m\} \to x$, *then* $g(x_m) \preccurlyeq g(x)$ *for all* $m \geq 0$*;*

(ii) if nonincreasing sequence $\{x_m\} \to x$, *then* $g(x) \preccurlyeq g(x_m)$ *for all* $m \geq 0$.

If there exist $x_0^1, x_0^2, x_0^3, ..., x_0^n \in X$ *such that*

$$\begin{cases} g(x_0^1) \preccurlyeq F(x_0^1, x_0^2, x_0^3, ..., x_0^n), \\ F(x_0^2, x_0^3, ..., x_0^n, x_0^1) \preccurlyeq g(x_0^2), \\ g(x_0^3) \preccurlyeq F(x_0^3, ..., x_0^n, x_0^1, x_0^2), \\ \vdots \\ F(x_0^n, x_0^1, x_0^2, ..., x_0^{n-1}) \preccurlyeq g(x_0^n), \end{cases} \tag{2}$$

then F and g have an n-tupled coincidence point in X.

Proof. Let $x_0^1, x_0^2, x_0^3, ..., x_0^n \in X$ such that (2) holds. Since $F(X^n) \subseteq g(X)$, we can choose $x_1^1, x_1^2, x_1^3, ..., x_1^n \in X$ such that

$$\begin{cases} g(x_1^1) = F(x_0^1, x_0^2, x_0^3, ..., x_0^n), \\ g(x_1^2) = F(x_0^2, x_0^3, ..., x_0^n, x_0^1), \\ g(x_1^3) = F(x_0^3, ..., x_0^n, x_0^1, x_0^2), \\ \vdots \\ g(x_1^n) = F(x_0^n, x_0^1, x_0^2, ..., x_0^{n-1}). \end{cases} \tag{3}$$

As earlier, one can also choose $x_2^1, x_2^2, x_2^3, ..., x_2^n \in X$ such that

$$\begin{cases} g(x_2^1) = F(x_1^1, x_1^2, x_1^3, ..., x_1^n), \\ g(x_2^2) = F(x_1^2, x_1^3, ..., x_1^n, x_1^1), \\ g(x_2^3) = F(x_1^3, ..., x_1^n, x_1^1, x_1^2), \\ \vdots \\ g(x_2^n) = F(x_1^n, x_1^1, x_1^2, ..., x_1^{n-1}). \end{cases}$$

Continuing this process, we can construct sequences $\{x_m^1\}, \{x_m^2\}, ..., \{x_m^n\}$, $(m \geq 0)$ such that

$$\begin{cases} g(x_{m+1}^1) = F(x_m^1, x_m^2, x_m^3, ..., x_m^n), \\ g(x_{m+1}^2) = F(x_m^2, x_m^3, ..., x_m^n, x_m^1), \\ g(x_{m+1}^3) = F(x_m^3, ..., x_m^n, x_m^1, x_m^2), \\ \vdots \\ g(x_{m+1}^n) = F(x_m^n, x_m^1, x_m^2, ..., x_m^{n-1}). \end{cases} \tag{4}$$

In what follows, we shall prove that for all $m \geq 0$,

$$gx_m^1 \preccurlyeq gx_{m+1}^1, gx_{m+1}^2 \preccurlyeq gx_m^2, gx_m^3 \preccurlyeq gx_{m+1}^3, ..., gx_{m+1}^n \preccurlyeq gx_m^n. \tag{5}$$

Owing to (2) and (3), we have

$$gx_0^1 \preccurlyeq gx_1^1, gx_1^2 \preccurlyeq gx_0^2, gx_0^3 \preccurlyeq gx_1^3, ..., gx_1^n \preccurlyeq gx_0^n,$$

that is, (5) holds for $m = 0$. Suppose that (5) holds for some $m > 0$. As F has the mixed g-monotone property, we have from (4) that

$$\begin{aligned} gx_{m+1}^1 = F(x_m^1, x_m^2, x_m^3, ..., x_m^n) &\preccurlyeq F(x_{m+1}^1, x_m^2, x_m^3, ..., x_m^n) \\ &\preccurlyeq F(x_{m+1}^1, x_{m+1}^2, x_m^3, ..., x_m^n) \\ &\preccurlyeq F(x_{m+1}^1, x_{m+1}^2, x_{m+1}^3, ..., x_m^n) \\ &\preccurlyeq F(x_{m+1}^1, x_{m+1}^2, x_{m+1}^3, ..., x_{m+1}^n) = gx_{m+2}^1. \end{aligned}$$

$$\begin{aligned} gx^2_{m+2} = F(x^2_{m+1}, x^3_{m+1}, ..., x^n_{m+1}, x^1_{m+1}) &\preccurlyeq F(x^2_{m+1}, x^3_{m+1}, ..., x^n_{m+1}, x^1_m) \\ &\preccurlyeq F(x^2_{m+1}, x^3_{m+1}, ..., x^n_m, x^1_m) \\ &\preccurlyeq F(x^2_{m+1}, x^3_m, ..., x^n_m, x^1_m) \\ &\preccurlyeq F(x^2_m, x^3_m, ..., x^n_m, x^1_m) = gx^2_{m+1}. \end{aligned}$$

Also for the same reason, we have

$$\begin{aligned} gx^3_{m+1} = F(x^3_m, ..., x^n_m, x^1_m, x^2_m) &\preccurlyeq F(x^3_{m+1}, ..., x^n_{m+1}, x^1_{m+1}, x^2_{m+1}) = gx^3_{m+2}, \\ &\vdots \\ gx^n_{m+2} = F(x^n_{m+1}, x^1_{m+1}, x^2_{m+1}, ..., x^{n-1}_{m+1}) &\preccurlyeq F(x^n_m, x^1_m, x^2_m, ..., x^{n-1}_m) = gx^n_{m+1}. \end{aligned}$$

Hence by mathematical induction it follows that (5) holds for all $m \geq 0$. Therefore

$$\begin{cases} gx^1_0 \preccurlyeq gx^1_1 \preccurlyeq gx^1_2 \preccurlyeq ... \preccurlyeq gx^1_m \preccurlyeq gx^1_{m+1} \preccurlyeq ... \\ ...gx^2_{m+1} \preccurlyeq gx^2_m \preccurlyeq ... \preccurlyeq gx^2_2 \preccurlyeq gx^2_1 \preccurlyeq gx^2_0 \\ gx^3_0 \preccurlyeq gx^3_1 \preccurlyeq gx^3_2 \preccurlyeq ... \preccurlyeq gx^3_m \preccurlyeq gx^3_{m+1} ... \\ \vdots \\ ....gx^n_{m+1} \preccurlyeq gx^n_m \preccurlyeq ... \preccurlyeq gx^n_2 \preccurlyeq gx^n_1 \preccurlyeq gx^n_0. \end{cases} \tag{6}$$

Let

$$R_m = \max\{d(gx^1_{m+1}, gx^1_m), d(gx^2_{m+1}, gx^2_m), ..., d(gx^n_{m+1}, gx^n_m)\}.$$

Using (6), we have,

$$\begin{aligned} \psi(d(gx^1_m, gx^1_{m+1})) &= \psi(d(F(x^1_{m-1}, x^2_{m-1}, x^3_{m-1}, ..., x^n_{m-1}), F(x^1_m, x^2_m, x^3_m, ..., x^n_m))) \\ &\leq f(\psi(\max\{d(gx^1_{m-1}, gx^1_m), d(gx^2_{m-1}, gx^2_m), ..., d(gx^n_{m-1}, gx^n_m)\}), \\ &\quad \varphi(\max\{d(gx^1_{m-1}, gx^1_m), d(gx^2_{m-1}, gx^2_m), ..., d(gx^n_{m-1}, gx^n_m)\})), \end{aligned}$$

$$\begin{aligned} \psi(d(gx^2_m, gx^2_{m+1})) &= \psi(d(F(x^2_{m-1}, x^3_{m-1}, ..., x^n_{m-1}, x^1_{m-1}), F(x^2_m, x^3_m, ..., x^n_m, x^1_m))) \\ &\leq f(\psi(\max\{d(gx^2_{m-1}, gx^2_m), ..., d(gx^n_{m-1}, gx^n_m), d(gx^1_{m-1}, gx^1_m)\}), \\ &\quad \varphi(\max\{d(gx^2_{m-1}, gx^2_m), ..., d(gx^n_{m-1}, gx^n_m), d(gx^1_{m-1}, gx^1_m)\})). \end{aligned}$$

Similarly, we can inductively write

$$\begin{aligned}\psi(d(gx_m^n, gx_{m+1}^n)) &= \psi(d(F(x_{m-1}^n, x_{m-1}^1, x_{m-1}^2, ..., x_{m-1}^{n-1}), F(x_m^n, x_m^1, x_m^2, ..., x_m^{n-1})))\\ &\leq f(\psi(\max\{d(gx_{m-1}^n, gx_m^n), d(gx_{m-1}^1, gx_m^1), ..., d(gx_{m-1}^{n-1}, gx_m^{n-1})\}),\\ &\quad \varphi(\max\{d(gx_{m-1}^n, gx_m^n), d(gx_{m-1}^1, gx_m^1), ..., d(gx_{m-1}^{n-1}, gx_m^{n-1})\})).\end{aligned}$$

From above inequalities and monotone property of ψ, we have

$$\begin{aligned}&\psi(\max\{d(gx_{m+1}^1, gx_m^1), d(gx_{m+1}^2, gx_m^2), ..., d(gx_{m+1}^n, gx_m^n)\})\\ =\ &\max\{\psi d(gx_{m+1}^1, gx_m^1), \psi d(gx_{m+1}^2, gx_m^2), ..., \psi d(gx_{m+1}^n, gx_m^n)\}\\ \leq\ &f(\psi(\max\{d(gx_{m-1}^1, gx_m^1), d(gx_{m-1}^2, gx_m^2), ..., d(gx_{m-1}^{n-1}, gx_m^{n-1})\}),\\ &\varphi(\max\{d(gx_{m-1}^1, gx_m^1), d(gx_{m-1}^2, gx_m^2), ..., d(gx_{m-1}^{n-1}, gx_m^{n-1})\})),\end{aligned}$$

that is,

$$\psi(R_m) \leq f(\psi(R_{m-1}), \varphi(R_{m-1})). \tag{7}$$

Using the property of ψ, we have $\psi(R_m) \leq \psi(R_{m-1})$, which implies that $R_m \leq R_{m-1}$ (by the property of ψ). Therefore $\{R_m\}$ is a monotonically decreasing sequence of nonnegative real numbers. Hence there exists $r \geq 0$ such that $R_m \to r$ as $m \to \infty$. Taking the limit as $m \to \infty$ in (7). Then by the continuities of ψ and φ, we have

$$\psi(r) \leq \psi(r) - \varphi(r),$$

which is a contradiction unless $r = 0$. Therefore

$$R_m \to 0 \text{ as } m \to \infty, \tag{8}$$

so that

$$\lim_{m\to\infty} d(gx_{m-1}^1, gx_m^1) = 0, \lim_{m\to\infty} d(gx_{m-1}^2, gx_m^2) = 0, ..., \lim_{m\to\infty} d(gx_{m-1}^n, gx_m^n) = 0.$$

Next, we show that $\{gx_m^1\}, \{gx_m^2\}, ..., \{gx_m^n\}$ are Cauchy sequences. If possible suppose that at least one of $\{gx_m^1\}, \{gx_m^2\}, ..., \{gx_m^n\}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{t(k)\}$ such that for all positive integers k, $t(k) > m(k) > k$,

$$D_k = \max\{d(gx^1_{m(k)}, gx^1_{t(k)}), d(gx^2_{m(k)}, gx^2_{t(k)}), ..., d(gx^n_{m(k)}, gx^n_{t(k)})\} \geq \varepsilon$$

and

$$\max\{d(gx^1_{m(k)}, gx^1_{t(k)-1}), d(gx^2_{m(k)}, gx^2_{t(k)-1}), ..., d(gx^n_{m(k)}, gx^n_{t(k)-1})\} < \varepsilon.$$

Now,

$$\begin{aligned}\varepsilon \leq D_k &= \max\{d(gx^1_{m(k)}, gx^1_{t(k)}), d(gx^2_{m(k)}, gx^2_{t(k)}), ..., d(gx^n_{m(k)}, gx^n_{t(k)})\} \\ &\leq \max\{d(gx^1_{m(k)}, gx^1_{t(k)-1})), d(gx^2_{m(k)}, gx^2_{t(k)-1})), ..., d(gx^n_{m(k)}, gx^n_{t(k)-1}))\} \\ &\quad + \max\{d(gx^1_{t(k)-1}, gx^1_{t(k)}), d(gx^2_{t(k)-1}, gx^2_{t(k)}), ..., d(gx^n_{t(k)-1}, gx^n_{t(k)})\},\end{aligned}$$

that is,

$$\varepsilon \leq D_k = \max\{d(gx^1_{m(k)}, gx^1_{t(k)}), d(gx^2_{m(k)}, gx^2_{t(k)}), ..., d(gx^n_{m(k)}, gx^n_{t(k)})\} \leq \varepsilon + R_{t(k)-1}.$$

Letting $k \to \infty$ in above inequality and using (8), we have

$$\lim_{k\to\infty} D_k = \lim_{k\to\infty} \max\{d(gx^1_{m(k)}, gx^1_{t(k)}), d(gx^2_{m(k)}, gx^2_{t(k)}), ..., d(gx^n_{m(k)}, gx^n_{t(k)})\} = \varepsilon. \tag{9}$$

Again,

$$\begin{aligned}D_{k+1} &= \max\{d(gx^1_{m(k)+1}, gx^1_{t(k)+1}), d(gx^2_{m(k)+1}, gx^2_{t(k)+1}), ..., d(gx^n_{m(k)+1}, gx^n_{t(k)+1})\} \\ &\leq \max\{d(gx^1_{m(k)+1}, gx^1_{m(k)}), d(gx^2_{m(k)+1}, gx^2_{m(k)}), ..., d(gx^n_{m(k)+1}, gx^n_{m(k)})\} \\ &\quad + \max\{d(gx^1_{m(k)}, gx^1_{t(k)}), d(gx^2_{m(k)}, gx^2_{t(k)}), ..., d(gx^n_{m(k)}, gx^n_{t(k)})\} \\ &\quad + \max\{d(gx^1_{t(k)}, gx^1_{t(k)+1}), d(gx^2_{t(k)}, gx^2_{t(k)+1}), ..., d(gx^n_{t(k)}, gx^n_{t(k)+1}\}) \\ &= R_{m(k)} + D_k + R_{t(k)}\end{aligned}$$

and

$$D_k \leq R_{m(k)} + D_{k+1} + R_{t(k)}.$$

Letting $k \to \infty$ in the preceding inequality, using (8) and (9) we have

$$\lim_{k\to\infty} D_{k+1} = \lim_{k\to\infty} \max\{d(gx^1_{m(k)+1}, gx^1_{t(k)+1}), ..., d(gx^n_{m(k)+1}, gx^n_{t(k)+1})\} = \varepsilon. \tag{10}$$

Since $t(k) > m(k)$ and

$$gx^1_{m(k)} \preccurlyeq gx^1_{t(k)}, gx^2_{t(k)} \preccurlyeq gx^2_{m(k)}, gx^3_{m(k)} \preccurlyeq gx^3_{t(k)}, \ldots, gx^n_{t(k)} \preccurlyeq gx^n_{m(k)},$$

therefore owing to (1) and (4), we have

$$\begin{aligned}\psi(d(gx^1_{m(k)+1}, gx^1_{t(k)+1})) &= \psi(d(F(x^1_{m(k)}, x^2_{m(k)}, \ldots, x^n_{m(k)}), F(x^1_{t(k)}, x^2_{t(k)}, \ldots, x^n_{t(k)}))) \\ &\leq f(\psi(\max\{d(gx^1_{m(k)}, gx^1_{t(k)}), \ldots, d(gx^n_{m(k)}, gx^n_{t(k)})\}), \\ &\quad \varphi(\max\{d(gx^1_{m(k)}, gx^1_{t(k)}), \ldots, d(gx^n_{m(k)}, gx^n_{t(k)})\})),\end{aligned}$$

that is,

$$\psi(d(gx^1_{m(k)+1}, gx^1_{t(k)+1})) \leq \psi(D_k) - \varphi(D_k). \tag{11}$$

Also,

$$\begin{aligned}\psi(d(gx^2_{m(k)+1}, gx^2_{t(k)+1})) &= \psi(d(F(x^2_{m(k)}, \ldots, x^n_{m(k)}, x^1_{m(k)}), F(x^2_{t(k)}, \ldots, x^n_{t(k)}, x^1_{t(k)}))) \\ &\leq f(\psi(\max\{d(gx^2_{m(k)}, gx^2_{t(k)}), \ldots, d(gx^1_{m(k)}, gx^1_{t(k)})\}), \\ &\quad \varphi(\max\{d(gx^2_{m(k)}, gx^2_{t(k)}), \ldots, d(gx^1_{m(k)}, gx^1_{t(k)})\})),\end{aligned}$$

that is,

$$\psi(d(gx^2_{m(k)+1}, gx^2_{t(k)+1})) \leq \psi(D_k) - \varphi(D_k). \tag{12}$$

Similarly, we have

$$\begin{aligned}\psi(d(gx^n_{m(k)+1}, gx^n_{t(k)+1})) &= \psi(d(F(x^n_{m(k)}, x^1_{m(k)}, \ldots, x^{n-1}_{m(k)}), F(x^n_{t(k)}, x^1_{t(k)}, \ldots, x^{n-1}_{t(k)}))) \\ &\leq f(\psi(\max\{d(gx^n_{m(k)}, gx^n_{t(k)}), \ldots, d(gx^{n-1}_{m(k)}, gx^{n-1}_{t(k)})\}), \\ &\quad \varphi(\max\{d(gx^n_{m(k)}, gx^n_{t(k)}), \ldots, d(gx^{n-1}_{m(k)}, gx^{n-1}_{t(k)})\})),\end{aligned}$$

that is,

$$\psi(d(gx^n_{m(k)+1}, gx^n_{t(k)+1})) \leq \psi(D_k) - \varphi(D_k). \tag{13}$$

Using (11)-(13) along with monotone property of ψ, we have,

$$\begin{aligned}
\psi(D_{k+1}) &= \psi(\max\{d(gx^1_{m(k)+1}, gx^1_{t(k)+1}), ..., d(gx^n_{m(k)+1}, gx^n_{t(k)+1})\}) \\
&= \max\{\psi d(gx^1_{m(k)+1}, gx^1_{t(k)+1}), ..., \psi d(gx^n_{m(k)+1}, gx^n_{t(k)+1})\} \\
&= f(\psi(D_k), \varphi(D_k)).
\end{aligned}$$

Letting $k \to \infty$ in the above inequality, using (9), (10) and the continuities of ψ and φ, we have

$$\psi(\varepsilon) \leq f(\psi(\varepsilon), \varphi(\varepsilon)),$$

therefore $\psi(\varepsilon) = 0$ or $\varphi(\varepsilon) = 0$, then $\varepsilon = 0$ which is a contradiction. Thus $\{gx^1_m\}, \{gx^2_m\}, ...,$ $\{gx^n_m\}$ are Cauchy sequences in X. From the completeness of X, there exist $x^1, x^2, ..., x^n \in X$ such that

$$\begin{cases}
\lim\limits_{m\to\infty} F(x^1_m, x^2_m, x^3_m, ..., x^n_m) = \lim\limits_{m\to\infty} g(x^1_m) = x^1, \\
\lim\limits_{m\to\infty} F(x^2_m, x^3_m, ..., x^n_m, x^1_m) = \lim\limits_{m\to\infty} g(x^2_m) = x^2, \\
\lim\limits_{m\to\infty} F(x^3_m, ..., x^n_m, x^1_m, x^2_m) = \lim\limits_{m\to\infty} g(x^3_m) = x^3, \\
\vdots \\
\lim\limits_{m\to\infty} F(x^n_m, x^1_m, x^2_m, ..., x^{n-1}_m) = \lim\limits_{m\to\infty} g(x^n_m) = x^n,
\end{cases} \tag{14}$$

for some $x^1, x^2, ..., x^n \in X$ are satisfied. Since F and g are compatible, we have from (14) that

$$\begin{cases}
\lim\limits_{m\to\infty} d(g(F(x^1_m, x^2_m, x^3_m, ..., x^n_m)), F(gx^1_m, gx^2_m, gx^3_m, ..., gx^n_m)) = 0, \\
\lim\limits_{m\to\infty} d(g(F(x^2_m, x^3_m, ..., x^n_m, x^1_m,)), F(gx^2_m, gx^3_m, ..., gx^n_m, gx^1_m)) = 0, \\
\lim\limits_{m\to\infty} d(g(F(x^3_m, ..., x^n_m, x^1_m, x^2_m)), F(gx^3_m, ..., gx^n_m, gx^1_m, gx^2_m)) = 0, \\
\vdots \\
\lim\limits_{m\to\infty} d(g(F(x^n_m, x^1_m, x^2_m, ..., x^{n-1}_m)), F(gx^n_m, gx^1_m, gx^2_m, ..., gx^{n-1}_m)) = 0.
\end{cases} \tag{15}$$

Let condition (a) holds. Then for all $m \geq 0$, we have

$$\begin{aligned}
d(gx^1, F(gx^1_m, gx^2_m, ..., gx^n_m)) &\leq d(gx^1, g(F(x^1_m, x^2_m, ..., x^n_m))) \\
&\quad + d(g(F(x^1_m, x^2_m, ..., x^n_m)), F(gx^1_m, gx^2_m, ..., gx^n_m)).
\end{aligned}$$

Taking $m \to \infty$ in above inequality, using (14), (15) and continuities of F and g, we have

$$d(gx^1, F(x^1, x^2, x^3, ..., x^n)) = 0; \text{ that is, } gx^1 = F(x^1, x^2, x^3, ..., x^n).$$

Continuing this process, we obtain that

$$d(gx^2, F(x^2, x^3, ..., x^n, x^1)) = 0; \text{ that is } gx^2 = F(x^2, x^3, ..., x^n, x^1).$$

$$\vdots$$

$$d(gx^n, F(x^n, x^1, x^2, ..., x^{n-1})) = 0; \text{ that is } gx^n = F(x^n, x^1, x^2, ..., x^{n-1}).$$

Hence the element $(x^1, x^2, ..., x^n) \in X^n$ is an n-tupled coincidence point of the mappings $F : X^n \to X$ and $g : X \to X$. Next, we suppose that condition (b) holds. From (6) and (14), we have

$$ggx^1_m \preccurlyeq gx^1, gx^2 \preccurlyeq ggx^2_m, ggx^3_m \preccurlyeq gx^3, ..., gx^n \preccurlyeq ggx^n_m. \tag{16}$$

Since F and g are compatible and g is continuous, by (14) and (15) we have

$$\begin{cases} \lim\limits_{m\to\infty} ggx^1_m = gx^1 = \lim\limits_{m\to\infty} d(g(F(x^1_m, x^2_m, ..., x^n_m)) = \lim\limits_{m\to\infty} F(gx^1_m, gx^2_m, ..., gx^n_m), \\ \lim\limits_{m\to\infty} ggx^2_m = gx^2 = \lim\limits_{m\to\infty} d(g(F(x^2_m, ..., x^n_m, x^1_m)) = \lim\limits_{m\to\infty} F(gx^2_m, ..., gx^n_m, gx^1_m), \\ \lim\limits_{m\to\infty} ggx^3_m = gx^3 = \lim\limits_{m\to\infty} d(g(F(x^3_m, ..., x^1_m, x^2_m)) = \lim\limits_{m\to\infty} F(gx^3_m, ..., gx^1_m, gx^2_m), \\ \vdots \\ \lim\limits_{m\to\infty} ggx^n_m = gx^n = \lim\limits_{m\to\infty} d(g(F(x^n_m, x^1_m, ..., x^{n-1}_m)) = \lim\limits_{m\to\infty} F(gx^n_m, gx^1_m, ..., gx^{n-1}_m). \end{cases} \tag{17}$$

Now, using triangle inequality, we have

$$d(F(x^1, x^2, ..., x^n), gx^1) \leq d(F(x^1, x^2, ..., x^n), ggx^1_{m+1}) + d(ggx^1_{m+1}, gx^1),$$

that is,

$$d(F(x^1, x^2, ..., x^n), gx^1) \leq d(F(x^1, x^2, ..., x^n), g(F(x^1_m, x^2_m, ..., x^n_m)) + d(ggx^1_{m+1}, gx^1).$$

Taking $m \to \infty$ in the above inequality and using (17) we have

$$\begin{aligned} d(F(x^1, x^2, ..., x^n), gx^1) &\leq \lim_{m\to\infty} d(F(x^1, x^2, ..., x^n), g(F(x^1_m, x^2_m, ..., x^n_m)) \\ &\quad + \lim_{m\to\infty} d(ggx^1_{m+1}, gx^1) \\ &= \lim_{m\to\infty} d(F(x^1, x^2, ..., x^n), F(gx^1_m, gx^2_m, ..., gx^n_m)). \end{aligned}$$

Since ψ is continuous and monotonically increasing, from the above inequality we have

$$\begin{aligned}\psi(d(F(x^1,x^2,...,x^n),gx^1)) &\leq \psi(\lim_{m\to\infty} d(F(x^1,x^2,...,x^n),F(gx_m^1,gx_m^2,...,gx_m^n)))\\ &= \lim_{m\to\infty}\psi(d(F(x^1,x^2,...,x^n),F(gx_m^1,gx_m^2,...,gx_m^n))).\end{aligned}$$

By (1) and (16), we have

$$\begin{aligned}\psi(d(F(x^1,x^2,...,x^n),gx^1)) &\leq \lim_{m\to\infty} f([\psi(max\{d(gx^1,ggx_m^1),...,d(gx^n,ggx_m^n)\}),\\ &\qquad \varphi(\max\{d(gx^1,ggx_m^1),d(gx^2,ggx_m^2),...,d(gx^n,ggx_m^n)\})]).\end{aligned}$$

Using (17) and the properties of ψ and φ we have

$$\psi(d(F(x^1,x^2,x^3,...,x^n),gx^1)) = 0,$$

which implies that

$$d(F(x^1,x^2,x^3,...,x^n),gx^1) = 0, \text{ that is } F(x^1,x^2,x^3,...,x^n) = gx^1.$$

Again, we have

$$d(F(x^2,...,x^n,x^1),gx^2) \leq d(F(x^2,...,x^n,x^1),ggx_{m+1}^2) + d(ggx_{m+1}^2,gx^2),$$

that is,

$$d(F(x^2,...,x^n,x^1),gx^2) \leq d(F(x^2,...,x^n,x^1),g(F(x_m^2,...,x_m^n,x_m^1))) + d(ggx_{m+1}^2,gx^2).$$

Taking $m\to\infty$ in the above inequality, using (17) we have

$$\begin{aligned}d(F(x^2,...,x^n,x^1),gx^2) &\leq \lim_{m\to\infty} d(F(x^2,...,x^n,x^1),g(F(x_m^2,...,x_m^n,x_m^1)))\\ &\quad + \lim_{m\to\infty} d(ggx_{m+1}^2,gx^2)\\ &= \lim_{m\to\infty} d(F(x^2,...,x^n,x^1),g(F(x_m^2,...,x_m^n,x_m^1))).\end{aligned}$$

Since ψ is continuous and monotonically increasing, from the above inequality we have

$$\begin{aligned}\psi(d(F(x^2,...,x^n,x^1),gx^2)) &\leq \psi(\lim_{m\to\infty} d(F(x^2,...,x^n,x^1),g(F(x_m^2,...,x_m^n,x_m^1))))\\ &= \lim_{m\to\infty}\psi(d(F(x^2,...,x^n,x^1),g(F(x_m^2,...,x_m^n,x_m^1)))).\end{aligned}$$

By (1) and (16), we have

$$\begin{aligned}\psi(d(F(x^2,...,x^n,x^1),gx^2)) &\leq \lim_{m\to\infty} f([\psi(max\{d(gx^2,ggx_m^2),...,d(gx^1,ggx_m^1)\}),\\ &\qquad \varphi(\max\{d(gx^2,ggx_m^2),...,d(gx^n,ggx_m^n),d(gx^1,ggx_m^1)\})]).\end{aligned}$$

Using (17) and the properties of ψ and φ, we have

$$\psi(d(F(x^2,...,x^n,x^1),gx^2)) = 0,$$

which implies that

$$d(F(x^2,...,x^n,x^1),gx^2) = 0, \text{ that is } F(x^2,...,x^n,x^1) = gx^2.$$

Continuing in this way, we get

$$d(F(x^n,x^1,x^2,...,x^{n-1}),gx^n) = 0, \text{ that is } F(x^n,x^1,x^2,...,x^{n-1}) = gx^n.$$

Hence the element $(x^1,x^2,...,x^n) \in X^n$ is an n-tupled coincidence point of mappings F and g. This completes the proof of the theorem.

Theorem 3.2. *In addition to the hypotheses of Theorem 3.1, suppose that for real* $(x^1,x^2,...,x^n),(y^1,y^2,...,y^n) \in X^n$ *there exists,* $(z^1,z^2,...,z^n) \in X^n$ *such that* $(F(z^1,z^2,...,z^n),F(z^2,...,z^n,z^1),...,F(z^n,z^1,...,z^{n-1}))$ *is comparable to* $(F(x^1,x^2,...,x^n),F(x^2,...,x^n,x^1),...,F(x^n,x^1,...,x^{n-1}))$ *and* $(F(y^1,y^2,...,y^n),F(y^2,...,y^n,y^1),...,F(y^n,y^1,...,y^{n-1}))$*. Then* F *and* g *have a unique n-tupled common fixed point.*

Proof. The set of n-tupled coincidence points of F and g is non-empty due to Theorem 3.1. Assume now, $(x^1,x^2,...,x^n),(y^1,y^2,...,y^n)$ are two n-tupled coincidence points, that is,

$$F(x^1,x^2,...,x^n) = g(x^1),\ F(y^1,y^2,...,y^n) = g(y^1),$$

$$F(x^2,...,x^n,x^1) = g(x^2),\ F(y^2,...,y^n,y^1) = g(y^2),$$

$$\vdots$$

$$F(x^n,x^1,...,x^{n-1}) = g(x^n),\ F(y^n,y^1,...,y^{n-1}) = g(y^n).$$

Now, we show that

$$g(x^1) = g(y^1), g(x^2) = g(y^2),...,g(x^n) = g(y^n). \tag{18}$$

By assumption, there exists $(z^1,z^2,...,z^n) \in X^n$ such that $(F(z^1,z^2,...,z^n),F(z^2,...,z^n,z^1),...,F(z^n,z^1,...,z^{n-1}))$ is comparable to $(F(x^1,x^2,...,x^n),F(x^2,...,x^n,x^1),...,F(x^n,x^1,...,x^{n-1}))$ and

$(F(y^1,y^2,...,y^n),F(y^2,...,y^n,y^1),...,F(y^n,y^1,...,y^{n-1}))$. Put $z_0^1=z^1,z_0^2=z^2,...,z_0^n=z^n$ and choose $z_1^1,z_1^2,...,z_1^n\in X$ such that

$$g(z_1^1)=F(z_0^1,z_0^2,z_0^3,...,z_0^n),$$

$$g(z_1^2)=F(z_0^2,z_0^3,...,z_0^n,z_0^1),$$

$$\vdots$$

$$g(z_1^n)=F(z_0^n,z_0^1,z_0^2,...,z_0^{n-1}).$$

Further define sequences $\{g(z_m^1)\},\{g(z_m^2)\},...,\{g(z_m^n)\}$ such that

$$g(z_{m+1}^1)=F(z_m^1,z_m^2,z_m^3,...,z_m^n),$$

$$g(z_{m+1}^2)=F(z_m^2,z_m^3,...,z_m^n,z_m^1),$$

$$\vdots$$

$$g(z_{m+1}^n)=F(z_m^n,z_m^1,z_m^2,...,z_m^{n-1}).$$

Further set $x_0^1=x^1,x_0^2=x^2,...,x_0^n=x^n$ and $y_0^1=y^1,y_0^2=y^2,...,y_0^n=y^n$. In the same way, define the sequences $\{g(x_m^1)\},\{g(x_m^2)\},...,\{g(x_m^n)\}$ and $\{g(y_m^1)\},\{g(y_m^2)\},...,\{g(y_m^n)\}$. Then it is easy to show that

$$g(x_{m+1}^1)=F(x_m^1,x_m^2,x_m^3,...,x_m^n),g(y_{m+1}^1)=F(y_m^1,y_m^2,y_m^3,...,y_m^n),$$

$$g(x_{m+1}^2)=F(x_m^2,x_m^3,...,x_m^n,x_m^1),g(y_{m+1}^2)=F(y_m^2,y_m^3,...,y_m^n,y_m^1),$$

$$\vdots$$

$$g(x_{m+1}^n)=F(x_m^n,x_m^1,x_m^2,...,x_m^{n-1}),g(y_{m+1}^n)=F(y_m^n,y_m^1,y_m^2,...,y_m^{n-1}).$$

Since $(F(x^1,...,x^n),F(x^2,...,x^n,x^1),...,F(x^n,x^1,...,x^{n-1}))=(g(x_1^1),...,g(x_1^n))=(g(x^1),...,g(x^n))$ and

$(F(z^1,z^2,...,z^n),F(z^2,...,z^n,z^1),...,F(z^n,z^1,...,z^{n-1}))=(g(z_1^1),g(z_1^2),...,g(z_1^n))$ are comparable, we have

$$g(x^1)\preccurlyeq g(z_1^1),g(z_1^2)\preccurlyeq g(x^2),g(x^3)\preccurlyeq g(z_1^3),...,g(z_1^n)\preccurlyeq g(x^n).$$

It is easy to show that $g(x_1^1),g(x_1^2),...,g(x_1^n)$ and $g(z_m^1),g(z_m^2),...,g(z_m^n)$ are comparable, that is, for all $m\geq 1$,

$$g(x^1)\preccurlyeq g(z_m^1),g(z_m^2)\preccurlyeq g(x^2),...,g(z_m^n)\preccurlyeq g(x^n).$$

From (1), we have

$$\begin{aligned}
\psi(d(g(x^1),g(z^1_{m+1}))) &= \psi(d(F(x^1,x^2,\ldots,x^n),F(z^1_m,z^2_m,\ldots,z^n_m)))\\
&\leq f(\psi(\max\{d(g(x^1),g(z^1_m)),\ldots,d(g(z^n_m),g(x^n))\}),\\
&\quad \varphi(\max\{d(g(x^1),g(z^1_m)),\ldots,d(g(z^n_m),g(x^n))\})),
\end{aligned}$$

$$\begin{aligned}
\psi(d(g(x^2),g(z^2_{m+1}))) &= \psi(d(F(x^2,\ldots,x^n,x^1),F(z^2_m,\ldots,z^n_m,z^1_m)))\\
&\leq f(\psi(\max\{d(g(z^2_m),g(x^2)),\ldots,d(g(x^1),g(z^1_m))\}),\\
&\quad \varphi(\max\{d(g(z^2_m),g(x^2)),\ldots,d(g(x^1),g(z^1_m))\})),
\end{aligned}$$

$$\begin{aligned}
\psi(d(g(x^n),g(z^n_{m+1}))) &= \psi(d(F(x^n,x^1,\ldots,x^{n-1}),F(z^n_m,z^1_m,\ldots,z^{n-1}_m)))\\
&\leq f(\psi(\max\{d(g(z^n_m),g(x^n)),\ldots,d(g(z^{n-1}_m),g(x^{n-1}))\}),\\
&\quad \varphi(\max\{d(g(z^n_m),g(x^n)),\ldots,d(g(z^{n-1}_m),g(x^{n-1}))\})).
\end{aligned}$$

From above inequalities and monotone property of ψ, we have

$$\begin{aligned}
&\psi(\max\{d(g(z^n_{m+1}),g(x^n)),d(g(x^1),g(z^1_{m+1})),\ldots,d(g(z^{n-1}_{m+1}),g(x^{n-1}))\})\\
&= \max\{\psi d(g(z^n_{m+1}),g(x^n)),\psi d(g(x^1),g(z^1_{m+1})),\ldots,\psi d(g(z^{n-1}_{m+1}),g(x^{n-1}))\})\\
&\leq f(\psi(\max\{d(g(z^n_m),g(x^n)),d(g(x^1),g(z^1_m)),\ldots,d(g(z^{n-1}_m),g(x^{n-1}))\}),\\
&\quad \varphi(\max\{d(g(z^n_m),g(x^n)),d(g(x^1),g(z^1_m)),\ldots,d(g(z^{n-1}_m),g(x^{n-1}))\})).
\end{aligned}$$

Let

$$R_m = \max\{d(g(z^1_{m+1}),g(x^1)),d(g(x^2),g(z^2_{m+1})),,\ldots,d(g(z^n_{m+1}),g(x^n))\}.$$

It follows that

$$\psi(R_m) \leq f(\psi(R_{m-1}),\varphi(R_{m-1})). \tag{19}$$

Using the property of ψ, we have

$$\psi(R_m) \leq \psi(R_{m-1}) \Rightarrow R_m \leq R_{m-1}.$$

Therefore $\{R_m\}$ is a monotone decreasing sequence of nonnegative real numbers. Hence there exists $r \geq 0$ such that $R_m \to r$ as $m \to \infty$. Taking the limit as $m \to \infty$ in (19), we have

$$\psi(r) \leq f(\psi(r),\varphi(r)),$$

which is a contradiction unless $r = 0$. Therefore $R_m \to 0$ as $m \to \infty$. Then

$$\lim_{m\to\infty} d(g(z^1_{m+1}), g(x^1)) = 0, \lim_{m\to\infty} d(g(x^2), g(z^2_{m+1})) = 0, ..., \lim_{m\to\infty} d(g(z^n_{m+1}), g(x^n)) = 0.$$

Similarly, we can prove that

$$\lim_{m\to\infty} d(g(z^1_{m+1}), g(y^1)) = 0, \lim_{m\to\infty} d(g(y^2), g(z^2_{m+1})) = 0, ..., \lim_{m\to\infty} d(g(z^n_{m+1}), g(y^n)) = 0.$$

On using the triangle inequality, we have

$$d(gx^1, gy^1) \leq d(gx^1, gz^1_{m+1}) + d(gz^1_{m+1}, gy^1) \to 0 \text{ as } m \to \infty,$$

$$d(gx^2, gy^2) \leq d(gx^2, gz^2_{m+1}) + d(gz^2_{m+1}, gy^2) \to 0 \text{ as } m \to \infty,$$

$$\vdots$$

$$d(gx^n, gy^n) \leq d(gx^n, gz^n_{m+1}) + d(gz^n_{m+1}, gy^n) \to 0 \text{ as } m \to \infty.$$

Hence, we have

$$gx^1 = gy^1, ..., gx^n = gy^n. \tag{20}$$

Since

$$F(x^1, x^2, ..., x^n) = g(x^1), F(x^2, ..., x^n, x^1) = g(x^2), ..., F(x^n, x^1, x^2, ..., x^{n-1}) = g(x^n),$$

and F and g are compatible, we have

$$F(gx^1, gx^2, ..., gx^n) = gg(x^1), F(gx^2, ..., gx^n, gx^1) = gg(x^2), ...,$$

$$F(gx^n, gx^1, ..., gx^{n-1}) = gg(x^n).$$

Writing $g(x^1) = a^1, g(x^2) = a^2, ..., g(x^n) = a^n$, we have

$$\begin{cases} g(a^1) = F(a^1, a^2, a^3, ..., a^n), \\ g(a^2) = F(a^2, a^3, ..., a^n, a^1), \\ \vdots \\ g(a^n) = F(a^n, a^1, a^2, ..., a^{n-1}). \end{cases} \tag{21}$$

Thus $(a^1,a^2,a^3,\dots,a^n)$ is an n-tupled coincidence point of F and g. Owing to (20) with $y^1 = a^1, y^2 = a^2,\dots,y^n = a^n$, it follows that

$$g(x^1) = g(a^1), g(x^2) = g(a^2), \dots, g(x^n) = g(a^n),$$

that is,

$$g(a^1) = a^1, g(a^2) = a^2, \dots, g(a^n) = a^n. \tag{22}$$

Using (21) and (22), we have

$$\begin{cases} a^1 = g(a^1) = F(a^1,a^2,a^3,\dots,a^n) \\ a^2 = g(a^2) = F(a^2,a^3,\dots,a^n,a^1) \\ \vdots \\ a^n = g(a^n) = F(a^n,a^1,a^2,\dots,a^{n-1}). \end{cases} \tag{23}$$

Thus $(a^1,a^2,a^3,\dots,a^n)$ is an n-tupled common fixed point of F and g. To prove the uniqueness, assume that $(b^1,b^2,\dots,b^n)$ is another n-tupled common fixed point of F and g. In view of (20), we have

$$b^1 = g(b^1) = g(a^1) = a^1,$$

$$b^2 = g(b^2) = g(a^2) = a^2,$$

$$\vdots$$

$$b^n = g(b^n) = g(a^n) = a^n.$$

This completes the proof of the theorem.

In Theorem 3.1, setting $f(s,t) = s - t,\ s,t \in (0,\infty)$, we obtain the following result.

Corollary 3.3. *Let $(X,d,\preccurlyeq)$ be a complete ordered metric space. Let φ be an ultra-altering distance function and ψ an altering distance function. Let $F : X^n \to X$ and $g : X \to X$ be two mappings such that F has the mixed g-monotone property on X and*

$$\psi(d(F(x^1,x^2,\dots,x^n),F(y^1,y^2,\dots,y^n))) \le \psi(\max\{d(gx^1,gy^1),\dots,d(gx^n,gy^n)\})$$
$$-\varphi(\max\{d(gx^1,gy^1),\dots,d(gx^n,gy^n)\}))$$

for all $x^1, x^2, ..., x^n, y^1, y^2, ..., y^n \in X$ *for which* $gy^1 \preccurlyeq gx^1, gx^2 \preccurlyeq gy^2, gy^3 \preccurlyeq gx^3, ..., gx^n \preccurlyeq gy^n$. *Suppose that* $F(X^n) \subseteq g(X)$, *g is continuous and F and g are compatible. Also, suppose that*

(a) F is continuous or

(b) X has the following properties:

(i) if nondecreasing sequence $\{x_m\} \to x$, *then* $g(x_m) \preccurlyeq g(x)$ *for all* $m \geq 0$;

(ii) if nonincreasing sequence $\{x_m\} \to x$, *then* $g(x) \preccurlyeq g(x_m)$ *for all* $m \geq 0$.

If there exist $x_0^1, x_0^2, x_0^3, ..., x_0^n \in X$ *such that (2) holds. Then F and g have an n-tupled coincidence point in X.*

In Theorem 3.1, setting $f(s,t) = \frac{s}{(1+t)^r}$, $r \in (0,\infty)$, $s,t \in (0,\infty)$, we obtain the following result.

Corollary 3.4. *Let* $(X, d, \preccurlyeq)$ *be a complete ordered metric space. Let* φ *be an ultra-altering distance function and* ψ *an altering distance function. Let* $F : X^n \to X$ *and* $g : X \to X$ *be two mappings such that F has the mixed g-monotone property on X and*

$$\psi(d(F(x^1,x^2,...,x^n),F(y^1,y^2,...,y^n))) \leq \frac{\psi(max\{d(gx^1,gy^1),...,d(gx^n,gy^n)\})}{(1+\varphi(max\{d(gx^1,gy^1),...,d(gx^n,gy^n)\}))^r}$$

for all $x^1, x^2, ..., x^n, y^1, y^2, ..., y^n \in X$ *and* $r \in (0,\infty)$ *for which* $gy^1 \preccurlyeq gx^1, gx^2 \preccurlyeq gy^2, gy^3 \preccurlyeq gx^3, ..., gx^n \preccurlyeq gy^n$. *Suppose that* $F(X^n) \subseteq g(X)$, *g is continuous and F and f are compatible. Also, suppose that*

(a) F is continuous or

(b) X has the following properties:

(i) if nondecreasing sequence $\{x_m\} \to x$, *then* $g(x_m) \preccurlyeq g(x)$ *for all* $m \geq 0$;

(ii) if nonincreasing sequence $\{x_m\} \to x$, *then* $g(x) \preccurlyeq g(x_m)$ *for all* $m \geq 0$.

If there exist $x_0^1, x_0^2, x_0^3, ..., x_0^n \in X$ *such that (2) holds. Then F and g have an n-tupled coincidence point in X.*

In Theorem 3.1, setting $f(s,t) = s\log_{a+t} a$, $a > 1$, $s,t \in (0,\infty)$ (f is a C-class function), we obtain the following result.

Corollary 3.5. *Let* $(X, d, \preccurlyeq)$ *be a complete ordered metric space. Let* φ *be an ultra-altering distance function and* ψ *an altering distance function. Let* $F : X^n \to X$ *and* $g : X \to X$ *be two* *mappings such that F has the mixed g-monotone property on X and*

$$\psi(d(F(x^1,x^2,...,x^n),F(y^1,y^2,...,y^n))) \leq \psi(max\{d(gx^1,gy^1),...,d(gx^n,gy^n)\})$$

$$\log_{a+\varphi(max\{d(gx^1,gy^1),...,d(gx^n,gy^n)\})} a$$

for all $x^1,x^2,...,x^n,y^1,y^2,...,y^n \in X$ for which $gy^1 \preccurlyeq gx^1, gx^2 \preccurlyeq gy^2, gy^3 \preccurlyeq gx^3,...,gx^n \preccurlyeq gy^n$. Suppose that $F(X^n) \subseteq g(X)$, g is continuous and F and g are compatible. Also, suppose that

(a) F is continuous or

(b) X has the following properties:

(i) if nondecreasing sequence $\{x_m\} \to x$, then $g(x_m) \preccurlyeq g(x)$ for all $m \geq 0$;

(ii) if nonincreasing sequence $\{x_m\} \to x$, then $g(x) \preccurlyeq g(x_m)$ for all $m \geq 0$.

If there exist $x_0^1,x_0^2,x_0^3,...,x_0^n \in X$ such that (2) holds. Then F and g have an n-tupled coincidence point in X.

Considering g to be an identity mapping in Theorem 3.1, we have the following result.

Corollary 3.6. *Let $(X,\preccurlyeq)$ be an ordered set. Suppose that there is a metric d on X such that (X,d) is a complete metric space. Let φ be an ultra-altering distance function and ψ be an altering distance function. Let $F: X^n \to X$ be a mapping having the mixed monotone property on X and f a C-class function and*

$$\begin{aligned}\psi(d(F(x^1,x^2,...,x^n),F(y^1,y^2,...,y^n))) \quad &\leq \quad f(\psi(max\{d(x^1,y^1),d(x^2,y^2),...,d(x^n,y^n)\}),\\ &\qquad \varphi(\max\{d(x^1,y^1),d(x^2,y^2),...,d(x^n,y^n)\}))\end{aligned}$$

for all $x^1,x^2,...,x^n,y^1,y^2,...,y^n \in X$ for which $y^1 \preccurlyeq x^1, x^2 \preccurlyeq y^2, y^3 \preccurlyeq x^3,...,x^n \preccurlyeq y^n$. Suppose that

(a) F is continuous or

(b) X has the following properties:

(i) if nondecreasing sequence $\{x_m\} \to x$, then $x_m \preccurlyeq x$ for all $m \geq 0$;

(ii) if nonincreasing sequence $\{x_m\} \to x$, then $x \preccurlyeq x_m$ for all $m \geq 0$.

If there exist $x_0^1, x_0^2, x_0^3, \dots, x_0^n \in X$ *such that*

$$\begin{cases} x_0^1 \preccurlyeq F(x_0^1, x_0^2, x_0^3, \dots, x_0^n), \\ F(x_0^2, x_0^3, \dots, x_0^n, x_0^1) \preccurlyeq x_0^2, \\ x_0^3 \preccurlyeq F(x_0^3, \dots, x_0^n, x_0^1, x_0^2), \\ \vdots \\ F(x_0^n, x_0^1, x_0^2, \dots, x_0^{n-1}) \preccurlyeq x_0^n, \end{cases} \tag{24}$$

then F *has an n-tupled fixed point in* X.

Considering ψ and g to be identity mappings in Theorem 3.1, we have the following result.

Corollary 3.7. *Let* $(X, \preccurlyeq)$ *be an ordered set. Suppose that there is a metric* d *on* X *such that* (X,d) *is a complete metric space. Let* φ *be an ultra-altering distance function and* f *a C-class function. Let* $F : X^n \to X$ *be a mapping having the mixed monotone property on* X *and*

$$\begin{aligned} d(F(x^1,x^2,\dots,x^n),F(y^1,y^2,\dots,y^n)) \leq\; & f(max\{d(x^1,y^1),d(x^2,y^2),\dots,d(x^n,y^n)\}, \\ & \varphi(\max\{d(x^1,y^1),d(x^2,y^2),\dots,d(x^n,y^n)\})) \end{aligned}$$

for all $x^1,x^2,\dots,x^n,y^1,y^2,\dots,y^n \in X$ *for which* $y^1 \preccurlyeq x^1, x^2 \preccurlyeq y^2, y^3 \preccurlyeq x^3, \dots, x^n \preccurlyeq y^n$. *Also in view of conditions (a) and (b) of Corollary 3.6, if (24) is satisfied, then* F *has an n-tupled fixed point in* X.

Considering ψ and g to be identity mappings, $f(s,t) = s - t$ and $\varphi(t) = (1-k)t$, where $0 \leq k < 1$ in Theorem 3.1, we have the following result.

Corollary 3.8. *Let* $(X, \preccurlyeq)$ *be an ordered set. Suppose that there is a metric* d *on* X *such that* (X,d) *is a complete metric space. Let* $F : X^n \to X$ *be a mapping having the mixed monotone property on* X. *Assume that there exists* $k \in [0,1)$ *with*

$$d(F(x^1,x^2,\dots,x^n),F(y^1,y^2,\dots,y^n)) \leq k\max\{d(x^1,y^1),d(x^2,y^2),\dots,d(x^n,y^n)\}$$

for all $x^1,x^2,\dots,x^n,y^1,y^2,\dots,y^n \in X$ *for which* $y^1 \preccurlyeq x^1, x^2 \preccurlyeq y^2, y^3 \preccurlyeq x^3, \dots, x^n \preccurlyeq y^n$. *Also in view of conditions (a) and (b) of Corollary 3.6, if (24) is satisfied, then* F *has an n-tupled fixed point in* X.

Remark 3.9. With $n = 2$, Theorem 3.1 and Corollaries 3.3-3.8 respectively yield the results of Choudhury *et al*. [9]. However, from Theorem 3.2, we can deduce a unique coupled common fixed point theorem.

Example 3.10. *Let $X = [0,1]$. Then $(X,\preccurlyeq)$ is an ordered set with the natural ordering of real numbers. Let $d(x,y) = |x-y|$ for all $x,y \in X$. Then (X,d) is a complete metric space with the required properties of Theorem 3.1. Define $g : X \to X$ by $g(x) = x^2$ for all $x \in X$ and $F : X^n \to X$ (wherein n is a fixed even integer) by*

$$F(x^1,x^2,...,x^n) = \begin{cases} \frac{(x^1)^2-(x^2)^2+(x^3)^2-.....+(x^{n-1})^2-(x^n)^2}{n+1}, \text{ if } x^{i+1} \preccurlyeq x^i, i = 1,3,...,n-1, \\ \\ 0 \qquad\qquad\qquad\qquad \text{otherwise,} \end{cases}$$

for all $x^1,x^2,...,x^n \in X$. Then F obeys the mixed g-monotone property. Now, define a function $f : [0,\infty)^2 \to \mathbb{R}$ by $f(s,t) = s-t$, $s,t \in [0,\infty)$. Then f is a C-class function. Let $\psi : [0,\infty) \to [0,\infty)$ and $\varphi : [0,\infty) \to [0,\infty)$ be defined respectively as follows:

$$\psi(t) = t^2 \text{ and } \varphi(t) = \frac{2n+1}{(n+1)^2}t^2, \text{ for } t \in [0,\infty).$$

Then ψ and φ have the properties mentioned in Theorem 3.1. Also F and f are compatible in X. Now choose $(x_0^1,x_0^2,\ldots,x_0^n) = (0,c,0,c,...,c)$ $(c > 0)$. Then

$$\begin{cases} g(x_0^1) = g(0) = 0 = F(x_0^1,x_0^2,x_0^3,...,x_0^n) = g(x_1^1), \\ g(x_1^2) = F(x_0^2,x_0^3,...,x_0^n,x_0^1) \preceq c^2 = g(c) = g(x_0^2), \\ g(x_0^3) = g(0) = 0 = F(x_0^3,...,x_0^n,x_0^1,x_0^2) = g(x_1^3), \\ \vdots \\ g(x_1^n) = F(x_0^n,x_0^1,x_0^2,...,x_0^{n-1}) \preceq c^2 = g(c) = g(x_0^n). \end{cases}$$

We next verify inequality (1) (of Theorem 3.1). We take $x^1,x^2,...,x^n$, $y^1,y^2,...,y^n \in X$ such that

$$gy^1 \preceq gx^1,\ gx^2 \preceq gy^2,\ gy^3 \preceq gx^3,...,gx^n \preceq gy^n.$$

Let

$$\begin{aligned} M &= \max\{d(gx^1,gy^1),d(gx^2,gy^2),d(gx^3,gy^3),...,d(gx^n,gy^n)\} \\ &= \max\{|(x^1)^2-(y^1)^2|,|(x^2)^2-(y^2)^2|,|(x^3)^2-(y^3)^2|,...,|(x^n)^2-(y^n)^2|\}. \end{aligned}$$

Then

$$M \geq |(x^1)^2-(y^1)^2|,\ M \geq |(x^2)^2-(y^2)^2|, M \geq |(x^3)^2-(y^3)^2|,..., M \geq |(x^n)^2-(y^n)^2|.$$

The following four cases arise:

Case I: Let $x^1,x^2,x^3,...,x^n,\ y^1,y^2,y^3,...,y^n \in X$ *such that* $x^{i+1} \preceq x^i, y^{i+1} \preceq y^i$ *for* $i = 1,3,...,n-1$. *Then*

$$\begin{aligned} & d(F(x^1,x^2,x^3,...,x^n),F(y^1,y^2,y^3,...,y^n)) \\ &= d\left(\frac{(x^1)^2-(x^2)^2+(x^3)^2-....-(x^n)^2}{n+1},\frac{(y^1)^2-(y^2)^2+(y^3)^2-....-(y^n)^2}{n+1}\right) \\ &= \left|\frac{(x^1)^2-(x^2)^2+(x^3)^2-....-(x^n)^2}{n+1}-\frac{(y^1)^2-(y^2)^2+(y^3)^2-....-(y^n)^2}{n+1}\right| \\ &= \left|\frac{((x^1)^2-(y^1)^2)-((x^2)^2-(y^2)^2)+((x^3)^2-(y^3)^2)-....-((x^n)^2-(y^n)^2)}{n+1}\right| \\ &\leq \frac{|(x^1)^2-(y^1)^2|+|(x^2)^2-(y^2)^2|+|(x^3)^2-(y^3)^2|+....+|(x^n)^2-(y^n)^2|}{n+1} \\ &\leq \frac{n}{n+1}M. \end{aligned}$$

Case II: Let $x^1,x^2,x^3,...,x^n,\ y^1,y^2,y^3,...,y^n \in X$ *such that* $x^{i+1} \preceq x^i$ *for* $i = 1,3,...,n-1$ *and* $y^i \preceq y^{i+1}$ *for at least one* i. *Then (for* $y^1 \preceq y^2$*),*

$$\begin{aligned} & d(F(x^1,x^2,x^3,...,x^n),F(y^1,y^2,y^3,...,y^n)) \\ &= d\left(\frac{(x^1)^2-(x^2)^2+(x^3)^2-....-(x^n)^2}{n+1},0\right) \\ &\leq \left|\frac{(x^1)^2-(x^2)^2+(x^3)^2-....-(x^n)^2+(y^2)^2-(y^1)^2}{n+1}\right| \end{aligned}$$

$$= \left| \frac{((x^1)^2 - (y^1)^2) - ((x^2)^2 - (y^2)^2) + (x^3)^2 - (x^4)^2 + ... - (x^n)^2}{n+1} \right|$$

$$\vdots$$

$$\leq \frac{|(x^1)^2 - (y^1)^2| + |(x^2)^2 - (y^2)^2| + |(x^3)^2 - (y^3)^2| + ... + |(x^n)^2 - (y^n)^2|}{n+1}$$

$$\leq \frac{n}{n+1}M.$$

Case III: Let $x^1, x^2, x^3, ..., x^n,\ y^1, y^2, y^3, ..., y^n \in X$ *such that* $x^i \preceq x^{i+1}$ *for at least one i and* $y^{i+1} \preceq y^i$ *for* $i = 1, 3, ..., n-1$. *Then arguing as in Case II, one verify inequality (1).*

Case IV: Let $x^1, x^2, x^3, ..., x^n,\ y^1, y^2, y^3, ..., y^n \in X$ *such that* $x^i \preceq x^{i+1}, y^i \preceq y^{i+1}$ *for at least one* i. *Then*

$$d(F(x^1, x^2, x^3, ..., x^n), F(y^1, y^2, y^3, ..., y^n)) = d(0,0) \leq \frac{n}{n+1}M.$$

In all above cases

$$\psi(d(F(x^1, x^2, x^3, ..., x^n), F(y^1, y^2, y^3, ..., y^n)))$$

$$\leq \frac{n^2}{(n+1)^2}M^2 = M^2 - \frac{2n+1}{(n+1)^2}M^2$$

$$= \psi(\max\{d(gx^1, gy^1), d(gx^2, gy^2), ..., d(gx^n, gy^n)\})$$

$$-\varphi(\max\{d(gx^1, gy^1), d(gx^2, gy^2), ..., d(gx^n, gy^n)\})$$

$$= f(\psi(\max\{d(gx^1, gy^1), d(gx^2, gy^2), ..., d(gx^n, gy^n)\}),$$

$$\varphi(\max\{d(gx^1, gy^1), d(gx^2, gy^2), ..., d(gx^n, gy^n)\})).$$

Hence all the conditions of Theorem 3.1 are satisfied and $(0,0,0,...,0)$ *is an n-tupled coincidence point of* F *and* g.

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