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# SOME RESULTS ON SPLIT VARIATIONAL INCLUSION AND FIXED POINT PROBLEMS IN HILBERT SPACES

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**Abstract.** The purpose of this paper is to study common solutions of split variational inclusion and fixed point problems of a nonexpansive mapping via a new viscosity iterative method with Meri-Keeler contractions. A strong convergence theorem is established in the framework of Hilbert spaces with mild restrictions imposed on the control sequences. The result presented in this paper are the supplement, extension and generalization of the previously known results in this area.

**Keywords.** Viscosity iterative method; Meri-Keeler contraction; Split variational inclusion problem; Fixed point problem; Nonexpansive mapping.

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## 1. Introduction-Preliminaries

Throughout the paper unless otherwise stated, we assume that $H_1$ and $H_2$ are real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let $C$ and $Q$ be nonempty closed convex subsets of $H_1$ and $H_2$, respectively. A mapping $S : H_1 \to H_1$ is called contraction, if there exists a constant $\alpha \in (0,1)$ such that $\| Sx - Sy \| \leq \alpha \| x - y \|$, $\forall x, y \in H_1$. If $\alpha = 1$, $S$ is called a nonexpansive mapping. Further, we consider the following fixed point problem (in short, FPP) for a nonexpansive mapping $S : H_1 \to H_1$: Find $x \in H_1$ such that $Sx = x$. The solution set of the FPP is denoted by $Fix(S)$. It is well known that if $C$ is closed convex and bounded, then $Fix(S)$ is nonempty, closed and convex.

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For every point $x \in H_1$, there exists a unique nearest point in $C$ denoted by $P_C x$ such that $\| x - P_C x \| \leq \| x - y \|$, $\forall y \in C$. $P_C$ is called the metric projection of $H_1$ onto $C$. It is well known that $P_C$ is nonexpansive mapping and satisfies $\langle x - y, P_C x - P_C y \rangle \geq \| P_C x - P_C y \|^2$, $\forall x, y \in H_1$. Moreover, $P_C x$ is characterized by the fact $P_C x \in C$ and $\langle x - P_C x, y - P_C x \rangle \leq 0$, and $\| x - y \|^2 \geq \| x - P_C x \|^2 + \| y - P_C x \|^2$, $\forall x \in H_1, y \in C$. In a real Hilbert space the following hold: $\| \lambda x + (1-\lambda) y \|^2 = \lambda \| x \|^2 + (1-\lambda) \| y \|^2 - \lambda(1-\lambda) \| x - y \|^2$ for all $x, y \in H_1$ and $\lambda \in (0,1)$. It is well known that every nonexpansive operator $T : H_1 \to H_1$ satisfies, for all $x, y \in H_1 \times H_1$, the inequality $\langle (x - T(x)) - (y - T(y)), T(y) - T(x) \rangle \leq \frac{1}{2} \| (T(x) - x) - (T(y) - y) \|^2$, and therefore, we get, for all $(x, y) \in H_1 \times Fix(T)$, $\langle x - T(x), y - T(y) \rangle \leq \frac{1}{2} \| T(x) - x \|^2$ . A mapping $T : H_1 \to H_1$ is said to be averaged if and only if it can be written as the average of the identity mapping and a nonexpansive mapping, i.e., $T := (1-\alpha)I + \alpha S$ where $\alpha \in (0,1)$ and $S : H_1 \to H_1$ is nonexpansive and $I$ is the identity operator on $H_1$. We note that averaged mappings are nonexpansive. Further, firmly nonexpansive mappings(in particular, projections on nonempty closed and convex subsets and resolvent operators of maximal monotone operators) are averaged.

If $T = (1-\alpha)S + \alpha V$, where $S : H_1 \to H_1$ is averaged, $V : H_1 \to H_1$ is nonexpansive and $\alpha \in (0,1)$, then $T$ is averaged. The composite of finitely many averaged mappings is averaged. If the mappings $\{T_i\}_{i=1}^N$ are averaged and have a nonempty common fixed point, then $\bigcap_{i=1}^N Fix(T_i) = Fix(T_1, T_2, \ldots, T_N)$. If $T$ is $\tau - ism$, then for $\gamma > 0$, $\gamma T$ is $\frac{\tau}{\gamma} - ism$. $T$ is averaged if and only if, its complement $I - T$ is $\tau - ism$ for some $\tau > \frac{1}{2}$.

Let $f$ be a Meir-Keeler contraction on (in short, MKC) $C$. Then for any $t \in (0,1)$, the mapping $S_t^f : x \mapsto t f(x) + (1-t) S x$ is also a MKC from $C$ into itself. By the Meir-Keeler fixed point theorem [1], $S_t^f$ has a unique fixed point $x_t$ in $C$, i.e., $x_t = t f(x_t) + (1-t) S(x_t)$, $t \in (0,1)$. The viscosity approximation methods are very important due to they be applied to convex optimization, linear programming, monotone type variational inequality, monotone inclusions, elliptic differential equations, and other applied science; see [2-5] and the references therein. Recall that a mapping $T : H_1 \to H_1$ is said to be

(i) monotone, if

$$\langle Tx - Ty, x - y \rangle \geq 0, \forall x, y \in H_1.$$

(ii) $\alpha$-strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha \| x - y \|^2, \forall x, y \in H_1.$$

(iii) $\beta$-inverse strongly monotone(or, $\beta$-ism), if there exists a constant $\beta > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \beta \parallel Tx - Ty \parallel^2, \forall x, y \in H_1.$$

(iv) firmly nonexpansive, if

$$\langle Tx - Ty, x - y\rangle \geq \parallel Tx - Ty \parallel^2, \forall x, y \in H_1.$$

A multi-valued mapping $M : H_1 \to 2^{H_1}$ is called monotone if for all $x, y \in H_1$, $u \in Mx$ and $v \in My$ such that $\langle x - y, u - v\rangle \geq 0$. A monotone mapping $M : H_1 \to 2^{H_1}$ is maximal if the Graph($M$) is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping $M$ is maximal if and only if for $(x, u) \in H_1 \times H_1$, $\langle x - y, u - v\rangle \geq 0$, for every $(y, v) \in Graph(M)$ implies that $u \in M(x)$. Let $M : H_1 \to 2^{H_1}$ be a multi-valued maximal monotone mapping. Then, the resolvent mapping $J_\lambda^M : H_1 \to H_1$ associated with $M$, is defined by $J_\lambda^M := (I + \lambda M)^{-1}(x), \forall x \in H_1$, for some $\lambda > 0$, where $I$ stands identity operator on $H_1$. We note that for all $\lambda > 0$ the resolvent operator $J_\lambda^M$ is single-valued, nonexpansive and firmly nonexpansive. Recently, Moudafi [6] introduced the following split monotone variational inclusion problem (in short, SMVIP): Find $x^* \in H_1$ such that

$$0 \in f_1(x^*) + B_1(x^*), \tag{1.1}$$

and such that

$$y^* = Ax^* \in H_2 \quad solves \quad 0 \in f_2(y^*) + B_2(y^*), \tag{1.2}$$

where $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are multi-valued maximal monotone mappings. Moudafi [6] introduced an iterative method for solving SMVIP (1.1)-(1.2), which can be seen an important generalization of an iterative method given by Censor et al. [7] for split variational inequality problem. As Moudafi motes in [6], SMVIP (1.4)-(1.5) includes as special cases, the split common fixed point problem, split variational inequality problem, split zero problem and split feasibility problem [8-11] which have already been studied and used in practice as a model in intensity-modulated radiation therapy treatment planning, see [12]. This formalism is also at the core of modeling of many inverse problems arising for phase retrieval and other real-world problems.

If $f_1 \equiv 0$ and $f_2 \equiv 0$ then SMVIP (1.1)-(1.2) reduces to the following split variational inclusion problem (in short, SVIP): Find $x^* \in H_1$ such that

$$0 \in B_1(x^*), \tag{1.3}$$

and such that

$$y^* = Ax^* \in H_2 \quad solves \quad 0 \in B_2(y^*). \tag{1.4}$$

When looked separately, (1.3) is the variational inclusion problem and we denoted its solution set by SOLVIP ($B_1$). The SVIP (1.3)-(1.4) constitutes a pair of variational inclusion problems which have to be solved so that the image $y^* = Ax^*$ under a given bounded linear operator $A$, of the solution $x^*$ of SVIP (1.3) in $H_1$ is the solution of another SVIP (1.4) in another space $H_2$, we denote the solution set of SVIP (1.4) by SOLVIP ($B_2$). The solution set of SVIP (1.3)-(1.4) is denoted by $\Gamma = \{x^* \in H_1 : x^* \in SOLVIP(B_1)$
$and \quad Ax^* \in SOLVIP(B_2)\}$. Motivated by the work of going on in this direction, we suggest and analyze an iterative method for approximating a common solution of SVIP (1.3)-(1.4) and a fixed point of nonexpansive mapping. Furthermore, we prove that the sequences generated by the iterative scheme converge strongly to a common solution of SVIP (1.3)-(1.4).

**Lemma 2.1.** [13] *Assume that $T$ is nonexpansive self mapping of a closed convex subset $C$ of a Hilbert space $H_1$. If $T$ has a fixed point, then $I-T$ is demiclosed, i.e., whenever $\{x_n\}$ is a sequence in $C$ converging weakly to some $x \in C$ and the sequence $\{(I-T)x_n\}$ converges strongly to some $y$, it follows that $(I-T)x = y$. Here $I$ is the identity mapping on $H_1$.*

**Lemma 2.2.** [14] *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\xi_n)a_n + \delta_n, n \geq 0,$$

*where $\{\xi_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $R$ such that*
*(i) $\Sigma_{n=1}^{\infty} \xi_n = \infty$;*
*(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\xi_n} \leq 0$ or $\sum_{n=1}^{\infty} \mid \delta_n \mid < \infty$.*
*Then $\lim_{n\to\infty} a_n = 0$.*

**Lemma 2.3.** [15] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Hilbert space $H_1$ and let $\{\beta_n\}$ be a sequence in (0, 1) with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and*

$$\limsup_{n\to\infty} (\| y_{n+1} - y_n \| - \| x_{n+1} - x_n \|) \leq 0.$$

*Then $\lim_{n\to\infty} \| y_n - x_n \| = 0$.*

## 2. Main results

**Theorem 2.1.** *Let $H_1$ and $H_2$ be two real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator and $f : H_1 \to H_2$ be a MKC. Let $S : H_1 \to H_2$ be a nonexpansive mapping such that $Fix(S) \cap \Gamma \neq \varnothing$. For a give $x_0 \in H_1$, arbitrarily, let the iterative sequences $\{u_n\}$ and $\{x_n\}$ be generated by*

$$\begin{aligned} u_n &= J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n), \\ x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n S u_n, \end{aligned} \tag{2.1}$$

*where $\lambda > 0$ and $\gamma \in (0, \frac{1}{L})$, $L$ is the spectral radius of the operator $A^*A$ and $A^*$ is the adjoint of $A$ and $\{\alpha_n\}$ ,$\{\beta_n\}$, $\{\gamma_n\}$ be real number sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$ and $\lim_{n\to\infty} \alpha_n = 0$, $\Sigma_{n=0}^{\infty} \alpha_n = \infty$, and $\Sigma_{n=0}^{\infty} \|\alpha_n - \alpha_{n-1}\| < \infty$,$0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Then the sequences $\{u_n\}$ and $\{x_n\}$ both converge strongly to $z \in Fix(S) \cap \Gamma$, where $z = P_{Fix(S)\cap\Gamma} f(z)$.*

**Proof** Letting $p \in Fix(S) \cap \Gamma$, we have $p = J_\lambda^{B_1} p$, $Ap = J_\lambda^{B_2}(Ap)$ and $Sp = p$. For a give $y_0 \in H_1$, let the iterative sequences $\{v_n\}$ and $\{y_n\}$ be generated by

$$\begin{aligned} v_n &= J_\lambda^{B_1}(y_n + \gamma A^*(J_\lambda^{B_2} - I)Ay_n), \\ y_{n+1} &= \alpha_n f(z) + \beta_n y_n + \gamma_n S v_n, \end{aligned} \tag{2.2}$$

We show that $\{y_n\}$ is a bounded sequence. Note that

$$\begin{aligned} \| v_n - p \|^2 &\le \| y_n + \gamma A^*(J_\lambda^{B_2} - I)Ay_n - p \|^2 \\ &\le \| y_n - p \|^2 + \gamma^2 \| A^*(J_\lambda^{B_2} - I)Ay_n \|^2 + 2\gamma\langle y_n - p, A^*(J_\lambda^{B_2} - I)Ay_n\rangle. \end{aligned} \tag{2.3}$$

Thus, we have

$$\begin{aligned} \| v_n - p \|^2 \le \| y_n - p \|^2 &+ \gamma^2 \langle (J_\lambda^{B_2} - I)Ay_n, AA^*(J_\lambda^{B_2} - I)Ay_n\rangle \\ &+ 2\gamma\langle y_n - p, A^*(J_\lambda^{B_2} - I)Ay_n\rangle. \end{aligned} \tag{2.4}$$

Therefore, we find that

$$\begin{aligned} \gamma^2 \langle (J_\lambda^{B_2} - I)Ay_n, AA^*(J_\lambda^{B_2} - I)Ay_n\rangle &\le L\gamma^2 \langle (J_\lambda^{B_2} - I)Ay_n, (J_\lambda^{B_2} - I)Ay_n\rangle \\ &= L\gamma^2 \| (J_\lambda^{B_2} - I)Ay_n \|^2 . \end{aligned} \tag{2.5}$$

Denoting $\Lambda = 2\gamma\langle y_n - p, A^*(J_\lambda^{B_2} - I)Ay_n\rangle$, we have

$$\begin{aligned}
\Lambda &= 2\gamma\langle y_n - p, A^*(J_\lambda^{B_2} - I)Ay_n\rangle \\
&= 2\gamma\langle A(y_n - p), (J_\lambda^{B_2} - I)Ay_n\rangle \\
&= 2\gamma\langle A(y_n - p) + (J_\lambda^{B_2} - I)Ay_n - (J_\lambda^{B_2} - I)Ay_n, (J_\lambda^{B_2} - I)Ay_n\rangle \\
&= 2\gamma\{\langle J_\lambda^{B_2}Ax_n - Ap, (J_\lambda^{B_2} - I)Ay_n\rangle - \| (J_\lambda^{B_2} - I)Ay_n \|^2\} \\
&\leq 2\gamma\{\frac{1}{2} \| (J_\lambda^{B_2} - I)Ay_n \|^2 - \| (J_\lambda^{B_2} - I)Ay_n \|^2\} \\
&\leq -\gamma \| (J_\lambda^{B_2} - I)Ay_n \|^2 .
\end{aligned} \tag{2.6}$$

Using (2.4), (2.5) and (2.6), we obtain

$$\| v_n - p \|^2 \leq \| y_n - p \|^2 + \gamma(L\gamma - 1) \| (J_\lambda^{B_2} - I)Ay_n \|^2 . \tag{2.7}$$

Since $\gamma \in (0, \frac{1}{L})$, we obtain

$$\| v_n - p \|^2 \leq \| y_n - p \|^2 . \tag{2.8}$$

Next, we estimate

$$\begin{aligned}
\| y_{n+1} - p \| &= \| \alpha_n f(z) + \beta_n y_n + \gamma_n S v_n - p \| \\
&= \| \alpha_n f(z) + \beta_n y_n + \gamma_n S v_n - (\alpha_n + \beta_n + \gamma_n)p \| \\
&\leq \alpha_n \| f(z) - p \| + \beta_n \| y_n - p \| + \gamma_n \| S v_n - p \| \\
&\leq \alpha_n \| f(z) - p \| + \beta_n \| y_n - p \| + \gamma_n \| v_n - p \| \\
&\leq \alpha_n \| f(z) - p \| + \beta_n \| y_n - p \| + \gamma_n \| y_n - p \| \\
&= \alpha_n \| f(z) - p \| + (1 - \alpha_n) \| y_n - p \| \\
&\leq \max\{\| f(z) - p \|, \| y_n - p \|\} \\
&\;\;\vdots \\
&\leq \max\{\| f(z) - p \|, \| y_0 - p \|\}.
\end{aligned} \tag{2.9}$$

Hence $\{y_n\}$ is bounded and consequently, we deduce that $\{v_n\}$, $\{Sv_n\}$ are bounded. Now, we show that the sequence $\{y_n\}$ is asymptotically regular, i.e., $\| y_{n+1} - y_n \| \to 0$ as $n \to \infty$. Let

$y_{n+1}=\beta_n y_n+(1-\beta_n)\rho_n$, then $\rho_n=\frac{\alpha_n f(z)+\gamma_n Sv_n}{1-\beta_n}$.

$$\begin{aligned}
\| \rho_{n+1}-\rho_n \| &=\| \frac{\alpha_{n+1}f(z)+(1-\alpha_{n+1}-\beta_{n+1})Sv_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n f(z)+(1-\alpha_n-\beta_n)Sv_n}{1-\beta_n} \| \\
&=\| \frac{\alpha_{n+1}(f(z)-Sv_{n+1})}{1-\beta_{n+1}}+Sv_{n+1}-\frac{\alpha_n(f(z)-Sv_n)}{1-\beta_n}-Sv_n \| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}} \| f(z)-Sv_{n+1} \| +\frac{\alpha_n}{1-\beta_n} \| f(z)-Sv_n \| \\
&\quad + \| Sv_{n+1}-Sv_n \| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}} \| f(z)-Sv_{n+1} \| +\frac{\alpha_n}{1-\beta_n} \| f(z)-Sv_n \| \\
&\quad + \| v_{n+1}-v_n \| .
\end{aligned} \tag{2.10}$$

Since, for $\gamma\in(0,\frac{1}{L})$, the mapping $J_\lambda^{B_1}(I+\gamma A^*(J_\lambda^{B_2}-I)A)$ is averaged and hence nonexpansive. Then we obtain

$$\begin{aligned}
\| v_{n+1}-v_n \| &\leq\| J_\lambda^{B_1}(y_{n+1}+\gamma A^*(J_\lambda^{B_2}-I)Ay_{n+1})-J_\lambda^{B_1}(y_n+\gamma A^*(J_\lambda^{B_2}-I)Ay_n) \| \\
&\leq\| J_\lambda^{B_1}(I+\gamma A^*(J_\lambda^{B_2}-I)A)y_{n+1}-J_\lambda^{B_1}(I+\gamma A^*(J_\lambda^{B_2}-I)A)y_n) \| \\
&\leq\| y_{n+1}-y_n \| .
\end{aligned} \tag{2.11}$$

Using (2.10) and (2.11), we obtain

$$\| \rho_{n+1}-\rho_n \| - \| y_{n+1}-y_n \| \leq \frac{\alpha_{n+1}}{1-\beta_{n+1}} \| f(z)-Sv_{n+1} \| +\frac{\alpha_n}{1-\beta_n} \| f(z)-Sv_n \| .$$

Since, for $\lim_{n\to\infty}\alpha_n=0$, we get

$$\limsup_{n\to\infty} \| \rho_{n+1}-\rho_n \| - \| y_{n+1}-y_n \|\leq 0.$$

It follows that $\lim_{n\to\infty} \| \rho_n-y_n \|=0$, which in turn implies that

$$\lim_{n\to\infty} \| y_{n+1}-y_n \|=0. \tag{2.12}$$

Now, we write

$$\begin{aligned}
y_{n+1}-y_n &= \alpha_n f(z)+\beta_n y_n+\gamma_n Sv_n-y_n \\
&= \alpha_n(f(z)-y_n)+\gamma_n(Sv_n-y_n).
\end{aligned}$$

It follows that

$$\gamma_n \| Sv_n-y_n \|\leq\| y_{n+1}-y_n \| +\alpha_n \| f(z)-y_n \| .$$

Since $\| y_{n+1} - y_n \| \to 0$ and $\alpha_n \to 0$ as $n \to \infty$, we obtain $\| Sv_n - y_n \| \to 0$ as $n \to \infty$. Note that

$$\begin{aligned}
\| y_{n+1} - p \|^2 &= \| \alpha_n f(z) + \beta_n y_n + \gamma_n S v_n - p \|^2 \\
&= \| \alpha_n f(z) + \beta_n y_n + \gamma_n S v_n - (\alpha_n + \beta_n + \gamma_n) p \|^2 \\
&\leq \alpha_n \| f(z) - p \|^2 + \beta_n \| y_n - p \|^2 + \gamma_n \| S v_n - p \|^2 \\
&\leq \alpha_n \| f(z) - p \|^2 + \beta_n \| y_n - p \|^2 + \gamma_n \| v_n - p \|^2 \\
&\leq \alpha_n \| f(z) - p \|^2 + \beta_n \| y_n - p \|^2 \\
&+ \gamma_n [\| y_n - p \|^2 + \gamma(L\gamma - 1) \| (J_\lambda^{B_2} - I) A y_n \|^2] \\
&= \alpha_n \| f(z) - p \|^2 + (\beta_n + \gamma_n) \| y_n - p \|^2 + \gamma_n \gamma (L\gamma - 1) \| (J_\lambda^{B_2} - I) A y_n \|^2 \\
&\leq \alpha_n \| f(z) - p \|^2 + \| y_n - p \|^2 + \gamma (L\gamma - 1) \| (J_\lambda^{B_2} - I) A y_n \|^2
\end{aligned} \tag{2.13}$$

Therefore,

$$\begin{aligned}
&\gamma(1 - L\gamma) \| (J_\lambda^{B_2} - I) A y_n \|^2 \\
&\quad \leq \alpha_n \| f(z) - p \|^2 + \| y_n - p \|^2 - \| y_{n+1} - p \|^2 \\
&\quad \leq \alpha_n \| f(z) - p \|^2 + \| y_{n+1} - y_n \| (\| y_n - p \| + \| y_{n+1} - p \|).
\end{aligned}$$

Since $(1 - L\gamma) > 0$, and $\alpha_n \to 0$, and $\| y_{n+1} - y_n \| \to 0$ as $n \to \infty$, we have

$$\| (J_\lambda^{B_2} - I) A y_n \| = 0. \tag{3.14}$$

Furthermore, using $\gamma \in (0, \frac{1}{L})$, we observe that

$$
\begin{aligned}
\| v_n - p \|^2 &= \| J_\lambda^{B_1}(y_n + \gamma A^*(J_\lambda^{B_2} - I)Ay_n) - p \|^2 \\
&= \| J_\lambda^{B_1}(y_n + \gamma A^*(J_\lambda^{B_2} - I)Ay_n) - J_\lambda^{B_1} p \|^2 \\
&\leq \langle v_n - p, y_n + \gamma A^*(J_\lambda^{B_2} - I)Ay_n - p \rangle \\
&= \frac{1}{2}\{ \| v_n - p \|^2 + \| y_n + \gamma A^*(J_\lambda^{B_2} - I)Ay_n - p \|^2 \\
&\quad - \| (v_n - p) - [y_n + \gamma A^*(J_\lambda^{B_2} - I)Ay_n - p] \|^2 \} \\
&\leq \frac{1}{2}\{ \| v_n - p \|^2 + \| y_n - p \|^2 + \gamma(L\gamma - 1) \| (J_\lambda^{B_2} - I)Ay_n \|^2 \\
&\quad - \| v_n - y_n - \gamma A^*(J_\lambda^{B_2} - I)Ay_n \|^2 \} \\
&\leq \frac{1}{2}\{ \| v_n - p \|^2 + \| y_n - p \|^2 - [\| v_n - y_n \|^2 \\
&\quad + \gamma^2 \| A^*(J_\lambda^{B_2} - I)Ay_n \|^2 - 2\gamma \langle v_n - y_n, A^*(J_\lambda^{B_2} - I)Ay_n \rangle ] \} \\
&\leq \frac{1}{2}\{ \| v_n - p \|^2 + \| y_n - p \|^2 - \| v_n - y_n \|^2 \\
&\quad + 2\gamma \| A(v_n - y_n) \| \| (J_\lambda^{B_2} - I)Ay_n \| \}.
\end{aligned}
$$

Hence, we obtain

$$
\| v_n - p \|^2 \leq \| y_n - p \|^2 - \| v_n - y_n \|^2 + 2\gamma \| A(v_n - y_n) \| \| (J_\lambda^{B_2} - I)Ay_n \| . \tag{2.15}
$$

It follows that from (2.13) and (2.15) that

$$
\begin{aligned}
\| y_{n+1} - p \|^2 &\leq \alpha_n \| f(z) - p \|^2 + \beta_n \| y_n - p \|^2 + \gamma_n \| v_n - p \|^2 \\
&\leq \alpha_n \| f(z) - p \|^2 + \beta_n \| y_n - p \|^2 \\
&\quad + \gamma_n [\| y_n - p \|^2 - \| v_n - y_n \|^2 + 2\gamma \| A(v_n - y_n) \| \| (J_\lambda^{B_2} - I)Ay_n \|] \\
&= \alpha_n \| f(z) - p \|^2 + (\beta_n + \gamma_n) \| y_n - p \|^2 \\
&\quad - \gamma_n \| v_n - y_n \|^2 + 2\gamma\gamma_n \| A(v_n - y_n) \| \| (J_\lambda^{B_2} - I)Ay_n \| \\
&\leq \alpha_n \| f(z) - p \|^2 + \| y_n - p \|^2 \\
&\quad - \gamma_n \| v_n - y_n \|^2 + 2\gamma \| A(v_n - y_n) \| \| (J_\lambda^{B_2} - I)Ay_n \| .
\end{aligned}
$$

Therefore, we arrive at

$$\begin{aligned}\gamma_n \parallel v_n - y_n \parallel^2 \leq& \alpha_n \parallel f(z) - p \parallel^2 + \parallel y_n - p \parallel^2 - \parallel y_{n+1} - p \parallel^2 \\ &+ 2\gamma \parallel A(v_n - y_n) \parallel \parallel (J_\lambda^{B_2} - I)Ay_n \parallel \\ \leq& \alpha_n \parallel f(z) - p \parallel^2 + (\parallel y_n - p \parallel + \parallel y_{n+1} - p \parallel) \parallel y_n - y_{n+1} \parallel \\ &+ 2\gamma \parallel A(v_n - y_n) \parallel \parallel (J_\lambda^{B_2} - I)Ay_n \parallel .\end{aligned}$$

Since $\alpha_n \to 0$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \parallel v_n - y_n \parallel = 0. \tag{2.16}$$

Now, we can write

$$\parallel Sv_n - v_n \parallel \leq \parallel Sv_n - y_n \parallel + \parallel y_n - v_n \parallel . \tag{2.17}$$

Since $\{v_n\}$ is bounded, we consider a weak cluster point $w$ of $\{v_n\}$. Hence, there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$, which converges weakly to $w$. Now, $S$ being nonexpnsive, by (2.17), we obtain that $w \in Fix(S)$. On the other hand, $v_{n_k} = J_\lambda^{B_1}(y_{n_k} + \gamma A^*(J_\lambda^{B_2} - I)Ay_{n_k})$ can be rewritten as

$$\frac{(y_{n_k} - v_{n_k}) + A^*(J_\lambda^{B_2} - I)Ay_{n_k}}{\lambda} \in B_1 v_{n_k}. \tag{3.18}$$

By passing to limit $k \to 0$ in (2.18) and by taking into account (2.14), (2.16) and the fact that the graph of a maximal monotone operator is weakly-strongly closed, we obtain $0 \in B_1(w)$, i.e., $w \in SOLVIP(B_1)$. Furthermore, since $\{y_n\}$ and $\{v_n\}$ have the same asymptotical behavior, $\{Ay_{n_k}\}$ weakly converges to $Aw$. Again, by (2.14) and the fact that the resolvent $J_\lambda^{B_2}$ is nonexpansive, we obtain that $Aw \in B_2(Aw)$, i.e., $Aw \in SOLVIP(B_2)$. Thus $w \in Fix(S) \cap \Gamma$. Next, we claim that $\limsup_{n\to\infty} \langle f(z) - z, y_n - z\rangle \leq 0$, where $z = P_{Fix(S)\cap\Gamma} f(z)$. Indeed, we have

$$\begin{aligned}\limsup_{n\to\infty} \langle f(z) - z, y_n - z\rangle &= \limsup_{n\to\infty} \langle f(z) - z, Sv_n - z\rangle \\ &\leq \limsup_{n\to\infty} \langle f(z) - z, v_n - z\rangle \\ &= \langle f(z) - z, w - z\rangle \\ &\leq 0,\end{aligned} \tag{2.19}$$

since $z = P_{Fix(S)\cap\Gamma} f(z)$. Note that

$$\begin{aligned}\| y_{n+1} - z \|^2 &= \langle \alpha_n f(z) + \beta_n y_n + \gamma_n S v_n - z, y_{n+1} - z \rangle \\ &\leq \alpha_n \langle f(z) - z, y_{n+1} - z \rangle + \beta_n \langle y_n - z \rangle + \gamma_n \langle v_n - z, y_{n+1} - z \rangle \\ &\leq \alpha_n \langle f(z) - z, y_{n+1} - z \rangle + \beta_n \langle y_n - z \rangle + \gamma_n \langle y_n - z, y_{n+1} - z \rangle \\ &= \alpha_n \langle f(z) - z, y_{n+1} - z \rangle + (1 - \alpha_n) \langle y_n - z, y_{n+1} - z \rangle \\ &\leq \alpha_n \langle f(z) - z, y_{n+1} - z \rangle + \frac{(1 - \alpha_n)}{2} \{ \| y_n - z \|^2 + \| y_{n+1} - z \|^2 \},\end{aligned}$$

which implies that

$$(1 + \alpha_n) \| y_{n+1} - z \|^2 \leq (1 - \alpha_n) \| y_n - z \|^2 + 2\alpha_n \langle f(z) - z, y_{n+1} - z \rangle.$$

It follows that $\| y_{n+1} - z \|^2 \leq (1 - \alpha_n) \| y_n - z \|^2 + 2\alpha_n \langle f(z) - z, y_{n+1} - z \rangle$. Now, by using (2.19), we deduce that $y_n \to z$. Further it follows from $\| v_n - y_n \| \to 0$, $v_n \rightharpoonup w \in Fix(S) \cap \Gamma$ and $y_n \to z$ as $n \to \infty$, that $z = w$. Finally, we prove that $x_n \to z (n \to \infty)$. To end this, we need to show that $x_n - y_n \to 0$ as $n \to \infty$. Assume that $a = \overline{\lim}_n \| x_n - y_n \| > 0$; then $\forall \varepsilon \in (0, a)$, we can choose $\eta > 0$ such that

$$\overline{\lim_n} \| x_n - y_n \| > \varepsilon + \eta. \tag{2.20}$$

For above $\varepsilon > 0$, using Suzuki [16], we know that there exists $\beta \in (0, 1)$ such that

$$\| f(x) - f(y) \| \leq \beta \| x - y \|, \tag{3.21}$$

for all $x, y \in H_1$ with $\| x - y \| \geq \varepsilon$, which implies that

$$\| f(x) - f(y) \| \leq \max\{\beta \| x - y \|, \varepsilon\}, \tag{2.22}$$

for all $x, y \in H_1$. Since $y_n \to z$ as $n \to \infty$, we see that there exists some integer $n_0 \geq 1$ such that

$$\| y_n - z \| \leq (1 - \beta)\eta, \tag{2.23}$$

for all $n \geq n_0$. We now consider tow possible cases.

Case 1. There exists some $n_1 \geq n_0$ such that

$$\| x_{n_1} - y_{n_1} \| \leq \varepsilon + \eta. \tag{2.24}$$

Therefore, we have

$$\begin{aligned}
\| x_{n_1+1} - y_{n_1+1} \| &= \| \alpha_{n_1} f(x_{n_1}) + \beta_{n_1} x_{n_1} + \gamma_{n_1} S u_{n_1} - (\alpha_{n_1} f(z) + \beta_{n_1} y_{n_1} + \gamma_{n_1} S v_{n_1}) \| \\
&\leq \alpha_{n_1} \| f(x_{n_1}) - f(z) \| + \beta_{n_1} \| x_{n_1} - y_{n_1} \| + \gamma_{n_1} \| S u_{n_1} - S v_{n_1} \| \\
&\leq \alpha_{n_1} \| f(x_{n_1}) - f(z) \| + \beta_{n_1} \| x_{n_1} - y_{n_1} \| + \gamma_{n_1} \| u_{n_1} - v_{n_1} \| \\
&\leq \alpha_{n_1} \| f(x_{n_1}) - f(y_{n_1}) \| + \alpha_{n_1} \| f(y_{n_1}) - f(z) \| \\
&\quad + \beta_{n_1} \| x_{n_1} - y_{n_1} \| + \gamma_{n_1} \| x_{n_1} - y_{n_1} \| \\
&= \alpha_{n_1} \| f(x_{n_1}) - f(y_{n_1}) \| + \alpha_{n_1} \| f(y_{n_1}) - f(z) \| \\
&\quad + (1 - \alpha_{n_1}) \| x_{n_1} - y_{n_1} \| \\
&\leq \alpha_{n_1} \max\{\beta \| x_{n_1} - y_{n_1} \|, \varepsilon\} \\
&\quad + \alpha_{n_1} \| f(y_{n_1}) - f(z) \| + (1 - \alpha_{n_1}) \| x_{n_1} - y_{n_1} \| \\
&\leq \max\{\alpha_{n_1}(\varepsilon\beta + \eta) + (1 - \alpha_{n_1})(\varepsilon + \eta), \\
&\qquad\qquad \alpha_{n_1}(\varepsilon + \eta - \beta\eta) + (1 - \alpha_{n_1})(\varepsilon + \eta)\} \\
&\leq \varepsilon + \eta.
\end{aligned}$$

Similarly, we can prove that $\| x_{n_1+2} - y_{n_1+2} \| \leq \varepsilon + \eta$. By induction, we have $\| x_{n_1+m} - y_{n_1+m} \| \leq \varepsilon + \eta$, for all $m \geq 1$, which implies that $\overline{\lim}_n \| x_n - y_n \| \leq \varepsilon + \eta$, which contradicts with (2.20). This contraction shows that $x_n - y_n \to 0$ as $n \to \infty$. Consequently, $x_n \to z$.

Case 2. $\| x_{n_1} - y_{n_1} \| > \varepsilon + \eta$ for all $n \geq n_1$.

We shall prove that the case is impossible. Suppose case 2 hold true. By virtue of (2.21), we have

$$\| f(x_n) - f(y_n) \| \leq \beta \| x_n - y_n \|, \tag{2.25}$$

for all $n \geq n_1$. It follows that

$$\begin{aligned}
\| x_{n+1} - y_{n+1} \| &\leq \alpha_n \| f(x_n) - f(y_n) \| + \alpha_n \| f(y_n) - f(z) \| \\
&\quad + (1 - \alpha_n) \| x_n - y_n \| \\
&\leq (1 - (1 - \beta)\alpha_n) \| x_n - y_n \| + \alpha_n \| y_n - z \|,
\end{aligned}$$

which yields to $x_n - y_n \to 0 (n \to \infty)$. Consequently, $0 \geq \varepsilon + \eta$ is a contradiction. This shows that case 2 is impossible. The proof is completed.

Since every contractive is MKC, we find from Theorem 2.1 the following result immediately.

**Corollary 2.2.** *Let $H_1$ and $H_2$ be two real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator and $f : H_1 \to H_2$ be a contraction. Let $S : H_1 \to H_2$ be a nonexpansive mapping such* *that $Fix(S)\cap\Gamma\neq\varnothing$. For a give $x_0\in H_1$, arbitrarily, let the iterative sequences $\{u_n\}$ and $\{x_n\}$ be generated by*

$$u_n = J_{\lambda}^{B_1}(x_n+\gamma A^*(J_{\lambda}^{B_2}-I)Ax_n),$$

$$x_{n+1} = \alpha_n f(x_n)+\beta_n x_n+\gamma_n S u_n,$$

*where $\lambda>0$ and $\gamma\in(0,\frac{1}{L})$, $L$ is the spectral radius of the operator $A^*A$ and $A^*$ is the adjoint of $A$ and $\{\alpha_n\}$ ,$\{\beta_n\}$, $\{\gamma_n\}$ be real number sequences in $(0,1)$ such that $\alpha_n+\beta_n+\gamma_n=1$ and* $\lim_{n\to\infty}\alpha_n=0$, *$\Sigma_{n=0}^{\infty}\alpha_n=\infty$, and $\Sigma_{n=0}^{\infty}\|\alpha_n-\alpha_{n-1}\|<\infty$,$0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$. Then the sequences $\{u_n\}$ and $\{x_n\}$ both converge strongly to $z\in Fix(S)\cap\Gamma$, where $z=P_{Fix(S)\cap\Gamma}f(z)$.*

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