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# NONLINEAR OPERATORS, EQUILIBRIUM PROBLEMS AND MONOTONE PROJECTION ALGORITHMS

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**Abstract.** The purpose of this paper is to study common fixed points of a countable family of nonlinear operators and solutions of a countable family of equilibrium problems based on a monotone projection algorithm. We establish a strong convergence theorems without any compact assumptions imposed on the operators.

**Keywords.** Equilibrium problem; Nonlinear operator; Monotone projection algorithm; Variational inequality.

**2010 Mathematics Subject Classification.** 47H05, 90C33.

## 1. Introduction

Equilibrium problems involving bifunctions provides us with a unified, natural, innovative and general framework to study a wide class of problems arising in finance, economics, network analysis, transportation, and elasticity. This theory has witnessed an explosive growth in theoretical advances and applications across all disciplines of pure and applied sciences. As a result of this interaction, we have a variety of techniques to study the existence results for the equilibrium problems, which include variational inequalities, saddle point problems and complementary problems as special cases, see [1-8] and the references therein. In this paper, we

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suggest and analyze a monotone projection algorithm for fixed points of nonlinear operators and solutions of equilibrium problems.

## 2. Preliminaries

Let $E$ be a real Banach space with the dual $E^*$. Recall that the normalized duality mapping $J$ from $E$ to $2^{E^*}$ is defined by $Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$, where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. Let $U_E = \{x \in E : \|x\| = 1\}$. The norm of $E$ is said to be Gâteaux differentiable if the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U_E$. In this case, $E$ is said to be smooth. The norm of $E$ is said to be uniformly Gâteaux differentiable if for each $y \in U_E$, the limit is attained uniformly for all $x \in U_E$. The norm of $E$ is said to be Fréchet differentiable if for each $x \in U_E$, the limit is attained uniformly for all $y \in U_E$. The norm of $E$ is said to be uniformly Fréchet differentiable if the limit is attained uniformly for all $x, y \in U_E$. It is well known that (uniform) Fréchet differentiability of the norm of $E$ implies (uniform) Fréchet differentiability of the norm of $E$. Let $\rho_E : [0, \infty) \to [0, \infty)$ be the modulus of smoothness of $E$ by $\rho_E(t) = \sup\{\frac{\|x+y\|-\|x-y\|}{2} - 1 : x \in U_E, \|y\| \leq t\}$. A Banach space $E$ is said to be uniformly smooth if $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$. It is known that $E$ is uniformly smooth if and only if the norm of $E$ is uniformly Fréchet differentiable. The modulus of convexity of $E$ is the function $\delta_E(\varepsilon) : (0, 2] \to [0, 1]$ defined by $\delta_E(\varepsilon) = \inf\{1 - \frac{\|x+t\|}{2} : \|x\| = \|y\| = 1, \|x-y\| \geq \varepsilon\}$. Recall that $E$ is said to be uniformly convex if $\delta_E(\varepsilon) > 0$ for any $\varepsilon \in (0, 2]$. Let $p > 1$. We say that $E$ is $p$-uniformly convex if there exists a constant $c_q > 0$ such that $\delta_E(\varepsilon) \geq c_p \varepsilon^p$ for any $\varepsilon \in (0, 2]$. In what follows, we use $\rightharpoonup$ and $\to$ to stand for the weak and strong convergence, respectively. Recall that $E$ enjoys Kadec-Klee property iff for any sequence $\{x_n\} \subset E$, and $x \in E$ with $x_n \rightharpoonup x$, and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$. It is well known that if $E$ is a uniformly convex Banach spaces, then $E$ enjoys Kadec-Klee property.

Let $E$ be a smooth Banach space. Let us consider the functional defined by $\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$, $\forall x, y \in E$. Observe that, in a Hilbert space $H$, the equality is reduced to $\phi(x, y) = \|x - y\|^2$, $x, y \in H$. As we all know that if $C$ is a nonempty closed convex subset of a Hilbert space $H$ and $P_C : H \to C$ is the metric projection of $H$ onto $C$, then $P_C$ is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [9] recently introduced a generalized projection operator $\Pi_C$ in a Banach space $E$ which is an analogue of the metric projection $P_C$ in Hilbert spaces. Recall that the generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x,y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem $\phi(\bar{x},x) = \min_{y\in C} \phi(y,x)$ existence and uniqueness of the operator $\Pi_C$ follows from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping $J$. If $E$ is a reflexive, strictly convex and smooth Banach space, then $\phi(x,y) = 0$ if and only if $x = y$. In Hilbert spaces, we know that $\Pi_C = P_C$.

Let $C$ be a nonempty subset of $E$ and let $T : C \to C$ be a mapping. In this paper, we use $F(T)$ to stand for the fixed point set of $T$. Recall that $T$ is said to be asymptotically regular on $C$ iff for any bounded subset $K$ of $C$, $\limsup_{n\to\infty}\{\|T^{n+1}x - T^n x\| : x \in K\} = 0$. Recall that $T$ is said to be closed iff for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then $Tx_0 = y_0$. Recall that a point $p$ in $C$ is said to be an asymptotic fixed point of $T$ iff $C$ contains a sequence $\{x_n\}$ which converges weakly to $p$ such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of $T$ will be denoted by $\widetilde{F}(T)$. $T$ is said to be relatively nonexpansive iff $\widetilde{F}(T) = F(T) \neq \emptyset$ and $\phi(p,Tx) \le \phi(p,x), \quad \forall x \in C, \forall p \in F(T)$. $T$ is said to be relatively asymptotically nonexpansive iff $\widetilde{F}(T) = F(T) \neq \emptyset$ and $\phi(p,T^n x) \le (1+\mu_n)\phi(p,x), \quad \forall x \in C$, $\forall p \in F(T), \forall n \ge 1$, where $\{\mu_n\} \subset [0,\infty)$ is a sequence such that $\mu_n \to 0$ as $n \to \infty$.

Recall that $T$ is said to be quasi-$\phi$-nonexpansive iff $F(T) \neq \emptyset$ and $\phi(p,Tx) \le \phi(p,x)$, $\forall x \in C, \forall p \in F(T)$. Recall that $T$ is said to be asymptotically quasi-$\phi$-nonexpansive iff there exists a sequence $\{\mu_n\} \subset [0,\infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that $F(T) \neq \emptyset, \quad \phi(p,T^n x) \le (1+\mu_n)\phi(p,x), \quad \forall x \in C, \forall p \in F(T), \forall n \ge 1$.

**Remark 2.1.** The class of asymptotically quasi-$\phi$-nonexpansive mappings is an extension of the class of quasi-$\phi$-nonexpansive mappings. The class of quasi-$\phi$-nonexpansive mappings and the class of asymptotically quasi-$\phi$-nonexpansive mappings are more general than the class of relatively nonexpansive mappings and the class of relatively asymptotically nonexpansive mappings. Quasi-$\phi$-nonexpansive mappings and asymptotically quasi-$\phi$-nonexpansive do not require the restriction $F(T) = \widetilde{F}(T)$.

Let $F$ be a bifunction from $C\times C$ to $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Recall that the following Ky Fan inequality, which is also known as "equilibrium problem". Find $p\in C$ such that $F(p,y)\geq 0$, $\forall y\in C$. We use $EP(F)$ to denote the solution set of the equilibrium problem. Given a mapping $Q:C\to E^*$, let $F(x,y)=\langle Qx,y-x\rangle$, $\forall x,y\in C$. Then $p\in EP(F)$ if and only if $p$ is a solution of the following variational inequality. Find $p$ such that $\langle Qp,y-p\rangle\geq 0$, $\forall y\in C$. Numerous problems in physics, optimization and economics reduce to find a solution of the equilibrium problem.

**Lemma 2.2** [1] *Let $C$ be a nonempty closed convex subset of a smooth Banach space $E$ and $x\in E$. Then, $x_0=\Pi_C x$ if and only if $\langle x_0-y,Jx-Jx_0\rangle\geq 0$, $\forall y\in C$.*

**Lemma 2.3** [11] *Let $E$ be a smooth and uniformly convex Banach space and let $r>0$. Then there exists a strictly increasing, continuous and convex function $g:[0,2r]\to R$ such that $g(0)=0$ and $\|\sum_{i=1}^{\infty}(\alpha_i x_i)\|^2\leq\sum_{i=1}^{\infty}(\alpha_i\|x_i\|^2)-\alpha_i\alpha_j g(\|x_i-x_j\|)$, $\forall i,j\in\{1,2,\ldots,\}$ for all $x_1,x_2,\ldots,\in B_r=\{x\in E:\|x\|\leq r\}$ and $\alpha_1,\alpha_2,\ldots,\in[0,1]$ such that $\sum_{i=1}^{\infty}\alpha_i=1$.*

**Lemma 2.4** [9] *Let $E$ be a reflexive, strictly convex and smooth Banach space, $C$ a nonempty closed convex subset of $E$ and $x\in E$. Then $\phi(\Pi_C x,x)+\phi(y,\Pi_C x)\leq\phi(y,x)$, $\forall y\in C$.*

**Lemma 2.5.** [10] *Let $E$ be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property and let $C$ be a nonempty closed and convex subset of $E$. Let $T:C\to C$ be a asymptotically quasi-$\phi$-nonexpansive mapping. Then $F(T)$ is convex and closed.*

**Lemma 2.6.** [4] *Let $C$ be a closed convex subset of a smooth, strictly convex and reflexive Banach space $E$. Let $F$ be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying $(A1)$-$(A4)$. Let $r>0$ and $x\in E$. Then there exists $z\in C$ such that $rF(z,y)+\langle y-z,Jz-Jx\rangle\geq 0$, $\forall y\in C$. Define a mapping $T_r:E\to C$ by $S_r x=\{z\in C: rf(z,y)+\langle y-z,Jz-Jx\rangle,\quad\forall y\in C\}$. Then $S_r$ is a single-valued quasi-$\phi$-nonexpansive mapping; $F(S_r)=EP(F)$ is closed and convex; $\phi(q,S_r x)+\phi(S_r x,x)\leq\phi(q,x)$, $\forall q\in F(S_r)$.*

## 3. Main results

Now, we are in a position to show our main results.

**Theorem 3.1.** *Let $E$ be a strictly convex and uniformly smooth Banach space which also has the Kadec-Klee property. Let $C$ be a nonempty closed and convex subset of $E$. Let $T_i : C \to C$ a asymptotically quasi-$\phi$-nonexpansive mapping, which is closed asymptotically regular on $C$ and let $F_i$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)$-$(A4)$ for each $i \geq 1$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E, \text{ chosen arbitrarily}, \\ C_{1,i} = C, C_1 = \cap_{i=1}^{\infty} C_{1,i}, \\ x_1 = \Pi_{C_1} x_0, \\ Jy_n = \sum_{i=1}^{\infty} \alpha_{n,i} J T_i^n x_n + \alpha_{n,0} J x_n, \\ u_{n,i} \in C \text{ such that } \langle y - u_{n,i}, J u_{n,i} - J y_n \rangle + r_{n,i} F_i(u_{n,i}, y) \geq 0, \quad \forall y \in C, \\ C_{n+1,i} = \{ z \in C_n : \phi(z, x_n) + \sum_{i=1}^{\infty} \mu_{n,i} M_n \geq \phi(z, u_{n,i}) \}, \\ C_{n+1} = \cap_{i=1}^{\infty} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases}$$

*where $\{\alpha_{n,i}\}$ is a real number sequence in $(0,1)$ $\{r_{n,i}\}$ is a real number sequence in $[r, \infty)$, where $r$ is some positive real number, and $M_n = \sup\{\phi(z, x_n) : z \in \cap_{i=1}^{\infty} F(T_i) \bigcap \cap_{i=1}^{\infty} EP(F_i)\}$. Assume that $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ and $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ for every $i \geq 1$. If $\mathfrak{S} = \cap_{i=1}^{\infty} F(T_i) \bigcap \cap_{i=1}^{\infty} EP(F_i)$ is nonempty and bounded, then $\{x_n\}$ converges strongly to $\Pi_{\mathfrak{S}} x_0$, where $\Pi_{\mathfrak{S}}$ is the generalized projection from $E$ onto $\mathfrak{S}$.*

**Proof.** First, we show that the sets $C_n$ is convex and closed. It suffices to show, for any fixed but arbitrary $i$ that $C_{n,i}$ is convex and closed. This can be proved by induction. It is clear that $C_{1,j} = C$ is convex and closed. Assume that $C_{m,j}$ is closed, and convex for some $m \geq 1$. We next prove that $C_{m+1,j}$ is convex and closed. It is clear that $C_{m+1,j}$ is closed. We only prove the are convex. Indeed, $\forall x, y \in C_{m+1,j}$, we find that $x, y \in C_{m,j}$,

$$\phi(x, x_m) + \sum_{i=1}^{N} \mu_{n,i} M_n \geq \phi(x, u_{m,j}),$$

and

$$\phi(y, x_m) + \sum_{i=1}^{N} \mu_{n,i} M_n \geq \phi(y, u_{m,j}).$$

Notice that the above two inequalities are equivalent to the following inequalities, respectively.

$$\|x_m\|^2-\|u_{m,j}\|^2+\sum_{i=1}^{N}\mu_{n,i}M_n\geq 2\langle x,Jx_m-Ju_{m,j}\rangle,$$

and

$$\|x_m\|^2-\|u_{m,j}\|^2+\sum_{i=1}^{N}\mu_{n,i}M_n\geq 2\langle y,Jx_m-Ju_{m,j}\rangle.$$

These imply that

$$\|x_m\|^2-\|u_{m,j}\|^2+\sum_{i=1}^{N}\mu_{n,i}M_n\geq 2\langle ax+(1-a)y,Jx_m-Ju_{m,j}\rangle,\quad \forall a\in(0,1).$$

Since $C_{m,j}$ is convex, we see that $ax+(1-a)y\in C_{m,j}$. Notice that the above inequality is equivalent to

$$\phi\big(ax+(1-a)y,x_m\big)+\sum_{i=1}^{N}\mu_{n,i}M_n\geq\phi\big(ax+(1-a)y,u_{m,j}\big).$$

This proves that $C_{m+1,j}$ is convex. This completes that $C_n$ is closed and convex. Note that $\mathfrak{S}\subset C_{1,j}=C$. Suppose that $\mathfrak{S}\subset C_{m,j}$ for some $m$. Then, for $\forall z\in\mathfrak{S}\subset C_{m,j}$, we have

$$\begin{aligned}
\phi(z,u_{m,j}) &\leq \|z\|^2-2\langle z,\alpha_{m,0}Jx_m+\sum_{i=1}^{\infty}\alpha_{m,i}JT_i^m x_m\rangle+\|\alpha_{m,0}Jx_m+\sum_{i=1}^{\infty}\alpha_{m,i}JT_i^m x_m\|^2\\
&\leq \|z\|^2-2\alpha_{m,0}\langle z,Jx_m\rangle-2\sum_{i=1}^{\infty}\alpha_{m,i}\langle z,JT_i^m x_m\rangle\\
&\quad+\alpha_{m,0}\|x_m\|^2+\sum_{i=1}^{\infty}\alpha_{m,i}\|T_i^m x_m\|^2\\
&=\alpha_{m,0}\phi(z,x_m)+\sum_{i=1}^{\infty}\alpha_{m,i}\phi(z,T_i^m x_m)\\
&\leq \alpha_{m,0}\phi(z,x_m)+\sum_{i=1}^{\infty}\alpha_{m,i}\phi(z,x_m)+\sum_{i=1}^{\infty}\alpha_{m,i}\mu_{m,i}\phi(z,x_m)+\sum_{i=1}^{\infty}\alpha_{m,i}\xi_m\\
&\leq \phi(z,x_m)+\sum_{i=1}^{\infty}\mu_{m,i}\phi(z,x_m)+\sum_{i=1}^{\infty}\alpha_{m,i}\xi_m\\
&\leq \phi(z,x_m)+\sum_{i=1}^{\infty}\mu_{m,i}M_m.
\end{aligned}$$

This implies $z\in C_{m+1,j}$.

Next, we show that $x_n\to p\in\mathfrak{S}$. It is clear that $\phi(x_n,x_0)$ is a bounded sequence. This in turn implies that $\{x_n\}$ is also bounded.

Since the framework of the space is uniform, we find that it is also reflexive. Without loss of generality, we assume that $x_n \rightharpoonup p$, where $p \in C_n$. Note that $\phi(p,x_0) \geq \phi(x_n,x_0)$. It follows that

$$\phi(p,x_0) \leq \liminf_{n\to\infty} \phi(x_n,x_0) \leq \limsup_{n\to\infty} \phi(x_n,x_0) \leq \phi(p,x_0).$$

This gives that $\lim_{n\to\infty} \phi(x_n,x_0) = \phi(p,x_0)$. Hence, we have $\lim_{n\to\infty} \|x_n\| = \|p\|$.

Since the space $E$ enjoys Kadec-Klee property, we find that $x_n \to p$ as $n \to \infty$. By the construction of $C_n$, we have that $C_{n+1} \subset C_n$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_n$, we see that

$$\begin{aligned}\phi(x_{n+1},x_0) - \phi(x_n,x_0) &= \phi(x_{n+1},x_0) - \phi(\Pi_{C_n} x_0,x_0) \\ &\geq \phi(x_{n+1},\Pi_{C_n} x_0) \\ &= \phi(x_{n+1},x_n).\end{aligned}$$

Letting $n \to \infty$, we find $\phi(x_{n+1},x_n) \to 0$. Since $x_{n+1} \in C_{n+1}$, one further obtains that

$$\phi(x_{n+1},x_n) + \sum_{i=1}^{\infty} \mu_{n,i} M_n \geq \phi(x_{n+1},u_{n,i}).$$

It follows that $\lim_{n\to\infty} \phi(x_{n+1},u_{n,i}) = 0$. On the other hand, one has $\lim_{n\to\infty} \|u_{n,i}\| = \|p\|$. It follows that $\lim_{n\to\infty} \|Ju_{n,i}\| = \|Jp\|$ This implies that $\{Ju_{n,i}\}$ is bounded. Note that $E$ is reflexive and $E^*$ is also reflexive. We may assume that $Ju_{n,i} \rightharpoonup u^{*,i} \in E^*$. Since $E$ is reflexive, we see that there exists an $u^i \in E$ such that $Ju^j = u^{*,i}$. It follows that $\phi(x_{n+1},u_n) = \|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_n\rangle + \|Ju_n\|^2$. Taking $\liminf_{n\to\infty}$ the both sides of equality above yields that

$$\begin{aligned}0 &\geq \|p\|^2 - 2\langle p, u^{*,i}\rangle + \|u^{*,i}\|^2 \\ &= \|p\|^2 - 2\langle p, Ju^i\rangle + \|Ju^i\|^2 \\ &= \|p\|^2 - 2\langle p, Ju^i\rangle + \|u^i\|^2 \\ &= \phi(p,u^i).\end{aligned}$$

It follows that $p = u^i$, which in turn implies that $Jp = u^{*,i}$. It follows that $Ju_{n,i} \rightharpoonup Jp \in E^*$. Since $E^*$ enjoys the Kadec-Klee property, we obtain that $Ju_{n,i} - Jp \to 0$ *as* $n \to \infty$. Since $J^{-1} : E^* \to E$ is demi-continuous. It follows that $u_{n,i} \rightharpoonup p$. Since $E$ enjoys the Kadec-Klee property, we obtain that $u_{n,i} \to p$ *as* $n \to \infty$. Note that $\|x_n - u_{n,i}\| \leq \|x_n - p\| + \|p - u_{n,i}\|$. This gives that $\lim_{n\to\infty} \|x_n - u_{n,i}\| = 0$. Since $J$ is uniformly norm-to-norm continuous on any bounded sets, we have $\lim_{n\to\infty}\|Jx_n - Ju_{n,i}\| = 0$. Notice that

$$\phi(z,x_n) - \phi(z,u_{n,i}) \le \|x_n - u_{n,i}\|(\|x_n\| + \|u_{n,i}\|) + 2\|z\|\|Jx_n - Ju_{n,i}\|.$$

It follows that $\lim_{n\to\infty}(\phi(z,x_n) - \phi(z,u_{n,i}) = 0$. This further yields that $\phi(z,y_n) \le \phi(z,x_n) + \sum_{i=1}^{\infty}\mu_{n,i}M_n$, where $z \in \mathfrak{S}$. In view of $u_{n,i} = S_{r_{n,i}}y_n$, we find that

$$\phi(u_{n,i},y_n) \le \phi(z,y_n) - \phi(z,S_{r_{n,i}}y_n) \le \phi(z,x_n) - \phi(z,u_{n,i}) + \sum_{i=1}^{\infty}\mu_{n,i}M_n.$$

Hence, we arrive at $\lim_{n\to\infty}\phi(u_{n,i},y_n) = 0$. This further implies that $\|u_{n,i}\| - \|y_n\| \to 0$ as $n \to \infty$. Since $u_{n,i} \to p$ as $n \to \infty$, we arrive at $\lim_{n\to\infty}\|y_n\| = \|p\|$. It follows that $\lim_{n\to\infty}\|Jy_n\| = \|Jp\|$. Since $E^*$ is also reflexive, we may assume that $Jy_n \rightharpoonup y^* \in E^*$. In view of $J(E) = E^*$, we see that there exists $y \in E$ such that $Jy = y^*$. It follows that $\phi(u_{n,i},y_n) = \|Jy_n\|^2 - 2\langle u_{n,i}, Jy_n\rangle + \|u_{n,i}\|^2$. Taking $\liminf_{n\to\infty}$ the both sides of equality above yields that $\phi(p,y) \le 0$. That is, $p = y$, which in turn implies that $y^* = Jp$. It follows that $Jy_n \rightharpoonup Jp \in E^*$. Since $E^*$ enjoys the Kadec-Klee property, we obtain that $Jy_n - Jp \to 0$ as $n \to \infty$. Note that $J^{-1} : E^* \to E$ is demi-continuous. It follows that $y_n \rightharpoonup p$. Since $E$ enjoys the Kadec-Klee property, we obtain that $y_n \to p$ as $n \to \infty$. Since

$$\|u_{n,i} - y_n\| \le \|u_{n,j} - p\| + \|p - y_n\|,$$

we find that that $\lim_{n\to\infty}\|u_{n,i} - y_n\| = 0$. Since $J$ is uniformly norm-to-norm continuous on any bounded sets, we have $\lim\limits_{n\to\infty}\|Ju_{n,i} - Jy_n\| = 0$. From the assumption $r_{n,i} \ge r$, we see that

$$\lim_{n\to\infty}\frac{\|Ju_{n,i} - Jy_n\|}{r_{n,i}} = 0.$$

Notice that

$$\langle y - u_{n,i}, Ju_{n,i} - Jy_n\rangle + r_{n,i}F_j(u_{n,i},y) \ge 0, \quad \forall y \in C.$$

Hence, we have $\|y - u_{n,i}\|\|Ju_{n,i} - Jy_n\| \ge \langle y - u_{n,i}, Ju_{n,i} - Jy_n\rangle \ge r_{n,i}F_i(y,u_{n,j}), \quad \forall y \in C$. Taking the limit as $n \to \infty$, we find that $F_i(y,p) \le 0$, $\forall y \in C$. For $0 < t_i < 1$ and $y \in C$, define $y_{t_i} = t_iy + (1-t_i)p$. It follows that $y_{t,ji} \in C$, which yields that $F_i(y_{t,i},p) \le 0$. It follows from the conditions $(A1)$ and $(A4)$ that $0 = F_i(y_{t,i},y_{t,i}) \le t_iF_i(y_{t,i},y) + (1-t_i)F_i(y_{t,i},p) \le t_iF_j(y_{t,i},y)$. This yields that $F_i(y_{t,i},y) \ge 0$. Letting $t_i \downarrow 0$, we find from the condition $(A3)$ that $F_i(p,y) \ge 0$, $\forall y \in C$. This implies that $p \in EP(F_i)$.

Since $E$ is uniformly smooth, we know that $E^*$ is uniformly convex. It follows that

$$\begin{aligned}
\phi(z,u_{n,i}) &\le \|z\|^2 - 2\langle z, \alpha_{n,0}Jx_n + \sum_{i=1}^{\infty}\alpha_{n,i}JT_i^n x_n\rangle + \|\alpha_{n,0}Jx_n + \sum_{i=1}^{\infty}\alpha_{n,i}JT_i^n x_n\|^2 \\
&\le \|z\|^2 - 2\alpha_{n,0}\langle z, Jx_n\rangle - 2\sum_{i=1}^{\infty}\alpha_{n,i}\langle z, JT_i^n x_n\rangle \\
&\quad + \alpha_{n,0}\|x_n\|^2 + \sum_{i=1}^{\infty}\alpha_{n,i}\|T_i^n x_n\|^2 - \alpha_{n,0}(1-\alpha_{n,i})g(\|Jx_n - JT_i^n x_n\|) \\
&\le \alpha_{n,0}\phi(z,x_m) + \sum_{i=1}^{\infty}\alpha_{n,i}\phi(z,x_m) + \sum_{i=1}^{\infty}\alpha_{n,i}\mu_{n,i}\phi(z,x_n) + \sum_{i=1}^{\infty}\alpha_{n,i}\xi_n \\
&\quad - \alpha_{n,0}(1-\alpha_{n,i})g(\|Jx_n - JT_i^n x_n\|) \\
&\le \phi(z,x_n) + \sum_{i=1}^{\infty}\mu_{n,i}M_n - \alpha_{n,0}(1-\alpha_{n,i})g(\|Jx_n - JT_i^n x_n\|).
\end{aligned}$$

In view of $\liminf_{n\to\infty}\alpha_{n,0}(1-\alpha_{n,i}) > 0$, we have $\lim_{n\to\infty}\|JT_i^n x_n - Jx_n\| = 0$. Since $x_n \to p$ as $n \to \infty$ and $J : E \to E^*$ is demi-continuous, we obtain that $Jx_n \rightharpoonup Jp \in E^*$. On the other hand, we have $\|Jx_n\| \to \|Jp\|$ as $n \to \infty$. Since $E^*$ enjoys the Kadec-Klee property, we see that $\lim_{n\to\infty}\|Jp - Jx_n\| = 0$. Hence, we have $\|Jx_n - JT_i^n x_n\| + \|Jp - Jx_n\| \ge \|JT_i^n x_n - Jp\| \ge 0$. This yields that $\lim_{n\to\infty}\|Jp - JT_i^n x_n\| = 0$. Since $J^{-1} : E^* \to E$ is demi-continuous, one sees that $T_i^n x_n \rightharpoonup p$. On the other hand, we have $\lim_{n\to\infty}\|T_i^n x_n\| = \|p\|$. It follows that $\lim_{n\to\infty}\|T_i^n x_n - p\| = 0$. Since $T$ is asymptotically regular, one sees that $\lim_{n\to\infty}\|p - T_i^{n+1}x_n\| = 0$. That is, $T_i T_i^n x_n - p \to 0$ as $n \to \infty$. It follows from the closedness of $T_i$ that $T_i p = p$ for every $i \le 1$. Since $x_n = \Pi_{C_n}x_0$, we see that $\langle x_n - w, Jx_0 - Jx_n\rangle \ge 0$, $\forall w \in C_n$. In view of $\mathfrak{S} \subset C_n$, we find that $\langle x_n - w, Jx_0 - Jx_n\rangle \ge 0$, $\forall w \in \mathfrak{S}$. Letting $n \to \infty$, we arrive at $\langle p - w, Jx_0 - Jp\rangle \ge 0$, $\forall w \in \mathfrak{S}$. This completes the proof.

In the framework of Hilbert spaces, we have the following result.

**Corollary 3.2.** *Let $E$ be a Hilbert space. Let $C$ be a nonempty closed and convex subset of $E$. Let $T_i : C \to C$ a asymptotically quasi-nonexpansive mapping, which is closed asymptotically regular on $C$ and let $F_i$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)$-$(A4)$ for each $i \ge 1$. Let*

*$\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E, \text{ chosen arbitrarily}, \\ C_{1,i} = C, C_1 = \cap_{i=1}^{\infty} C_{1,i}, \\ x_1 = Proj_{C_1} x_0, \\ y_n = \sum_{i=1}^{\infty} \alpha_{n,i} T_i^n x_n + \alpha_{n,0} x_n, \\ u_{n,i} \in C \text{ such that } \langle y - u_{n,i}, u_{n,i} - y_n \rangle + r_{n,i} F_i(u_{n,i}, y) \geq 0, \quad \forall y \in C, \\ C_{n+1,i} = \{z \in C_n : \|z - x_n\|^2 + \sum_{i=1}^{\infty} \mu_{n,i} M_n \geq \|z - u_{n,i}\|^2\}, \\ C_{n+1} = \cap_{i=1}^{\infty} C_{n+1,i}, \\ x_{n+1} = Proj_{C_{n+1}} x_0, \end{cases}$$

*where $\{\alpha_{n,i}\}$ is a real number sequence in $(0,1)$ $\{r_{n,i}\}$ is a real number sequence in $[r,\infty)$, where $r$ is some positive real number, and $M_n = \sup\{\|z - x_n\|^2 : z \in \cap_{i=1}^{\infty} F(T_i) \bigcap \cap_{i=1}^{\infty} EP(F_i)\}$. Assume that $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ and $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ for every $i \geq 1$. If $\mathfrak{S} = \cap_{i=1}^{\infty} F(T_i) \bigcap \cap_{i=1}^{\infty} EP(F_i)$ is nonempty and bounded, then $\{x_n\}$ converges strongly to $Proj_{\mathfrak{S}} x_0$, where $Proj_{\mathfrak{S}}$ is the metric projection from $E$ onto $\mathfrak{S}$.*

**Remark 3.3.** Theorem 3.1 mainly improves the corresponding results in Takahashi and Zembayashi [3], Qin, Cho and Kang [4], Wu and Wang [7], Kim [8], and Qin, Cho and Kang [10].

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