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ON SOME w-WEIGHTED FRACTIONAL INTEGRAL INEQUALITIES INVOLVING THE SAIGO FRACTIONAL INTEGRAL OPERATORS

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Abstract. Some weighted fractional integral inequalities are obtained using the Saigo fractional integral operators. Fractional q-integral inequalities are also established based on the Saigo fractional q-integral operators.

Keywords. Saigo fractional integral operator; Saigo fractional q-integral operator; q-integral inequality; Integral inequality.

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1. Introduction

By applying the different fractional integral operators, such as, the Riemann-Liouville fractional integral operators, the Hadamard fractional operators, the Saigo fractional integral operators and fractional q-integral operators, many researchers have obtained a lot of fractional integral inequalities and fractional q-integral inequalities; see [1, 2, 3, 4, 5, 6, 7, 8] and the references therein. Recently, Dahmani and Bedjaoui [9] gave the following integral inequality based on the Riemann-Liouville fractional integrals.

Let f and h be two positive and continuous functions on $[a,b]$ such that f is decreasing and h is increasing on $[a,b]$. Then for all $\alpha > 0, \sigma > 0$, $\delta \geq \theta > 0$, we have

$$I_a^\alpha \left[f^{\sigma+\delta}(t) \right]_a^\alpha \left[h^\sigma(t) f^\theta(t) \right] \geq I_a^\alpha \left[h^\sigma(t) f^\delta(t) \right] I_a^\alpha \left[f^{\sigma+\theta}(t) \right], \ a < t \leq b, \tag{1.1}$$

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Dahmani [10] established the following fractional integral inequalities which are generalizations of the inequalities (1), by using the Riemann-Liouville fractional integrals. Let f_i, $i=1,...,n$ and h be positive continuous functions on $[a,b]$ such that f_i, $i=1,...,n$ are decreasing and h is increasing on $[a,b]$. Then

$$\begin{gathered} I_a^{\alpha}\left[f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] I_a^{\alpha}\left[h^{\sigma}(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] \\ \geq I_a^{\alpha}\left[h^{\sigma}(t) f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] I_a^{\alpha}\left[\prod_{i=1}^{n} f_i^{\theta_i}(t)\right],\ a<t\leq b, \end{gathered} \tag{1.2}$$

for all $\sigma>0, \delta\geq\theta_k>0, k\in\{1,...,n\}$. Recently, Chinchane and Pachpatte [11] established some new fractional integral inequalities by using Saigo fractional integral operators. In [12], Chinchane and Pachpatte obtained some new integral inequalities for the Hadamard fractional integral operators. Dahmani and Pachpatte [13] derived certain integral inequalities involving the fractional $q-$integral operators. Motivated by [9, 10], the main aim of this paper is to establish some weighted fractional integral inequalities for (1.1) and (1.2) involving the Saigo fractional integral operators. Also, fractional $q-$integral inequalities are presented using the Saigo fractional $q-$integral operators.

2. Weighted integral inequalities involving the Saigo fractional integral

In this section, we introduce some definitions and properties concerning the Saigo fractional integral. For more details, we refer the reader to [4, 14, 15].

Definition 2.1. A real valued function $f(t)$ is said to be in the space $\mathbb{C}_{\mu}(0,\infty)$, $\mu\in\mathbb{R}$, if there exists a real number $p>\mu$ such that $f(t)=t^p f_1(t)$, where $f_1(t)\in\mathbb{C}(0,\infty)$.

Definition 2.2. Let $\alpha>0, \beta,\eta\in\mathbb{R}$. Then the Saigo fractional integral $I_{0,t}^{\alpha,\beta,\eta}$ of order α for a real valued continuous function $f(t)$ is defined by

$$I_{0,t}^{\alpha,\beta,\eta}[f(t)]=\frac{t^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^t (t-x)^{\alpha-1}\ {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}\right) f(x)\,dx, \tag{2.1}$$

where, the function ${}_2F_1(.)$ in the right-hand side of (2.1) is the Gaussian hypergeometric function defined by

$${}_2F_1(\varepsilon,\varepsilon;\kappa;t)=\sum\frac{(\varepsilon)_n(\varepsilon)_n\ t^n}{(\kappa)_n\ n!},$$

and $(\varepsilon)_n$ is the Pochhammer symbol $(\varepsilon)_n = \varepsilon(\varepsilon+1)...(\varepsilon+n-1)$, $(\varepsilon)_0 = 1$.

The integral operator $I_{0,t}^{\alpha,\beta,\eta}$ includes both the Riemann-Liouville and the Erdelyi-Kober fractional integral operators given by the following relationships

$$I^{\alpha}[f(t)] = I_{0,t}^{\alpha,-\alpha,\eta}[f(t)] = \frac{1}{\Gamma(\alpha)}\int_0^t (t-x)^{\alpha-1} f(x)\,dx,\ \alpha>0,$$

and

$$I^{\alpha,\eta}[f(t)] = I_{0,t}^{\alpha,0,\eta}[f(t)] = \frac{t^{-\alpha-\eta}}{\Gamma(\alpha)}\int_0^t (t-x)^{\alpha-1} x^{\eta} f(x)\,dx,\ \alpha>0,\ \eta\in\mathbb{R},$$

where $\Gamma(\alpha) := \int_0^{\infty} e^{-u}u^{\alpha-1}du$. For $f(t) = t^{\varpi}$ in (2.1), we get the known formula

$$I_{0,t}^{\alpha,\beta,\eta} t^{\varpi} = \frac{\Gamma(\varpi+1)\Gamma(\varpi+1-\beta+\eta)}{\Gamma(\varpi+1-\beta)\Gamma(\varpi+1+\alpha+\eta)} t^{\varpi-\beta},$$

for all $t>0$, $\alpha > \min(\varpi, \varpi-\beta+\eta) > -1$.

First, we present some weighted integral inequalities involving the Saigo fractional integral operators.

Theorem 2.1. *Let f and h be two positive and continuous functions on $[0,\infty)$, such that f is decreasing and g is increasing on $[0,\infty)$, $w:[0,\infty)\to\mathbb{R}^{+}$. Then we have*

$$\begin{aligned} & I_{0,t}^{\alpha,\beta,\eta}\left[w(t)h^{\sigma}(t)f^{\theta}(t)\right] I_{0,t}^{\alpha,\beta,\eta}\left[w(t)f^{\delta}(t)\right] \\ & \geq I_{0,t}^{\alpha,\beta,\eta}\left[w(t)h^{\sigma}(t)f^{\delta}(t)\right] I_{0,t}^{\alpha,\beta,\eta}\left[w(t)f^{\theta}(t)\right], \end{aligned} \tag{2.2}$$

for all $t>0$, $\alpha>\max(0,-\beta)$, $\beta<1$, $\beta-1<\eta<0$, $\sigma>0$, $\delta\geq\theta>0$.

Proof. Let us consider

$$\begin{aligned} F(t,x) &= \frac{t^{-\alpha-\beta}(t-x)^{\alpha-1}}{\Gamma(\alpha)}\, {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}\right),\ x\in(0,t);\ t>0, \\ &= \frac{1}{\Gamma(\alpha)}t^{-\alpha-\beta}(t-x)^{\alpha-1} + \frac{(\alpha+\beta)(-\eta)}{\Gamma(\alpha+1)}t^{-\alpha-\beta-1}(t-x)^{\alpha} \\ &\quad + \frac{(\alpha+\beta)(-\eta)(\alpha+\beta+1)(-\eta+1)}{\Gamma(\alpha+2)}t^{-\alpha-\beta-2}(t-x)^{\alpha+1} + ... \end{aligned}$$

We observe that the continuous function $F(t,x)$ remains positive, for all $x\in(0,t)$, $t>0$ since each term of the above series is positive in view of the conditions stated with Theorem 2.1. Since f and h are two positive and continuous on $[0,\infty)$ such that f is decreasing and h is increasing on $[0,\infty)$, we have

$$\left(h^{\sigma}(y)-h^{\sigma}(x)\right)\left(f^{\delta-\theta}(x)-f^{\delta-\theta}(y)\right)\geq 0,$$

for all $\sigma>0,\delta\geq\theta>0,\ x,y\in(0,t)\,,\ t>0$, which implies that

$$h^{\sigma}(y)f^{\delta-\theta}(x)+f^{\delta-\theta}(y)h^{\sigma}(x)\geq h^{\sigma}(y)f^{\delta-\theta}(y)+h^{\sigma}(x)f^{\delta-\theta}(x)\,. \tag{2.3}$$

Multiplying both sides of (2.3) by $F(t,x)w(x)f^{\theta}(x)\,,\ x\in(0,t)\,,\ t>0$, where $w:[0,\infty)\to\mathbb{R}^{+}$ is positive continuous function and integrating the resulting inequality with respect to x from 0 to t, we get

$$\begin{aligned}&h^{\sigma}(y)I_{0,t}^{\alpha,\beta,\eta}\left[w(t)f^{\delta}(t)\right]+f^{\delta-\theta}(y)I_{0,t}^{\alpha,\beta,\eta}\left[w(t)h^{\sigma}(t)f^{\theta}(t)\right]\\&\geq h^{\sigma}(y)f^{\delta-\theta}(y)I_{0,t}^{\alpha,\beta,\eta}\left[w(t)f^{\theta}(t)\right]+I_{0,t}^{\alpha,\beta,\eta}\left[w(t)h^{\sigma}(t)f^{\delta}(t)\right].\end{aligned} \tag{2.4}$$

Next, on multiplying both sides of (2.4) by $F(t,y)w(y)f^{\theta}(y)\,,\ y\in(0,t)\,,\ t>0$, and integrating the resulting inequality with respect to y over $(0,t)$, we can write

$$\begin{aligned}&I_{0,t}^{\alpha,\beta,\eta}\left[w(t)h^{\sigma}(t)f^{\theta}(t)\right]I_{0,t}^{\alpha,\beta,\eta}\left[w(t)f^{\delta}(t)\right]+I_{0,t}^{\alpha,\beta,\eta}\left[w(t)f^{\delta}(t)\right]I_{0,t}^{\alpha,\beta,\eta}\left[w(t)h^{\sigma}f^{\theta}(t)\right]\\&\geq I_{0,t}^{\alpha,\beta,\eta}\left[w(t)h^{\sigma}(t)f^{\delta}(t)\right]I_{0,t}^{\alpha,\beta,\eta}\left[w(t)f^{\theta}(t)\right]+I_{0,t}^{\alpha,\beta,\eta}\left[w(t)h^{\sigma}f^{\delta}(t)\right]I_{0,t}^{\alpha,\beta,\eta}\left[w(t)f^{\theta}(t)\right]\end{aligned}$$

which implies (2.2).

Theorem 2.2. *Suppose that f and h are two positive and continuous functions on $[0,\infty)$, such that f is decreasing and h is increasing on $[0,\infty)$, $w:[0,\infty)\to\mathbb{R}^{+}$. Then we have*

$$\begin{aligned}&I_{0,t}^{\omega,\lambda,\gamma}\left[w(t)h^{\sigma}(t)f^{\theta}(t)\right]I_{0,t}^{\alpha,\beta,\eta}\left[w(t)f^{\delta}(t)\right]\\&\quad+I_{0,t}^{\omega,\lambda,\gamma}\left[w(t)f^{\delta}(t)\right]I_{0,t}^{\alpha,\beta,\eta}\left[w(t)h^{\sigma}(t)f^{\theta}(t)\right]\\&\geq I_{0,t}^{\omega,\lambda,\gamma}\left[w(t)h^{\sigma}(t)f^{\delta}(t)\right]I_{0,t}^{\alpha,\beta,\eta}\left[w(t)f^{\theta}(t)\right]\\&\quad+I_{0,t}^{\alpha,\beta,\eta}\left[w(t)h^{\sigma}(t)f^{\delta}(t)\right]I_{0,t}^{\omega,\lambda,\gamma}\left[w(t)f^{\theta}(t)\right],\end{aligned} \tag{2.5}$$

for all $t>0,\alpha>\max(0,-\beta)\,,\omega>\max(0,-\lambda)\,,\beta,\lambda<1,\ \beta-1<\eta<0,\lambda-1<\gamma<0$, $\delta\geq\theta>0,\sigma>0$.

Proof. Multiplying both sides of (2.4) by the quantity $G(t,y)w(y)f^{\theta}(y)$, where

$$G(t,y)=\frac{t^{-\omega-\lambda}(t-y)^{\omega-1}}{\Gamma(\omega)}\,{}_2F_1\left(\omega+\lambda,-\gamma;\omega;1-\frac{y}{t}\right),\ y\in(0,t)\,,t>0,$$

and using the arguments mentioned above in the proof of Theorem 2.1, we see that the function $G(t,y)$ remains positive under the conditions stated with Theorem 2.2. Integrating the resulting inequality obtained with respect to y from 0 to t, we obtain

$$\begin{aligned}
&I_{0,t}^{\omega,\lambda,\gamma}\left[w(t)h^{\sigma}(t)f^{\theta}(t)\right]I_{0,t}^{\alpha,\beta,\eta}\left[w(t)f^{\delta}(t)\right]+I_{0,t}^{\omega,\lambda,\gamma}\left[w(t)f^{\delta}(t)\right]I_{0,t}^{\alpha,\beta,\eta}\left[w(t)h^{\sigma}(t)f^{\theta}(t)\right]\\
&\geq I_{0,t}^{\omega,\lambda,\gamma}\left[w(t)h^{\sigma}(t)f^{\delta}(t)\right]I_{0,t}^{\alpha,\beta,\eta}\left[w(t)f^{\theta}(t)\right]\\
&\quad+I_{0,t}^{\alpha,\beta,\eta}\left[w(t)h^{\sigma}(t)f^{\delta}(t)\right]I_{0,t}^{\omega,\lambda,\gamma}\left[w(t)f^{\theta}(t)\right].
\end{aligned}$$

Hence, we have (2.5).

Remark 2.3. If $\alpha=\omega$, $\beta=\lambda$ and $\eta=\gamma$ in Theorem 2.1, then we obtain Theorem 2.2.

Next, we generalize the previous theorems by using a family of n positive functions defined on $[0,\infty)$.

Theorem 2.4. *Let f_i, $i=1,...,n$ and h be positive continuous functions on $[0,\infty)$, such that h is increasing and f_i, $i=1,...,n$ are decreasing on $[0,\infty)$, $w:[0,\infty)\to\mathbb{R}^{+}$. Then, the following inequality*

$$\begin{aligned}
&I_{0,t}^{\alpha,\beta,\eta}\left[w(t)f_k^{\delta}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]I_{0,t}^{\alpha,\beta,\eta}\left[w(t)h^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]\\
&\geq I_{0,t}^{\alpha,\beta,\eta}\left[w(t)h^{\sigma}(t)f_k^{\delta}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]I_{0,t}^{\alpha,\beta,\eta}\left[w(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]
\end{aligned}\tag{2.6}$$

holds for any $t>0$, $\alpha>\max(0,-\beta)$, $\beta<1$, $\beta-1<\eta<0$, $\sigma>0$, $\delta\geq\theta_k>0$, $k\in\{1,...,n\}$.

Proof. Letting $x,y\in(0,t)$, $t>0$, we have

$$h^{\sigma}(y)f_k^{\delta-\theta_k}(x)+f_k^{\delta-\theta_k}(y)h^{\sigma}(x)\geq h^{\sigma}(y)f_k^{\delta-\theta_k}(y)+h^{\sigma}(x)f_k^{\delta-\theta_k}(x),\tag{2.7}$$

for any $\sigma>0$, $\delta\geq\theta_k>0$, $k\in\{1,2,...,n\}$. Multiplying both sides of (2.7) by

$$F(t,x)w(x)\prod_{i=1}^{n}f_i^{\theta_i}(x),\ x\in(0,t),t>0,$$

and integrating the resulting inequality with respect to x from 0 to t, we obtain

$$\begin{aligned}
&h^{\sigma}(y)\, I_{0,t}^{\alpha,\beta,\eta}\left[w(t)\, f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] + f_k^{\delta-\theta_k}(y)\, I_{0,t}^{\alpha,\beta,\eta}\left[w(t)\, h^{\sigma}(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right]\\
&\geq h^{\sigma}(y)\, f_k^{\delta-\theta_k}(y)\, I_{0,t}^{\alpha,\beta,\eta}\left[w(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] \qquad (2.8)\\
&\quad + I_{0,t}^{\alpha,\beta,\eta}\left[w(t)\, h^{\sigma}(t)\, f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right].
\end{aligned}$$

Now, Multiplying both sides of (2.8) by $F(t,y)\, w(y)\prod_{i=1}^{n} f_i^{\theta_i}(y)\,,\, y\in(0,t)\,,\, t>0$, and integrating the resulting inequality with respect to y from 0 to t, we have

$$\begin{aligned}
&I_{0,t}^{\alpha,\beta,\eta}\left[w(t)\, h^{\sigma}(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] I_{0,t}^{\alpha,\beta,\eta}\left[w(t)\, f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right]\\
&\quad + I_{0,t}^{\alpha,\beta,\eta}\left[w(t)\, f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] I_{0,t}^{\alpha,\beta,\eta}\left[w(t)\, h^{\sigma}(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right]\\
&\geq I_{0,t}^{\alpha,\beta,\eta}\left[w(t)\, h^{\sigma} f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] I_{0,t}^{\alpha,\beta,\eta}\left[w(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right]\\
&\quad + I_{0,t}^{\alpha,\beta,\eta}\left[w(t)\, h^{\sigma}(t)\, f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] I_{0,t}^{\alpha,\beta,\eta}\left[w(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right],
\end{aligned}$$

which implies (2.6). This completes proof.

Theorem 2.5. *Let f_i, $i=1,\ldots,n$ and h be positive continuous functions on $[0,\infty)$, such that h is increasing and f_i, $i=1,\ldots,n$ are decreasing on $[0,\infty)\,,\, w:[0,\infty)\to\mathbb{R}^{+}$. Then, for all $t>0$, we have*

$$\begin{aligned}
&I_{0,t}^{\omega,\lambda,\gamma}\left[w(t)\, h^{\sigma}(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] I_{0,t}^{\alpha,\beta,\eta}\left[w(t)\, f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right]\\
&\quad + I_{0,t}^{\alpha,\beta,\eta}\left[w(t)\, h^{\sigma}(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] I_{0,t}^{\omega,\lambda,\gamma}\left[w(t)\, f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right]\\
&\geq I_{0,t}^{\omega,\lambda,\gamma}\left[w(t)\, h^{\sigma}(t)\, f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] I_{0,t}^{\alpha,\beta,\eta}\left[w(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] \qquad (2.9)\\
&\quad + I_{0,t}^{\alpha,\beta,\eta}\left[w(t)\, h^{\sigma}(t)\, f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] I_{0,t}^{\omega,\lambda,\gamma}\left[w(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right],
\end{aligned}$$

with $\alpha>\max(0,-\beta)\,,\,\omega>\max(0,-\lambda)\,,\,\beta,\lambda<1,\ \beta-1<\eta<0, \lambda-1<\gamma<0,\ \delta\geq\theta>0, \sigma>0, \sigma>0, \delta\geq\theta_k>0,\ k\in\{1,\ldots,n\}$.

Proof. Multiplying both sides of (2.8) by $G(t,y)\,w(y)\prod_{i=1}^{n} f_i^{\theta_i}(y)$, $y \in (0,t)$, $t > 0$, and integrating the resulting inequality with respect to y from 0 to t, we obtain

$$\begin{aligned}
& I_{0,t}^{\omega,\lambda,\gamma}\left[w(t)\,h^{\sigma}(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] I_{0,t}^{\alpha,\beta,\eta}\left[w(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] \\
& +I_{0,t}^{\omega,\lambda,\gamma}\left[w(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] I_{0,t}^{\alpha,\beta,\eta}\left[w(t)\,h^{\sigma}(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] \\
\geq\ & I_{0,t}^{\omega,\lambda,\gamma}\left[w(t)\,h^{\sigma}(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] I_{0,t}^{\alpha,\beta,\eta}\left[w(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] \\
& +I_{0,t}^{\omega,\lambda,\gamma}\left[w(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] I_{0,t}^{\alpha,\beta,\eta}\left[w(t)\,h^{\sigma}(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right].
\end{aligned}$$

This ends the proof of Theorem 2.5.

Remark 2.6. Applying Theorem 2.5 for $\alpha = \omega$, $\beta = \lambda$ and $\eta = \gamma$, we obtain Theorem 2.4 immediately.

3. $q-$Integral inequalities involving the Saigo fractional $q-$integral

We give some necessary definitions and mathematical preliminaries of fractional $q-$calculus. More details, one can consult [16, 17, 18, 19].

For any complex number $\alpha \in \mathbb{C}$, we define$[\alpha]_q = \frac{1-q^{\alpha}}{1-q}, q \neq 1; [n]_q! = [n]_q [n-1]_q \dots [2]_q [1]_q$, $n \in \mathbb{N}$ and $\left([\vartheta]_q\right)_n = [\vartheta]_q [\vartheta+1]_q \dots [\vartheta+n-1]_q$, $n \in \mathbb{N}$ with $[0]_q! = 1$ and the $q-$shifted factorial is defined for as a product of n factors by

$$(\alpha;q)_n = 1, n = 0;\ \ (\alpha;q)_n = (1-\alpha)(1-\alpha q)\dots\left(1-\alpha q^{n-1}\right),\ n \in \mathbb{N}, \tag{3.1}$$

and in terms of the basic analogue of the gamma function

$$(q^{\alpha};q)_n = \frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)},\ n > 0,$$

where the $q-$gamma function is defined by

$$\Gamma_q(z) = \frac{(q;q)_{\infty}(1-q)^{1-z}}{(q^z;q)_{\infty}},\ 0 < q < 1.$$

We note that

$$\Gamma_q(1+z) = \frac{(1-q)^z\,\Gamma_q(z)}{1-q},$$

and if $|q|<1$, the definition (3.1) remains meaningful for $n=\infty$, as a convergent infinite product given by$(\alpha;q)_{\infty}=\prod_{i=0}^{\infty}\left(1-\alpha q^{i}\right)$. Also, the $q-$binomial expansion is given by

$$(\tau-\rho)_{\upsilon}=\tau^{\upsilon}\left(\frac{-\rho}{\tau};q\right)_{\upsilon}=\tau^{\upsilon}\prod_{i=0}^{\infty}\left(\frac{1-\left(\frac{\rho}{\tau}\right)q^{i}}{1-\left(\frac{\rho}{\tau}\right)q^{\upsilon+i}}\right).$$

Letting $t_0\in\mathbb{R}$, we define a specific time scale

$$T_{t_0}=\{t;t=t_0q^{n},\ n\in\mathbb{N}\}\cup\{0\},\ 0<q<1.$$

The Jackson's $q-$derivative and $q-$integral of a function f defined on T_{t_0} are, respectively, given by

$$D_{q,t}\left[f(t)\right]=\frac{f(t)-f(qt)}{t(1-q)},\ t\neq0,\ q\neq1,$$

and

$$\int_0^t f(x)\,d_qx=t(1-q)\sum_{j=0}^{\infty}q^{j}f\left(tq^{j}\right).$$

Definition 3.1. The Riemann-Liouville fractional $q-$integral operator of a function $f(t)$ of order α is given by

$$I_q^{\alpha}\left[f(t)\right]=\frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^t\left(\frac{qx}{t};q\right)_{\alpha-1}f(x)\,d_qx,\ \alpha>0,\ 0<q<1,$$

where

$$(a;q)_{\alpha}=\frac{(a;q)_{\infty}}{(aq^{\alpha};q)_{\infty}},\ \alpha\in\mathbb{R}.$$

Definition 3.2. For $\alpha>0$ and $\eta>0$, the basic analogue of the Kober fractional integral operator is given by

$$I_q^{\alpha,\eta}\left[f(t)\right]=\frac{t^{-\eta-1}}{\Gamma_q(\alpha)}\int_0^t\left(\frac{qx}{t};q\right)_{\alpha-1}x^{\eta}f(x)\,d_qx,\ 0<q<1.$$

Definition 3.3. For $\alpha>0,\beta\in\mathbb{R}$ a basic analogue of the Saigo's fractional integral operator is given for $\left|\frac{x}{t}\right|<1$ by

$$I_q^{\alpha,\beta,\eta}\left[f(t)\right]=\frac{t^{-\beta-1}q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)}\int_0^t\left(\frac{qx}{t};q\right)_{\alpha-1}\mathscr{F}_{q,\frac{q^{\alpha+1}x}{t}}\left({}_2\Omega_1\left[q^{\alpha+\beta},q^{-\eta};q^{\alpha};q,q\right]\right)f(x)\,d_qx,\tag{3.2}$$

where η is any non-negative integer, and the function ${}_2\Omega_1(.)$ and the $q-$translation operator occurring in theright-hand side of (3.2) are, respectively, defined by

$$\left({}_2\Omega_1\left[a,b;c;q,t\right]\right)=\sum_{n=0}^{\infty}\frac{(a;q)_n(b,q)_n}{(c;q)_n(q,q)_n}t^{n},\ \ |q|<1,|t|<1,$$

and

$$\mathscr{F}_{q,x}(f(t)) = \sum_{-\infty}^{\infty} A_n t^n \left(\frac{x}{t};q\right)_n,$$

where $(A_n)_{n\in\mathbb{Z}}$ $(\mathbb{Z} = 0, \pm 1, \pm 2, ...)$ is any bounded sequence of real or complex numbers. For $f(t) = t^{\varpi}$, we get the known formula

$$I_q^{\alpha,\beta,\eta}\left[t^{\varpi}\right] = \frac{\Gamma_q(\varpi+1)\Gamma_q(\varpi+1-\beta+\eta)}{\Gamma_q(\varpi+1-\beta)\Gamma_q(\varpi+1+\alpha+\eta)} t^{\varpi-\beta},$$

for all $t > 0$, $\min(\varpi, \varpi-\beta+\eta) > -1$, $0 < q < 1$.

Firstly, we prove some $q-$integral inequalities concerning the Saigo fractional $q-$integral operators.

Theorem 3.4. *Let f and h be two positive and continuous functions on T_{t_0}, such that f is decreasing and h is increasing on T_{t_0}. Then, for all $t > 0$, we have*

$$I_q^{\alpha,\beta,\eta}\left[h^{\sigma}(t) f^{\theta}(t)\right] I_q^{\alpha,\beta,\eta}\left[f^{\delta}(t)\right] \geq I_q^{\alpha,\beta,\eta}\left[h^{\sigma}(t) f^{\delta}(t)\right] I_q^{\alpha,\beta,\eta}\left[f^{\theta}(t)\right], \tag{3.3}$$

where $1 < q < 1$, $\alpha > \max(0, -\beta)$, $\beta < 1$, $\eta - \beta > -1$, $\sigma > 0$, $\delta \geq \theta > 0$.

Proof. Consider

$$F^*(t,x) = \frac{t^{-\beta-1} q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)} (qx/t;q)_{\alpha-1} \Phi_{q,\frac{q^{\alpha+1}x}{t}}\left({}_2\Omega_1\left[q^{\alpha+\beta}, q^{-\eta}; q^{\alpha}; q, q\right]\right),$$

for $x \in (0,t)$, $t > 0$. We note that the function $F^*(t,x)$ remains positive for all values of $x \in (0,t)$, $t > 0$, and under the conditions imposed with Theorem 3.4. Since f and h are two positive and continuous on T_{t_0}, such that f is decreasing and g is increasing on T_{t_0}, for $x \in (0,t)$, $t > 0$, one sees that the inequality (2.3) is satisfied. Now on multiplying both sides of (2.3) by $F^*(t,x) f^{\theta}(x)$, and taking $q-$integration with respect to x from 0 to t, we get

$$\begin{aligned} & h^{\sigma}(y) I_q^{\alpha,\beta,\eta}\left[f^{\delta}(t)\right] + f^{\delta-\theta}(y) I_q^{\alpha,\beta,\eta}\left[h^{\sigma}(t) f^{\theta}(t)\right] \\ & \geq h^{\sigma}(y) f^{\delta-\theta}(y) I_q^{\alpha,\beta,\eta}\left[f^{\theta}(t)\right] + I_q^{\alpha,\beta,\eta}\left[h^{\sigma}(t) f^{\delta}(t)\right]. \end{aligned} \tag{3.4}$$

Next, multiplying both sides of (3.4) by $F^*(t,y) f^{\theta}(y)$, and noting that the function $F^*(t,y)$ is also positive for all $y \in (0,t)$, $t > 0$ and under the conditions imposed with Theorem 3.1, and taking $q-$integration with respect to y from 0 to t, we obtain

$$\begin{aligned} & I_q^{\alpha,\beta,\eta}\left[h^{\sigma}(t) f^{\theta}(t)\right] I_q^{\alpha,\beta,\eta}\left[f^{\delta}(t)\right] + I_q^{\alpha,\beta,\eta}\left[f^{\delta}(t)\right] I_q^{\alpha,\beta,\eta}\left[h^{\sigma}(t) f^{\theta}(t)\right] \\ \geq\ & I_q^{\alpha,\beta,\eta}\left[h^{\sigma}(t) f^{\delta}(t)\right] I_q^{\alpha,\beta,\eta}\left[f^{\theta}(t)\right] + I_q^{\alpha,\beta,\eta}\left[h^{\sigma}(t) f^{\delta}(t)\right] I_q^{\alpha,\beta,\eta}\left[f^{\theta}(t)\right]. \end{aligned}$$

The proof is done.

Theorem 3.5. *Suppose that f and h are two positive and continuous functions on T_{t_0}, such that f is decreasing and h is increasing on T_{t_0}. Then we have*

$$\begin{aligned} & I_q^{\omega,\lambda,\gamma}\left[h^{\sigma}(t) f^{\theta}(t)\right] I_q^{\alpha,\beta,\eta}\left[f^{\delta}(t)\right]+I_q^{\alpha,\beta,\eta}\left[h^{\sigma}(t) f^{\theta}(t)\right] I_q^{\omega,\lambda,\gamma}\left[f^{\delta}(t)\right] \\ & \geq I_q^{\alpha,\beta,\eta}\left[h^{\sigma}(t) f^{\delta}(t)\right] I_q^{\omega,\lambda,\gamma}\left[f^{\theta}(t)\right]+I_q^{\omega,\lambda,\gamma}\left[h^{\sigma}(t) f^{\delta}(t)\right] I_q^{\alpha,\beta,\eta}\left[f^{\theta}(t)\right], \end{aligned} \tag{3.5}$$

for all $t>0, 0<q<1, \alpha>\max(0,-\beta), \omega>\max(0,-\lambda), \beta, \lambda<1$, $\eta-\beta, \gamma-\lambda>-1$, $\delta \geq \theta>0, \sigma>0$.

Proof. Multiplying both sides of (2.3) by $G^{*}(t,y) f^{\theta}(y)$, where

$$G^{*}(t,y)=\frac{t^{-\lambda-1} q^{-\gamma(\omega+\lambda)}}{\Gamma_q(\omega)}\left(\frac{qy}{t};q\right)_{\omega-1} \Phi_{q,\frac{q^{\omega+1}y}{t}}\left({}_2\Omega_1\left[q^{\omega+\lambda}, q^{-\gamma}; q^{\omega}; q, q\right]\right),$$

for $y \in(0,t)$, $t>0$, we can see that the function $G^{*}(t,y)$ remains positive under the conditions stated with Theorem 3.4. Integrating the resulting inequality obtained with respect to y from 0 to t, we have

$$\begin{aligned} & f^{\delta-\theta}(x) I_q^{\omega,\lambda,\gamma}\left[h^{\sigma}(t) f^{\theta}(t)\right]+h^{\sigma}(x) I_q^{\omega,\lambda,\gamma}\left[f^{\delta}(t)\right] \\ & \geq I_q^{\omega,\lambda,\gamma}\left[h^{\sigma}(t) f^{\delta}(t)\right]+h^{\sigma}(x) f^{\delta}(x) I_q^{\omega,\lambda,\gamma}\left[f^{\theta}(t)\right]. \end{aligned} \tag{3.6}$$

Multiplying both sides of (3.6) by $F^{*}(t,x) f^{\theta}(x)$, and integrating the resulting inequality with respect to x from 0 to t, we obtain

$$\begin{aligned} & I_q^{\omega,\lambda,\gamma}\left[h^{\sigma}(t) f^{\theta}(t)\right] I_q^{\alpha,\beta,\eta}\left[f^{\delta}(t)\right]+I_q^{\omega,\lambda,\gamma}\left[f^{\delta}(t)\right] I_q^{\alpha,\beta,\eta}\left[h^{\sigma}(t) f^{\theta}(t)\right] \\ & \geq I_q^{\omega,\lambda,\gamma}\left[h^{\sigma}(t) f^{\delta}(t)\right] I_q^{\alpha,\beta,\eta}\left[f^{\theta}(t)\right]+I_q^{\omega,\lambda,\gamma}\left[f^{\theta}(t)\right] I_q^{\alpha,\beta,\eta}\left[h^{\sigma}(t) f^{\delta}(t)\right]. \end{aligned}$$

This ends proof of Theorem 3.5.

Remark 3.6. For $\alpha=\omega$, $\beta=\lambda$ and $\eta=\gamma$, Theorem 3.5 immediately reduces to Theorem 3.4.

Next, by using the Saigo fractional $q-$integral, we generate new class of the Saigo fractional $q-$integral inequalities involving a family of n positive functions defined on T_{t_0}.

Theorem 3.6. *Let f_i, $i=1,...,n$ and h be positive continuous functions on T_{t_0}, such that h is increasing and f_i, $i=1,...,n$ are decreasing on T_{t_0}. Then, for all $t>0, 0<q<1$, we have*

$$\begin{aligned} &I_q^{\alpha,\beta,\eta}\left[f_k^{\delta}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]I_q^{\alpha,\beta,\eta}\left[h^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]\\ &\geq I_q^{\alpha,\beta,\eta}\left[h^{\sigma}(t)f_k^{\delta}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]I_q^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n}f_i^{\theta_i}(t)\right], \end{aligned} \tag{3.7}$$

where $\alpha>\max(0,-\beta), \beta<1, \eta-\beta>-1, \sigma>0, \delta\geq\theta_k>0, k\in\{1,...,n\}$.

Proof. Multiplying both sides of (2.7) by $F^*(t,x)\prod_{i=1}^{n}f_i^{\theta_i}(x)$, $x\in(0,t)$, $t>0$, and integrating the resulting inequality with respect to x over $(0,t)$, we obtain

$$\begin{aligned} &h^{\sigma}(y)I_q^{\alpha,\beta,\eta}\left[f_k^{\delta}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]+f_k^{\delta-\theta_k}(y)I_q^{\alpha,\beta,\eta}\left[h^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]\\ &\geq h^{\sigma}(y)f_k^{\delta-\theta_k}(y)I_q^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]+I_q^{\alpha,\beta,\eta}\left[h^{\sigma}(t)f_k^{\delta}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]. \end{aligned} \tag{3.8}$$

Now, multiplying both sides of (3.8) by $F^*(t,y)\prod_{i=1}^{n}f_i^{\theta_i}(y)$, $y\in(0,t)$, $t>0$, and integrating the resulting inequality with respect to y from 0 to t, we have

$$\begin{aligned} &2I_q^{\alpha,\beta,\eta}\left[f_k^{\delta}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]I_q^{\alpha,\beta,\eta}\left[h^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]\\ &\geq 2I_q^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]I_q^{\alpha,\beta,\eta}\left[h^{\sigma}(t)f_k^{\delta}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]. \end{aligned}$$

This completes proof of Theorem 3.6.

Theorem 3.7. *Let f_i, $i=1,...,n$ and h be positive continuous functions on T_{t_0}, such that h is increasing and f_i, $i=1,...,n$ are decreasing on T_{t_0}. Then, for all $t>0, 0<q<1$, we have*

$$\begin{aligned} &I_q^{\alpha,\beta,\eta}\left[f_k^{\delta}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]I_q^{\omega,\lambda,\gamma}\left[h^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]\\ &+I_q^{\omega,\lambda,\gamma}\left[f_k^{\delta}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]I_q^{\alpha,\beta,\eta}\left[h^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]\\ &\geq I_q^{\omega,\lambda,\gamma}\left[h^{\sigma}(t)f_k^{\delta}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]I_q^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]\\ &+I_q^{\alpha,\beta,\eta}\left[h^{\sigma}(t)f_k^{\delta}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]I_q^{\omega,\lambda,\gamma}\left[\prod_{i=1}^{n}f_i^{\theta_i}(t)\right], \end{aligned} \tag{3.9}$$

where $\alpha > \max(0,-\beta), \omega > \max(0,-\lambda), \beta, \lambda < 1, \eta-\beta, \gamma-\lambda > -1, \delta \geq \theta > 0, \sigma > 0, \sigma > 0, \delta \geq \theta_k > 0, k \in \{1,...,n\}$.

Proof. Multiplying both sides of (3.8) by $G^*(t,y)\prod_{i=1}^{n} f_i^{\theta_i}(y), y \in (0,t), t > 0$, and integrating with respect to y over $(0,t)$, we obtain

$$\begin{aligned}
& I_q^{\alpha,\beta,\eta}\left[f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] I_q^{\omega,\lambda,\gamma}\left[h^{\sigma}(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] \\
&+ I_q^{\alpha,\beta,\eta}\left[h^{\sigma}(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] I_q^{\omega,\lambda,\gamma}\left[f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] \\
\geq\, & I_q^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] I_q^{\omega,\lambda,\gamma}\left[h^{\sigma}(t) f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] \\
&+ I_q^{\alpha,\beta,\eta}\left[h^{\sigma}(t) f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] I_q^{\omega,\lambda,\gamma}\left[\prod_{i=1}^{n} f_i^{\theta_i}(t)\right].
\end{aligned}$$

Theorem 3.7 is thus proved.

Remark 3.8. Putting $\alpha = \omega$, $\beta = \lambda$ and $\eta = \gamma$ in Theorem 3.7, we obtain Theorem 3.6 immediately.

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