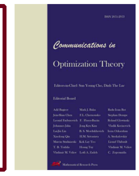
Commun. Optim. Theory 2017 (2017), Article ID 32 https://doi.org/10.23952/cot.2017.32

MATHRES

Communications in Optimization Theory

Available online at http://cot.mathres.org



DHAGE ITERATION METHOD IN THE THEORY OF ORDINARY NONLINEAR PBVPS OF FIRST ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

BAPURAO C. DHAGE

Kasubai, Gurukul Colony, Ahmepur-413515, Dist. Latur, Maharashtra, India

Abstract. In this paper, we prove the existence and uniqueness results for approximate solutions of a nonlinear periodic boundary value problem of first order nonlinear functional differential equations via construction of an algorithm. The main results rely on the Dhage iteration method embodied in a recent hybrid fixed point principle of Dhage (2014) in a partially ordered normed linear space. Examples are also furnished to illustrate the hypotheses and the abstract results of this paper.

Keywords. Periodic boundary value problem; Functional differential equation; Dhage iteration method; Hybrid fixed point principle.

2010 Mathematics Subject Classification. 34A12, 34A45, 47H07.

1. Introduction and Preliminaries

Given the real numbers $r > 0$ and $T > 0$, consider the closed and bounded intervals $I_0 = [-r, 0]$ and $I = [0, T]$ in $\mathbb{R}$ and let $J = [-r, T]$. By $\mathscr{C} = C(I_0, \mathbb{R})$ we denote the space of continuous real-valued functions defined on I_0. We equip the space $\mathscr{C}$ with the norm

$$\|x\|_{\mathscr{C}} = \sup_{-r\leq\theta\leq 0} |x(\theta)|. \tag{1.1}$$

Clearly, $\mathscr{C}$ is a Banach space with this supremum norm and it is called the history space of the functional differential equation in question. For any continuous function $x : J \to \mathbb{R}$ and for

E-mail address: bcdhage@gmail.com.

Received May 29, 2017; Accepted October 9, 2017.

any $t \in I$, we denote by x_t the element of the space $\mathscr{C}$ defined by

$$x_t(\theta) = x(t+\theta), \ -r \le \theta \le 0. \tag{1.2}$$

The differential equations involving the history of the dynamic systems are called functional differential equations and it has been recognized long back the importance of such problems in the theory of differential equations. Since then, several classes of nonlinear functional differential equations have been discussed in the literature for different qualitative properties of the solutions. A special class of functional differential equations has been discussed in Dhage [1, 2] and Dhage and Dhage [3, 4] for the existence and approximation of solutions via the new Dhage iteration method. Therefore, it is desirable to extend this method to other functional differential equations involving delay. In this paper, we consider the periodic boundary value problem (in short PBVP) of nonlinear first order functional differential equations

$$\left.\begin{aligned} x'(t) + \lambda x(t) = f(t, x_t), \ t \in I, \\ x(0) = \phi(0) = x(T), \\ x_0 = \phi, \end{aligned}\right\} \tag{1.3}$$

for some $\lambda > 0$, where $\phi \in \mathscr{C}$ and $f : I \times \mathscr{C} \to \mathbb{R}$ is a continuous function.

Definition 1.1. A function $x \in C(J, \mathbb{R})$ is said to be a *solution* of the PBVP (1.3) on J if

(i) $x_0 = \phi$,
(ii) $x_t \in \mathscr{C}$ for each $t \in I$, and
(iii) x is continuously differentiable on I and satisfies the equations in (1.3),

where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on J.

The PBVP (1.3) is well-known and extensively discussed in the literature for different aspects of the solutions; see Hale [5] and the references therein. There is a vast literature on nonlinear functional differential equations for different aspects of the solutions via different approaches and methods. The method of the upper and lower solution or the monotone method is interesting and well-known, however it requires the existence of both the lower as well as upper solutions as well as certain inequality involving monotonicity of the nonlinearity. In this paper, we prove the existence and approximate solutions for PBVP (1.3) via the Dhage iteration method which does not require the existence of both upper and lower solutions as well the related monotonic inequality. We also obtain the algorithm for the solutions. The novelty of the present paper lies in its method which is completely new and yields the monotonic successive approximations for the solutions under some well-known natural conditions.

The rest of the paper is organized as follows. Section 2 deals with some definitions and auxiliary results that play an important role in this article. The main results are given in Sections 3 and 4. Illustrative examples are also furnished at the end of each section.

2. Auxiliary results

Throughout this paper, unless otherwise mentioned, let $(E, \preceq, \|\cdot\|)$ denote a partially ordered normed linear space. Two elements x and y in E are said to be comparable if either the relation $x \preceq y$ or $y \preceq x$ holds. A non-empty subset C of E is called a chain or totally ordered if all the elements of C are comparable. It is known that E is regular if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E and $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. A few details of a partially ordered normed linear space can be found in Dhage [6] and the conditions guaranteeing the regularity of E can be found in Guo and Lakshmikantham [7], Heikkilä and Lakshmikantham [8], Zeidler [9] and the references therein. A mapping $\mathscr{T} : E \to E$ is called isotone or nondecreasing if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathscr{T}x \preceq \mathscr{T}y$ for all $x, y \in E$. Similarly, $\mathscr{T}$ is called nonincreasing if $x \preceq y$ implies $\mathscr{T}x \succeq \mathscr{T}y$ for all $x, y \in E$. Finally, $\mathscr{T}$ is called monotonic or simply monotone if it is either nondecreasing or nonincreasing on E. A mapping $\mathscr{T} : E \to E$ is called partially continuous at a point $a \in E$ if for $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|\mathscr{T}x - \mathscr{T}a\| < \varepsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathscr{T}$ called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathscr{T}$ is partially continuous on E, then it is continuous on every chain C contained in E and vice versa. A non-empty subset S of the partially ordered Banach space E is called partially bounded if every chain C in S is bounded. An operator $\mathscr{T}$ on a partially normed linear space E into itself is called partially bounded if $\mathscr{T}(E)$ is a partially bounded subset of E. A non-empty subset S of the partially ordered Banach space E is called partially compact if every chain C in S is a relatively compact subset of E. A mapping $\mathscr{T} : E \to E$ is called partially compact if $\mathscr{T}(E)$ is a partially relatively compact subset of E. $\mathscr{T}$ is called partially totally bounded if for any bounded subset S of E, $\mathscr{T}(S)$ is a partially relatively compact subset of E. If $\mathscr{T}$ is partially continuous and partially totally bounded, then it is called partially completely continuous on E.

Remark 2.1. Suppose that $\mathscr{T}$ is a nondecreasing operator on E into itself. Then $\mathscr{T}$ is partially bounded or partially compact if $\mathscr{T}(C)$ is a bounded or relatively compact subset of E for each chain C in E.

Definition 2.2. The order relation $\preceq$ and the metric d on a non-empty set E are said to be $\mathscr{D}$-compatible if $\{x_n\}$ is a monotone sequence, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the original sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E,\preceq,\|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be $\mathscr{D}$-compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are $\mathscr{D}$-compatible. A subset S of E is called Janhavi if the order relation $\preceq$ and the metric d or the norm $\|\cdot\|$ are $\mathscr{D}$-compatible in it. In particular, if $S=E$, then E is called a Janhavi metric or Janhavi Banach space.

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function $|\cdot|$ has this property. Again, every finite dimensional Euclidean space $\mathbb{R}^n$ with usual componentwise order relation and the standard norm possesses the $\mathscr{D}$-compatibility property and so is a Janhavi Banach space. Similarly, the following fundamental results show that every ordered Banach space E ordered by a cone K in E is regular and every partially compact subset S of E is Janhavi. We recall that a non-empty closed and convex subset K of the Banach space E is called a cone if i) $K+K\subseteq K$, ii) $\lambda K\subseteq K$ for $\lambda\in\mathbb{R}$, $\lambda>0$, and iii) $\{-K\}\bigcap K=\{\theta\}$, where θ is a zero element of E. The details of cones and their properties may be found in Guo and Lakshmikantham [7], Heikkilä and Lakshmikantham [8] and references therein. We define an order relation $\leq$ in E by

$$x\leq y \iff y-x\in K, \quad \forall x,y\in E. \tag{2.1}$$

The Banach space E together with the order relation $\leq$ becomes a partially ordered Banach space and it is denoted by (E,K).

Lemma 2.3. *Every partially ordered Banach space (E,K) is regular.*

Proof. Let $\{x_n\}$ be a monotone nondecreasing sequence of points in a partially ordered Banach space (E,K). Then $x_1\leq x_2\leq\cdots\leq x_n\leq\cdots$. Suppose that the sequence $\{x_n\}$ converges to a point x^*, that is, $x_n\to x^*$ as $n\to\infty$. Then, every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to the same limit point x^*, that is, $x_{n_k}\to x^*$ as $k\to\infty$. Since $\{x_n\}$ is nondecreasing, for any given positive integer n, we have $x_n\leq x_{n_k}$ for each $k\geq n\in\mathbb{N}$. This further by definition of the order relation $\leq$ implies that $x_{n_k} - x_n \in K$. As the cone K is closed and convex set in E, one has

$$\lim_{k\to\infty} \left(x_{n_k} - x_n \right) = x^* - x_n \in K$$

for each $n \in \mathbb{N}$. Therefore, $x_n \leq x^*$ for all $n \in \mathbb{N}$. Similarly, if $\{x_n\}$ is monotone noincreasing sequence of points in E, then using similar arguments, it can be proved that $x^* \leq x_n$ for all $n \in \mathbb{N}$. As a result, (E,K) is a regular partially ordered Banach space and the proof of the lemma is complete.

Lemma 2.4. *Every partially compact subset S of a partially ordered Banach space (E,K) is Janhavi.*

Proof. Let C be an arbitrary chain in a partially compact subset S of an ordered Banach space E. Then $\bar{C}$ is compact. Let $\{x_n\}$ be a monotone nondecreasing sequence of points in the chain C, that is, $x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots$. Then $\{x_n\}$ is a relatively compact set in E. Therefore $\{x_n\}$ has a convergent subsequence, say $\{x_{n_k}\}$ converging to a point x^*. We show that $\{x_n\}$ converges to x^*. Suppose not. Then for $\varepsilon > 0$ there exists a subsequence $\{x_{n_i}\}$ such that

$$\|x_{n_i} - x^*\| \geq \varepsilon \quad \forall i = 1,2,\ldots. \tag{2.2}$$

Now, by relative compactness of $\{x_{n_i}\}$, there is a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{i_j}} \to x'$ as $j \to \infty$. Hence for any given positive integer k, by nondecreasing nature of $\{x_n\}$ it follows that when j is large enough (e.g. $j \geq k$), we have that $x_{n_k} \leq x_{n_{i_j}}$. Then $x_{n_{i_j}} - x_{n_k} \in K$. As K is closed and convex, taking the limit first as $j \to \infty$ and then as $k \to \infty$, we obtain

$$x' - x^* \in K \quad \Rightarrow \quad x^* \leq x'.$$

Similarly, it can be shown that $x' \leq x^*$. As a result, we have $x' = x^*$ and that $x_{n_{i_j}} \to x^*$ as $j \to \infty$. Therefore, we get $\|x_{n_{i_j}} - x^*\| < \varepsilon$ for large j. This is a contradiction to (2.2) and the proof of the lemma is complete.

The concepts of the regularity of partially ordered Banach space and the Janhavi compact sets are often times employed in the hybrid fixed point theory in a partially ordered metric or Banach space which is applicable to the problems of nonlinear differential and integral equations for proving the different aspects of the solutions; see [1, 2, 3, 4, 10] and the references therein. Therefore, the above concepts are of fundamental importance in the study of nonlinear differential and integral equations in $\mathbb{R}$ or $\mathbb{R}^n$ or in abstract spaces.

Definition 2.5. An upper semi-continuous and monotone nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathscr{D}$-function provided $\psi(0) = 0$. An operator $\mathscr{T} : E \to E$ is called partial nonlinear $\mathscr{D}$-contraction if there exists a $\mathscr{D}$-function ψ such that $\|\mathscr{T}x - \mathscr{T}y\| \leq \psi\big(\|x-y\|\big)$ for all comparable elements $x, y \in E$, where $0 < \psi(r) < r$ for $r > 0$. In particular, if $\psi(r) = kr$, $k > 0$, $\mathscr{T}$ is called a partial Lipschitz operator with a Lipschitz constant k and moreover, if $0 < k < 1$, $\mathscr{T}$ is called a partial linear contraction on E with a contraction constant k.

Remark 2.6. Note that every partial nonlinear contraction mapping $\mathscr{T}$ is partially continuous but the converse may not be true.

The Dhage iteration method embodied in the following applicable hybrid fixed point principle of Dhage [11] in a partially ordered normed linear space is used as a key tool for our work contained in this paper. The details of other hybrid fixed point theorems involving the Dhage iteration principle and method are given in [1, 2, 6, 11, 12] and the references therein.

Theorem 2.7. (Dhage [11, 13]) *Let $\big(E, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear space such that every compact chain C in E is Janhavi. Let $\mathscr{T} : E \to E$ be a partially continuous, nondecreasing and partially compact operator. If there exists an element $x_0 \in E$ such that $x_0 \preceq \mathscr{T}x_0$ or $\mathscr{T}x_0 \preceq x_0$, then the operator equation $\mathscr{T}x = x$ has a solution x^* in E and the sequence $\{\mathscr{T}^n x_0\}$ of successive iterations converges monotonically to x^*.*

Theorem 2.8. (Dhage [11, 13]) *Let $(E, \preceq, \|\cdot\|)$ be a partially ordered Banach space and let $\mathscr{T} : E \to E$ be a nondecreasing and partial nonlinear $\mathscr{D}$-contraction. Suppose that there exists an element $x_0 \in E$ such that $x_0 \preceq \mathscr{T}x_0$ or $x_0 \succeq \mathscr{T}x_0$. If $\mathscr{T}$ is continuous or E is regular, then $\mathscr{T}$ has a unique comparable fixed point x^* and the sequence $\{\mathscr{T}^n x_0\}$ of successive iterations converges monotonically to x^*. Moreover, the fixed point x^* is unique if every pair of elements in E has a lower and an upper bound.*

Remark 2.9. The condition that every compact chain of E is Janhavi holds if every partially compact subset of E possesses the compatibility property with respect to the order relation $\preceq$ and the norm $\|\cdot\|$ in it. This simple fact is used to prove the main existence results of this paper.

Remark 2.10. The regularity of E in above Theorems 2.7 and 2.8 may be replaced with a stronger continuity condition of the operator $\mathscr{T}$ on E which is a result proved in Dhage [6].

3. Main results

In this section, we prove an existence and approximation result for the PBVP (1.1) on a closed and bounded interval $J=[-r,T]$ under mixed partial Lipschitz and partial compactness type conditions on the nonlinearities involved in it. We place the PBVP (1.1) in the function space $C(J,\mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J,\mathbb{R})$ by

$$\|x\| = \sup_{t\in J}|x(t)| \tag{3.1}$$

and

$$x \leq y \iff x(t)\leq y(t) \quad \text{for all } \ t\in J. \tag{3.2}$$

Clearly, $C(J,\mathbb{R})$ is a Banach space with respect to above supremum norm and also partially ordered w.r.t. the above partially order relation $\leq$. It is known that the partially ordered Banach space $C(J,\mathbb{R})$ is regular and lattice so that every pair of elements of E has a lower and an upper bound; see [6, 11, 13] and the references therein. The following useful lemma concerning the Janhavi subsets of $C(J,\mathbb{R})$ follows immediately from the Arzelá-Ascoli theorem for compactness or the cones in a Banach space $C(J,\mathbb{R})$.

Lemma 3.1. *Let $\big(C(J,\mathbb{R}),\leq,\|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leq$ defined by (3.1) and (3.2) respectively. Then every partially compact subset of $C(J,\mathbb{R})$ is Janhavi.*

Proof. The proof of this lemma is well-known and appears in the papers of Dhage [13] via the Arzelá-Ascoli theorem for compactness. Here we give the proof of the lemma using somewhat different arguments via cones in a Banach space $C(J,\mathbb{R})$. Define a subset K of $C(J,\mathbb{R})$ by

$$K=\{x\in C(J,\mathbb{R}) \mid x(t)\geq 0 \ \text{ for all } \ t\in J\}. \tag{3.3}$$

Clearly K is a non-empty, closed and convex subset of the Banach space $C(J,\mathbb{R})$ satisfying the properties (i)- (iii) of a cone. So K is a cone in $C(J,\mathbb{R})$. Now, the order relation $\leq$ given by (3.2) is equivalent to the order relation $\leq$ defined by the cone K in $C(J,\mathbb{R})$. Therefore, the desired conclusion follows by an application of Lemma 2.4. This completes the proof.

We introduce an order relation $\leq_{\mathscr{C}}$ in $\mathscr{C}$ induced by the order relation $\leq$ defined in $C(J,\mathbb{R})$. Thus, for any $x,y\in\mathscr{C}$, $x\leq_{\mathscr{C}} y$ implies $x(\theta)\leq y(\theta)$ for all $\theta\in I_0$. Note that if $x,y\in C(J,\mathbb{R})$ and $x\leq y$, then $x_t\leq_{\mathscr{C}} y_t$ for all $t\in I$.

We also need the following definition in what follows.

Definition 3.2. A differentiable function $u \in C(J,\mathbb{R})$ is said to be a lower solution of equation (1.3) if

(i) $u_t \in \mathscr{C}$ for each $t \in I$, and

(ii) u is continuously differentiable on I and satisfies

$$\left.\begin{aligned} u'(t)+\lambda u(t) \le f(t,u_t),\ t\in I,\\ u(0)=\phi(0)\le u(T),\\ u_0 \le_{\mathscr{C}} \phi. \end{aligned}\right\}$$

Similarly, a differentiable function $v \in C(J,\mathbb{R})$ is called an upper solution of the PBVP (1.3) if the above inequality is satisfied with reverse sign.

The following useful lemma is obvious and may be found in Dhage [14] and the references therein.

Lemma 3.3. For any function $\sigma \in L^1(I,\mathbb{R})$, x is a solution to the differential equation

$$\left.\begin{aligned} x'(t)+\lambda x(t)=\sigma(t),\ t\in I,\\ x(0)=x(T), \end{aligned}\right\}$$

if and only if it is a solution of the integral equation

$$x(t)=\int_0^T G_\lambda(t,s)\,\sigma(s)\,ds \tag{3.4}$$

where, the green's function $G(t,s)$ is given by

$$G_\lambda(t,s)=\begin{cases} \dfrac{e^{\lambda s-\lambda t+\lambda T}}{e^{\lambda T}-1}, & \text{if}\quad 0\le s\le t\le T,\\[2ex] \dfrac{e^{\lambda s-\lambda t}}{e^{\lambda T}-1}, & \text{if}\quad 0\le t< s\le T. \end{cases} \tag{3.5}$$

Notice that the Green's function G_λ is continuous and nonnegative on $I\times I$ and therefore, the number $K_\lambda := \max\{\,|G_\lambda(t,s)| : t,s\in[0,T]\,\}$ exists for all $\lambda\in\mathbb{R}^+$. For the sake of convenience, we write $G_\lambda(t,s)=G(t,s)$ and $K_\lambda = K$.

The following lemma can be found in Dhage and Dhage [3]. For the sake of completeness, we still give the proof.

Lemma 3.4. *If there exists a differentiable function $u \in C(I,\mathbb{R})$ such that*

$$\left.\begin{aligned} u'(t)+\lambda u(t) &\leq \sigma(t),\ t\in I,\\ u(0) &\leq u(T), \end{aligned}\right\} \tag{3.6}$$

for all $t \in I$, where $\lambda \in \mathbb{R}$, $\lambda > 0$ and $\sigma \in L^1(I,\mathbb{R})$, then

$$u(t) \leq \int_0^T G(t,s)\,\sigma(s)\,ds,$$

for all $t \in I$, where $G(t,s)$ is a Green's function given by (3.5).

Proof. Suppose that the function $u \in C(I,\mathbb{R})$ satisfies the inequalities given in (3.6). Multiplying the first inequality in (3.6) by $e^{\lambda t}$, $\left(e^{\lambda t}u(t)\right)' \leq e^{\lambda t}\sigma(t)$. A direct integration of above inequality from 0 to t yields

$$e^{\lambda t}u(t) \leq u(0) + \int_0^t e^{\lambda s}\sigma(s)\,ds, \tag{3.7}$$

for all $t \in I$. Therefore, in particular,

$$e^{\lambda T}u(T) \leq u(0) + \int_0^T e^{\lambda s}\sigma(s)\,ds. \tag{3.8}$$

It follows from $u(0) \leq u(T)$ that

$$u(0)e^{\lambda T} \leq u(T)e^{\lambda T}. \tag{3.9}$$

From (3.8) and (3.9), one has $e^{\lambda T}u(0) \leq u(0) + \int_0^T e^{\lambda s}\sigma(s)\,ds$ which further yields

$$u(0) \leq \int_0^T \frac{e^{\lambda s}}{(e^{\lambda T}-1)}\sigma(s)\,ds. \tag{3.10}$$

Substituting (3.10) into (3.7), we obtain $u(t) \leq \int_0^T G(t,s)\sigma(s)\,ds$, for all $t \in I$. This completes the proof.

We consider the following set of assumptions in what follows:

(H_1) There exists a constant $M_f > 0$ such that $|f(t,x)| \leq M_f$ for all $t \in I$ and $x \in \mathscr{C}$.

(H_2) $f(t,x)$ is nondecreasing in x for each $t \in I$.

(H_3) There exists $\mathscr{D}$-function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $0 \leq f(t,x) - f(t,y) \leq \varphi(\|x-y\|_{\mathscr{C}})$ for all $t \in I$ and $x,y \in \mathscr{C}$, $x \geq_{\mathscr{C}} y$.

(H_4) PBVP (1.3) has a lower solution $u \in C(J,\mathbb{R})$.

Lemma 3.5. *A function $x \in C(J,\mathbb{R})$ is a solution of the PBVP (1.3) if and only if it is a solution of the nonlinear integral equation*

$$x(t) = \begin{cases} \displaystyle\int_0^T G(t,s) f(s,x_s)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases}$$

where the function $G(t,s)$ is given by (3.5) with $\lambda = 1$.

Theorem 3.6. *Suppose that hypotheses (H_1)-(H_2) and (H_4) hold. Then the PBVP (1.3) has a solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by*

$$x_0 = u,$$
$$x_{n+1}(t) = \begin{cases} \displaystyle\int_0^T G(t,s) f(s,x_s^n)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0, \end{cases}$$

where $x_s^n(\theta) = x_n(s+\theta)$, $\theta \in I_0$, converges monotonically to x^*.

Proof. Set $E = C(J,\mathbb{R})$. From Lemma 3.1, one sees that every compact chain C in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ so that every compact chain C is Janhavi in E. Define an operator $\mathscr{T}$ on E by

$$\mathscr{T}x(t) = \begin{cases} \displaystyle\int_0^T G(t,s) f(s,x_s)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases}$$

From the continuity of the Green's function G, it follows that $\mathscr{T}$ defines the operator $\mathscr{T} : E \to E$. Applying Lemma 3.5, the PBVP (1.3) is equivalent to the operator equation $\mathscr{T}x(t) = x(t)$, $t \in J$.

Now, we show that the operators $\mathscr{T}$ satisfies all the conditions of Theorem in a series of following steps.

Step I: *$\mathscr{T}$ is nondecreasing on E.*

Let $x, y \in E$ be such that $x \geq y$. Then $x_t \geq_{\mathscr{C}} y_t$ for all $t \in I$ and by hypothesis (H_2), we obtain

$$\mathscr{T}x(t) = \begin{cases} \displaystyle\int_0^T G(t,s) f(s, x_s)\, ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases}$$

$$\geq \begin{cases} \displaystyle\int_0^T G(t,s) f(s, y_s)\, ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases}$$

$$= \mathscr{T}y(t),$$

for all $t \in J$. This shows that the operator that the operator $\mathscr{T}$ is also nondecreasing on E.

Step II: *$\mathscr{T}$ is partially continuous on E.*

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a chain C such that $x_n \to x$ as $n \to \infty$. Then $x_s^n \to x_s$ as $n \to \infty$. Since the f is continuous, we have

$$\lim_{n\to\infty} \mathscr{T}x_n(t) = \begin{cases} \displaystyle\int_0^T G(t,s) \left[\lim_{n\to\infty} f\left(s, x_s^n\right)\right] ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases}$$

$$= \begin{cases} \displaystyle\int_0^T G(t,s) f(s, x_s)\, ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases}$$

$$= \mathscr{T}x(t),$$

for all $t \in J$. This shows that $\mathscr{T}x_n$ converges to $\mathscr{T}x$ pointwise on J.

Now we show that $\{\mathscr{T}x_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence of functions in E. Now there are three cases:

Case I: Let $t_1,t_2 \in J$ with $t_1 > t_2 \geq 0$. Then we have

$$
\begin{aligned}
|\mathscr{T}x_n(t_2) - \mathscr{T}x_n(t_1)| &= \left| \int_0^T |G(t_2,s) - G(t_1,s)| \left| f\left(s,x_s^n\right) \right| ds \right| \\
&\leq M_f \int_0^T |G(t_2,s) - G(t_1,s)|\, ds \\
&\to 0 \quad \text{as} \quad t_2 \to t_1,
\end{aligned}
$$

uniformly for all $n \in \mathbb{N}$.

Case II: Let $t_1,t_2 \in J$ with $t_1 < t_2 \leq 0$. Then we have $|\mathscr{T}x_n(t_2) - \mathscr{T}x_n(t_1)| = |\phi(t_2) - \phi(t_1)| \to 0$ as $t_2 \to t_1$, uniformly for all $n \in \mathbb{N}$.

Case III: Let $t_1,t_2 \in J$ with $t_1 < 0 < t_2$. Then we have

$$
|\mathscr{T}x_n(t_2) - \mathscr{T}x_n(t_1)| \leq |\mathscr{T}x_n(t_2) - \mathscr{T}x_n(0)| + |\mathscr{T}x_n(0) - \mathscr{T}x_n(t_1)| \to 0 \; as \; t_2 \to t_1.
$$

Thus in all three cases, we obtain $|\mathscr{T}x_n(t_2) - \mathscr{T}x_n(t_1)| \to 0$ as $t_2 \to t_1$, uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathscr{T}x_n \to \mathscr{T}x$ is uniform and that $\mathscr{T}$ is a partially continuous operator on E into itself.

Step III: *$\mathscr{T}$ is partially compact operator on E.*

Let C be an arbitrary chain in E. We show that $\mathscr{B}(C)$ is uniformly bounded and equicontinuous set in E. First we show that $\mathscr{T}(C)$ is uniformly bounded. Let $y \in \mathscr{T}(C)$ be any element. Then there is an element $x \in C$ such that $y = \mathscr{T}x$. By hypothesis (H_2)

$$
\begin{aligned}
|y(t)| &= |\mathscr{T}x(t)| \\
&\leq \begin{cases} \displaystyle\int_0^T G(t,s)|f(s,x_s)|\,ds, & \text{if } t \in I, \\[2ex] |\phi(t)|, & \text{if } t \in I_0. \end{cases} \\
&\leq \|\phi\| + KM_f T = r,
\end{aligned}
$$

for all $t \in J$. Taking the supremum over t, we obtain $\|y\| \leq \|\mathscr{T}x\| \leq r$ for all $y \in \mathscr{T}(C)$. Hence $\mathscr{T}(C)$ is a uniformly bounded subset of E. Next, we show that $\mathscr{T}(C)$ is an equicontinuous set in E. Let $t_1,t_2 \in J$ be arbitrary with $t_1 < t_2$. Then proceeding with the arguments that given in Step II it can be shown that $\left|y(t_2) - y(t_1)\right| = |\mathscr{T}x(t_2) - \mathscr{T}x(t_1)| \to 0$ as $t_1 \to t_2$ uniformly for all $y \in \mathscr{T}(C)$. This shows that $\mathscr{T}(C)$ is an equicontinuous subset of E. Now, $\mathscr{T}(C)$ is a uniformly bounded and equicontinuous subset of functions in E and hence it is compact in view of Arzelá-Ascoli theorem. Consequently $\mathscr{T} : E \to E$ is a partially compact operator on E into itself.

Step IV: *u satisfies the inequality $u \leq \mathscr{T}u$.*

By hypothesis (H_4), the PBVP (1.3) has a lower solution u defined on J. Then we have

$$\begin{cases} u'(t) + \lambda u(t) \leq f(t, u_t), \ t \in I, \\ u(0) = \phi(0) \leq x(T) \\ u_0 \leq_{\mathscr{C}} \phi. \end{cases}$$

Using Lemmas 3.3 and 3.4, we obtain

$$u(t) \leq \begin{cases} \int_0^T G(t,s) f(s, u_s)\, ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases}$$
$$= \mathscr{T}u(t)$$

for all $t \in J$. As a result we have that $u \leq \mathscr{T}u$. Thus, $\mathscr{T}$ satisfies all the conditions of Theorem 2.7 and so the operator equation $\mathscr{T}x = x$ has a solution. Consequently the integral equation and the equation (1.3) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.5) converges monotonically to x^*. This completes the proof.

Remark 3.6. The conclusion of Theorems 3.6 also remains true if we replace the hypothesis (H_4) with the following ones:

(H_4') The PBVP (1.3) has an upper solution $v \in C(J, \mathbb{R})$.

The proof of Theorem 3.6 under this new hypothesis is similar and can be obtained by closely observing the same arguments with appropriate modifications.

Example 3.7. Given the closed and bounded intervals $I_0 = \left[-\frac{\pi}{2}, 0\right]$ and $I = [0,1]$, consider the PBVP

$$\left.\begin{aligned} x'(t) + x(t) = f_1(t, x_t),\ t \in I, \\ x(0) = \phi(0) = x(1), \\ x_0 = \phi, \end{aligned}\right\}$$

where $\phi \in \mathscr{C}$ and $f_1 : I \times \mathscr{C} \to \mathbb{R}$ are continuous functions given by

$$\phi(\theta) = \sin\theta, \quad \theta \in \left[-\frac{\pi}{2}, 0\right],$$

and

$$f_1(t,x) = \begin{cases} \tanh(\|x\|_{\mathscr{C}}) + 1, & \text{if } x \leq_{\mathscr{C}} 0, x \neq 0, \\ 1, & \text{if } x \leq_{\mathscr{C}} 0, \end{cases}$$

for all $t \in I$. Clearly, f is bounded on $I \times \mathscr{C}$ with $M_{f_1} = 2$. Furthermore, let $x, y \in \mathscr{C}$ be such that $x \geq_{\mathscr{C}} y \geq_{\mathscr{C}} 0$. Then $\|x\|_{\mathscr{C}} \geq \|y\|_{\mathscr{C}} \geq 0$ and therefore, we have

$$f_1(t,x) = \tanh(\|x\|_{\mathscr{C}}) + 1 \geq \tanh(\|y\|_{\mathscr{C}}) + 1 = f_1(t,y)$$

for all $t \in I$. Again, if $x, y \in \mathscr{C}$ be such that $x \leq_{\mathscr{C}} y \leq_{\mathscr{C}} 0$, then we obtain $f_1(t,x) = 1 = f_1(t,y)$ for all $t \in I$. This shows that the function $f_1(t,x)$ is nondecreasing in x for each $t \in I$. Finally, it can be verified that the function

$$u(t) = \begin{cases} \displaystyle\int_0^1 G(t,s)(1-s)\,ds, & \text{if } t \in I, \\ \sin t, & \text{if } t \in I_0, \end{cases}$$

is a lower solution of the PBVP (3.6) defined on J, where $G(t,s)$ is a Green's function given by (3.5) with $\lambda = 1$.

Thus, f_1 satisfies the hypotheses (H_1), (H_2) and (H_4). Hence we apply Theorem 3.6 and conclude that the PBVP (3.6) has a solution x^* on J and the sequence $\{x_n\}$ of successive approximation defined by

$$x_0(t) = \begin{cases} \displaystyle\int_0^1 G(t,s)(1-s)\,ds, & \text{if } t \in [0,1], \\ \sin t, & \text{if } t \in \left[-\frac{\pi}{2}, 0\right], \end{cases}$$

$$x_{n+1}(t) = \begin{cases} \displaystyle\int_0^1 G(t,s) f_1(s, x_s^n)\,ds, & \text{if } t \in [0,1], \\ \sin t, & \text{if } t \in \left[-\frac{\pi}{2}, 0\right], \end{cases}$$

converges monotonically to x^*.

Remark 3.8. The conclusion in Example 3.7 is also true if we replace the lower solution u with the upper solution v given by

$$v(t) = \begin{cases} \displaystyle\int_0^1 G(t,s)(2+s)\,ds, & \text{if } t \in I, \\ \sin t, & \text{if } t \in I_0, \end{cases}$$

Theorem 3.8. *Suppose that hypotheses (H_3) and (H_4) hold. Then the PBVP (1.3) has a unique solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by (3.4) converges monotonically to x^*.*

Proof. Set $E = C(J,\mathbb{R})$. Clearly, E is a lattice w.r.t. the order relation $\leq$ and so the lower and the upper bound exist for every pair of elements in E. Define the operator $\mathscr{T}$ by (3.5). Then, the PBVP (1.3) is equivalent to the operator equation (3.7). We shall show that $\mathscr{T}$ satisfies all the conditions of Theorem 2.8 in E. Clearly, $\mathscr{T}$ is a nondecreasing operator on E into itself. We shall simply show that the operator $\mathscr{T}$ is a partially nonlinear $\mathscr{D}$-contraction on E. Let $x, y \in E$ be any two elements such that $x \geq y$. Then, by hypothesis (H_4),

$$\begin{aligned} |\mathscr{T}x(t) - \mathscr{T}y(t)| &\leq \int_0^T G(t,s)|f(s,x_s) - f(s,y_s)|\,ds \\ &\leq K\int_0^T \varphi(\|x_s - y_s\|_{\mathscr{C}})\,ds \\ &\leq KT\varphi(\|x-y\|) \end{aligned} \tag{3.11}$$

for all $t \in J$. Taking the supremum over t, we obtain $\|\mathscr{T}x - \mathscr{T}y\| \leq KT\psi(\|x-y\|)$ for all $x, y \in E$, $x \geq y$, where $\psi(r) = KT\varphi(r) < r$ for $r > 0$. As a result $\mathscr{T}$ is a partially nonlinear $\mathscr{D}$-contraction on E in view of Remark . Furthermore, it can be shown as in the proof of Theorem 3.6 that the function u given in hypothesis (H_3) satisfies the operator inequality $u \leq \mathscr{T}u$ on J. Now a direct application of Theorem 2.8 yields that the PBVP (1.3) has a unique solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by (3.5) converges monotonically to x^*.

Remark 3.9. The conclusion of Theorems 3.8 also remains true if we replace the hypothesis (H_4) with the following ones:

(H_4') The PBVP (1.3) has an upper solution $v \in C(J,\mathbb{R})$.

The proof of Theorem 3.8 under this new hypothesis is similar and can be obtained by closely observing the same arguments with appropriate modifications.

Example 3.10. Given the closed and bounded intervals $I_0 = \left[-\frac{\pi}{2}, 0\right]$ and $I = [0,1]$, consider the PBVP

$$\left.\begin{aligned} x'(t) + x(t) = f_2(t,x_t),\ t \in I, \\ x(0) = \phi(0) = x(1), \\ x_0 = \phi, \end{aligned}\right\}$$

where $\phi \in \mathscr{C}$ and $f_2 : I \times \mathscr{C} \to \mathbb{R}$ is a continuous functions given by

$$\phi(\theta) = \sin\theta, \quad \theta \in \left[-\frac{\pi}{2}, 0\right],$$

and

$$f_2(t,x) = \begin{cases} \dfrac{\|x\|_{\mathscr{C}}}{1+\|x\|_{\mathscr{C}}} + 1, & \text{if } x \leq_{\mathscr{C}} 0, x \neq 0, \\ 1, & \text{if } x \leq_{\mathscr{C}} 0, \end{cases}$$

for all $t \in I$.

Clearly, f_2 is continuous on $I \times \mathscr{C}$. We show that f_2 satisfies the hypotheses (H$_3$) and (H$_4$). Let $x, y \in \mathscr{C}$ be such that $x \geq_{\mathscr{C}} y \geq_{\mathscr{C}} 0$. Then $\|x\|_{\mathscr{C}} \geq \|y\|_{\mathscr{C}} \geq 0$ and therefore, we have

$$0 \leq f_2(t,x) - f_2(t,x) = \frac{\|x\|_{\mathscr{C}}}{1+\|x\|_{\mathscr{C}}} - \frac{\|y\|_{\mathscr{C}}}{1+\|y\|_{\mathscr{C}}} \leq \varphi(\|x-y\|_{\mathscr{C}})$$

for all $t \in I$, where $\varphi(r) = \dfrac{r}{1+r} < r$, $r > 0$. Again, if $x, y \in \mathscr{C}$ be such that $x \leq_{\mathscr{C}} y \leq_{\mathscr{C}} 0$, then we obtain

$$0 \leq f_2(t,x) - f_2(t,x) \leq \varphi(\|x-y\|_{\mathscr{C}})$$

for all $t \in I$. This shows that the function $f(t,x)$ is nondecreasing in x for each $t \in I$ and satisfies the hypothesis (H$_3$). Finally,

$$u(t) = \begin{cases} \displaystyle\int_0^1 G(t,s), & \text{if } t \in I, \\ \sin t, & \text{if } t \in I_0, \end{cases}$$

is a lower solution of the PBVP (3.11) defined on J, where $G(t,s)$ is a Green's function defined by (3.5) with $\lambda = 1$. Thus, f satisfies the hypotheses (H$_3$) and (H$_4$). Hence we apply Theorem 3.8 and conclude that the PBVP (3.11) has a solution x^* on J and the sequence $\{x_n\}$ of successive approximation defined by

$$x_0(t) = \begin{cases} \displaystyle\int_0^1 G(t,s), & \text{if } t \in I, \\ \sin t, & \text{if } t \in I_0, \end{cases}$$

$$x_{n+1}(t) = \begin{cases} \displaystyle\int_0^1 G(t,s) f_2(s, x_s^n)\, ds, & \text{if } t \in I, \\ \sin t, & \text{if } t \in I_0, \end{cases}$$

converges monotonically to x^*.

Remark 3.11. The conclusion in Example 3.2 is also true if we replace the lower solution u with the upper solution v given by

$$v(t) = \begin{cases} 2\int_0^1 G(t,s), & \text{if } t \in I, \\ \\ \sin t, & \text{if } t \in I_0, \end{cases}$$

4. Linear perturbation of first type

Now, with the usual notations in Section 1, we consider the PBVPs of hybrid differential equations with linear perturbation of first type, namely,

$$\left.\begin{aligned} x'(t) + \lambda x(t) = f(t,x_t) + g(t,x_t),\ t \in I, \\ x(0) = \phi(0) = x(T), \\ x_0 = \phi, \end{aligned}\right\} \tag{4.1}$$

where $\phi \in \mathscr{C}$ and $f, g : I \times \mathscr{C} \to \mathbb{R}$ are continuous functions.

Definition 4.1. A differentiable function $u \in C(J,\mathbb{R})$ is said to be a lower solution of the equation (1.3) if

(i) $x_0 = \phi$,

(ii) $x_t \in \mathscr{C}$ for each $t \in I$, and

(iii) x is continuously differentiable on I and satisfies the equations in (4.1).

where $C(J,\mathbb{R})$ is the space of continuous real-valued functions defined on J.

The PBVP (4.1) is well-known in the literature and studied via different methods for existence of solutions; see [14] and references therein. The novelty of present study lies in its study of the new Dhage iteration method for proving the existence as well as approximation of the solutions. As a result of our new approach, we obtain an algorithm for the solutions of PBVP (4.1) on J. We use the Dhage iteration method embodied in the following hybrid fixed point principle of Dhage [11]; see also [13] for the related results.

Theorem 4.2. *Let $\big(E, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear space such that every compact chain C of E is Janhavi. Let $\mathscr{A}, \mathscr{B} : E \to E$ be two nondecreasing operators such that*

(a) *$\mathscr{A}$ is a partially bounded and partially nonlinear $\mathscr{D}$-contraction,*

(b) *$\mathscr{B}$ is partially continuous and partially compact,*

(c) *there exists an element $\alpha_0 \in X$ such that $\alpha_0 \preceq \mathscr{A}\alpha_0 + \mathscr{B}\alpha_0$ or $\alpha_0 \succeq \mathscr{A}\alpha_0 + \mathscr{B}\alpha_0$.*

Then the operator equation $\mathscr{A}x + \mathscr{B}x = x$ has a solution x^* and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathscr{A}x_n + \mathscr{B}x_n$, $n = 0, 1, \ldots$; converges monotonically to x^*.

We need the following definition in what follows.

Definition 4.3. A differentiable function $u \in C(J,\mathbb{R})$ is said to be a lower solution of the equation (1.3) if

(i) $u_t \in \mathscr{C}$ for each $t \in I$, and

(ii) u is continuously differentiable on I and satisfies

$$\left.\begin{aligned} u'(t) + \lambda u(t) \leq f(t,u_t) + g(t,u_t),\ t \in I, \\ u(0) = \phi(0) \leq x(T), \\ u_0 \leq_{\mathscr{C}} \phi. \end{aligned}\right\}$$

Similarly, a differentiable function $v \in C(J,\mathbb{R})$ is called an upper solution of the PBVP (1.3) if the above inequality is satisfied with reverse sign.

We consider the following set of hypotheses in what follows.

(H$_5$) There exists a constant $M_g > 0$ such that $|g(t,x)| \leq M_g$ for all $t \in I$ and $x \in \mathscr{C}$;

(H$_6$) $g(t,x)$ is nondecreasing in x for each $t \in I$.

(H$_7$) PBVP (4.1) has a lower solution $u \in C(J,\mathbb{R})$.

Our main existence and approximation result for the PBVP (4.1) is as follows.

Theorem 4.4. *Suppose that hypotheses (H$_1$), (H$_3$) and (H$_5$)-(H$_7$) hold. Then the PBVP (4.1) has a solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by*

$$x_0 = u,$$

$$x_{n+1}(t) = \begin{cases} \displaystyle\int_0^T G(t,s) f(s,x_s^n)\,ds + + \int_0^T G(t,s) g(s,x_s^n)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0, \end{cases}$$

where $x_s^n(\theta) = x_n(s+\theta)$, $\theta \in I_0$, converges monotonically to x^*.

Proof. Set $E = C(J,\mathbb{R})$. Then, in view of Lemma 3.1, every partially compact subset S of E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ so that every compact chain C in E is Janhavi.

Define two operators $\mathscr{A}$ and $\mathscr{B}$ on E by

$$\mathscr{A}x(t) = \begin{cases} \displaystyle\int_0^T G(t,s)f(s,x_s)\,ds, & \text{if } t \in I, \\ 0, & \text{if } t \in I_0, \end{cases}$$

and

$$\mathscr{B}x(t) = \begin{cases} \displaystyle\int_0^T G(t,s)g(s,x_s)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases} \tag{4.2}$$

From the continuity of the integral, it follows that $\mathscr{A}$ and $\mathscr{B}$ define the operator $\mathscr{A},\mathscr{B} : E \to E$. Applying Lemma 4.1, the PBVP (4.1) is equivalent to the operator equation $\mathscr{A}x(t)+\mathscr{B}x(t) = x(t)$, $t \in J$. Now, we show that the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy all the conditions of Theorem 2.7. Now proceeding with the arguments that given in Theorem 4.2, it can be shown that $\mathscr{B}$ is a partially continuous and compact operator on E into itself. By hypothesis (H_1), $\mathscr{A}$ is a bounded operator on E. Again following the arguments that given in Theorem 3.8 is shown that $\mathscr{A}$ is a partial nonlinear contraction on E into itself. Now a direct application of Theorem 4.2 yields that the PBVP (4.1) has a solution x^* and the sequence $\{x_n\}$ of successive approximations defined by (4.2) converges to x^*. This completes the proof.

Remark 4.5. The conclusion of Theorem 4.2 also remains true if we replace the hypothesis (H_7) with the following one.

(H'_7) The PBVP (4.1) has an upper solution $v \in C(J,\mathbb{R})$.

The proof of Theorem 4.2 under this new hypothesis is similar and can be obtained by closely observing the same arguments with appropriate modifications.

Example 4.6. Given the closed and bounded intervals $I_0 = \left[-\frac{\pi}{2}, 0\right]$ and $I = [0,1]$ and given a function $\phi \in \mathscr{C}(I_0,\mathbb{R})$, we consider the PBVP

$$\left.\begin{aligned} x'(t)+x(t) = f_1(t,x_t)+f_2(t,x_t),\ t \in I, \\ x(0) = \phi(0) = x(1), \\ x_0 = \phi, \end{aligned}\right\}$$

where $\phi \in \mathscr{C}$, and $f_1, f_2 : I \times \mathscr{C} \to \mathbb{R}$ are continuous functions given by

$$\phi(\theta) = \sin\theta, \quad \theta \in \left[-\frac{\pi}{2}, 0\right],$$

$$f_1(t,x) = \begin{cases} \dfrac{\|x\|_{\mathscr{C}}}{1+\|x\|_{\mathscr{C}}} + 1, & \text{if } x \leq_{\mathscr{C}} 0, x \neq 0, \\ 1, & \text{if } x \leq_{\mathscr{C}} 0, \end{cases}$$

and

$$f_2(t,x) = \begin{cases} \tanh(\|x\|_{\mathscr{C}}) + 1, & \text{if } x \leq_{\mathscr{C}} 0, x \neq 0, \\ 1, & \text{if } x \leq_{\mathscr{C}} 0, \end{cases}$$

for all $t \in I$.

Clearly the functions f_1 and f_2 satisfy the hypotheses (H_1) and (H_5) with $M_{f_1} = 2 = M_{f_2}$. Next, it can be show as in Theorem the nonlinearity f_1 satisfies the hypothesis (H_3). Similarly, the nonlinearity f_2 satisfies the hypothesis (H_6). Again, it can be verified that

$$u(t) = \begin{cases} 2\displaystyle\int_0^1 G(t,s)\,ds, & \text{if } t \in I, \\ \sin t, & \text{if } t \in I_0, \end{cases}$$

is a lower solution of the PBVP (4.1) defined on J, where $G(t,s)$ is a Green's function defined by (3.5) with $\lambda = 1$. Thus, f_1 and f_2 satisfy all the hypotheses of Theorem 4.2. Hence the PBVP (4.1) has a solution x^* and the sequence $\{x_n\}$ of successive approximations defined by

$$x_0(t) = \begin{cases} 2\displaystyle\int_0^1 G(t,s)\,ds, & \text{if } t \in [0,1], \\ \sin t, & \text{if } t \in \left[-\frac{\pi}{2}, 0\right], \end{cases}$$

$$x_{n+1}(t) = \begin{cases} \displaystyle\int_0^1 G(t,s) f_1(s, x_s^n)\,ds + + \int_0^1 G(t,s) f_2(s, x_s^n)\,ds, & \text{if } t \in [0,1], \\ \sin t, & \text{if } t \in \left[-\frac{\pi}{2}, 0\right], \end{cases}$$

converges monotonically to x^*, where $x_s^n(\theta) = x_n(s+\theta)$, $\theta \in \left[-\frac{\pi}{2}, 0\right]$.

Remark 4.7. The conclusion in Example 4.6 is also true if we replace the lower solution u with the upper solution v given by

$$v(t) = \begin{cases} \displaystyle\int_0^1 G(t,s)(s+4)\,ds, & \text{if } t \in [0,1], \\ \sin t, & \text{if } t \in \left[-\frac{\pi}{2}, 0\right]. \end{cases}$$

Remark 4.8. We note that if the PBVPs (1.3) and (4.1) have a lower solution u as well as an upper solution v such that $u \leq v$, then under the given conditions of Theorems 3.6 and 4.2 it has corresponding solutions x_* and x^* and these solutions satisfy $x_* \leq x^*$. Hence they are the minimal and maximal solutions of the PBVPs (1.3) and (4.1) respectively in the vector segment $[u,v]$ of the Banach space $E = C(J,\mathbb{R})$, where the vector segment $[u,v]$ is a set in $C(J,\mathbb{R})$ defined by

$$[u,v] = \{x \in C(J,\mathbb{R}) \mid u \leq x \leq v\}.$$

This is because the order relation $\leq$ defined by (3.2) is equivalent to the order relation defined by the order cone $K = \{x \in C(J,\mathbb{R}) \mid x \geq \theta\}$ which is a non-empty closed set in $C(J,\mathbb{R})$.

5. Remarks and Conclusion

In this paper, we discussed a very simple PBVP nonlinear first order ordinary functional differential equation via the Dhage iteration method by constructing an algorithm for the solutions. However, other several PBVPs nonlinear functional differential equations could also be studied for the existence and approximation of the solutions using the Dhage iteration method in an analogous way with appropriate modifications. We note that the existence result for the PBVPs (1.3) and (4.1) can also be proved using the Schauder fixed point principle (see Dhage [15]), but in that case we do not get an algorithm for approximating the solutions. Again, our discussion is limited to proving the existence and approximation theorem for the functional differential equations under consideration, but other qualitative aspects such as maximal and minimal solutions and comparison principle etc. could also be studied by constructing the algorithm via the Dhage iteration method on the lines of Dhage [2] and the references therein. It is known that the comparison principle is a very much useful result in the theory of nonlinear functional differential equations for proving the qualitative properties of the solutions. We note that the approximation of the solution to the PBVPs (1.3) and (4.1) may also be obtained via monotone iterative techniques blending with the existence of lower and upper solutions, but in that case one needs to prove a comparison theorem which is not required here in the present approach of the Dhage iteration method. This shows the advantage of the Dhage iteration method over that of monotone iterative techniques for nonlinear differential equations. Therefore, we claim that the Dhage iteration method is a powerful method in the theory of nonlinear differential and integral equations. Again the conclusion of Theorems 3.6 and 4.2 also remains true if we replace the continuity of the nonlinearities f and g in the PBVPs (1.3) and (4.1) with a weaker Carathe̒́odory condition.

REFERENCES

[1] B.C. Dhage, Approximating solutions of nonlinear periodic boundary value problems with maxima, Cogent Math. 3 (2016), Article ID 1206699.

[2] B.C. Dhage, Dhage iteration method for nonlinear first order ordinary hybrid differential equations with mixed perturbation of second type with maxima, J. Nonlinear Funct. Anal. 2016 (2016), Article ID 31.

[3] S.B. Dhage, B.C. Dhage, Dhage iteration method for Approximating positive solutions of PBVPs of nonlinear hybrid differential equations with maxima, Int. J. Anal. Appl. 10 (2016), 101-111.

[4] S.B. Dhage, B.C. Dhage, Dhage iteration method for approximating positive solutions of nonlinear first order ordinary quadratic differential equations with maxima, Nonlinear Anal. Forum 16 (2016), 87-100.

[5] J.K. Hale, Theory of Functional Differential Equations, Springer-Verlag, New York-Berlin, 1977.

[6] B. C. Dhage, Hybrid fixed point theory in partially ordered normed linear spaces and applications to fractional integral equations, Differ. Equ. Appl. 5 (2013), 155-184.

[7] D. Guo, V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, New York, London 1988.

[8] S. Heikkilä, V. Lakshmikantham, Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations, Marcel Dekker inc., New York 1994.

[9] E. Zeidler, Nonlinear Functional Analysis and Its Applications, Vol. I: Fixed Point Theorems, Springer, New York, 1986.

[10] B.C. Dhage, S.B. Dhage, Approximating positive solutions of PBVPs of nonlinear first order ordinary hybrid differential equations, Appl. Math. Lett. 46 (2015), 133-142.

[11] B.C. Dhage, Partially condensing mappings in partially ordered normed linear spaces and applications to functional integral equations, Tamkang J. Math. 45 (2014), 397-427.

[12] B.C. Dhage, S.B. Dhage, J.R. Graef, Dhage iteration method for initial value problems for nonlinear first order hybrid integrodifferential equations, J. Fixed Point Theory Appl. 18 (2016), 309-326.

[13] B.C. Dhage, Nonlinear $\mathscr{D}$-set-contraction mappings in partially ordered normed linear spaces and applications to functional hybrid integral equations, Malaya J. Math. 3 (2015), 62-85.

[14] B.C. Dhage, Periodic boundary value problems of first order Carathéodory and discontinuous differential equations, Nonlinear Funct. Anal. Appl. 13 (2008), 323-352.

[15] B.C. Dhage, Applicable fixed point theory in functional differential equations on unbounded intervals, Dynamic Systems Appl. 18 (2009), 701-724.