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SOME NEW INTEGRAL INEQUALITIES FOR THE GENERALIZED (s,m,φ)-PREINVEX GODUNOVA-LEVIN FUNCTIONS OF THE SECOND KIND

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Abstract. In this paper, we introduce a new classes of generalized (s,m,φ)-preinvex Godunova-Levin functions of the second kind. Some characterization theorems and some new integral inequalities for the left-hand side of the Gauss-Jacobi type quadrature formula involving generalized (s,m,φ)-preinvex Godunova-Levin functions of the second kind along with the beta function are established. Some extensions integral inequalities involving generalized $(h,(m,\varphi))$-preinvex functions are also given.

Keywords. Hölder's inequality; Power mean inequality; (s,m)-Godunova-Levin function; h-convex function; P-function.

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1. Introduction and Preliminaries

Recently, the theory of convex functions has received special attention from many researchers and the classical concepts of convex functions have been extended and generalized in various different directions using novel and innovative ideas and techniques. In [1] and [2], Dragomir introduced and investigated a new class of Godunova-Levin functions which is called s-Godunova-Levin functions of the second kind. In [3], Noor *et al.* extended the class of the Godunova-Levin functions and introduced the classes of s-Godunova-Levin functions of first

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kind, logarithmic s-Godunova-Levin functions of the first and second kinds. The interrelationship between theory of convex functions and theory of inequalities led many researchers to extend various classical inequalities known in the literature for these newly developed generalizations of classical convex functions. Convexity plays an important role in economics, management science, engineering, finance and optimization theory. Many interesting generalizations and extensions of classical convexity have been used in optimization and mathematical inequalities. The following notation is used throughout this paper. We use I,J to denote intervals on the real line $\mathbb{R}=(-\infty,+\infty)$ and I°,J° to denote the interior of I and J. For any subset $K\subseteq\mathbb{R}^n$, K° is used to denote the interior of K. $\mathbb{R}^n$ is used to denote a n-dimensional vector space and $\mathbb{R}^n_+$ is used to denote an n-dimensional nonnegative vector space. The nonnegative real numbers are denoted by $\mathbb{R}_\circ=[0,+\infty)$. The set of integrable functions on the interval $[a,b]$ is denoted by $L([a,b])$.

First, let us recall some definitions of various convex functions.

Definition 1.1. (see [4]) A nonnegative function $f:I\subseteq\mathbb{R}\longrightarrow\mathbb{R}_\circ$ is said to be a P-function or P-convex if

$$f(tx+(1-t)y)\leq f(x)+f(y),\quad \forall x,y\in I,\ t\in[0,1].$$

Definition 1.2. (see [5]) A function $f:I\subseteq\mathbb{R}\longrightarrow\mathbb{R}_\circ$ is said to be a Godunova-Levin function or $f\in Q(I)$, if f is nonnegative and for all $x,y\in I$, $t\in(0,1)$, we have

$$f(tx+(1-t)y)\leq\frac{f(x)}{t}+\frac{f(y)}{1-t}.$$

The class $Q(I)$ was first described in [6] by Godunova and Levin. Some further properties are given in [4], [7] and [8]. It is noted that nonnegative monotone and nonnegative convex functions belong to this class of functions.

Definition 1.3. (see [9]) A function $f:I\subseteq\mathbb{R}\longrightarrow\mathbb{R}_\circ$ is said to be (s,m)-Godunova-Levin functions of the first kind or $f\in Q^1_{(s,m)}$, if $\forall s,m\in(0,1]$, we have

$$f(tx+m(1-t)y)\leq\frac{1}{t^s}f(x)+m\left(\frac{1}{1-t^s}\right)f(y),\quad \forall x,y\in I,\ t\in(0,1).$$

We here mention that Definition 1.3 is also introduced and studied by Li, Wang and Wei [10] independently. Putting $m=1$ in Definition 1.3, we have the definition of s-Godunova-Levin functions of the first kind, which is introduced and investigated by Noor *et al.* see [3].

Definition 1.4. (see [9]) A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}_\circ$ is said to be a (s,m)-Godunova-Levin function of the second kind or $f \in Q^2_{(s,m)}$, if $s \in [0,1]$, $m \in (0,1]$, we have

$$f(tx+m(1-t)y) \leq \frac{1}{t^s} f(x) + m\left(\frac{1}{(1-t)^s}\right) f(y), \quad \forall x,y \in I,\ t \in (0,1).$$

It is obvious that for $s=0, m=1$, (s,m)-Godunova-Levin functions of the second kind reduce to Definition 1.1 of P-functions. If $s=1, m=1$, then it reduces to the Godunova-Levin functions. For $m=1$, we have the definition of s-Godunova-Levin function of the second kind introduced and studied by Dragomir; see [1], [2] and the references therein.

Definition 1.5. (see [11]) Let $h : J \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a positive function. We say that $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is a h-convex function, or that f belongs to the class $SX(h,I)$, if f is nonnegative and for all $x,y \in I$ and $t \in (0,1)$, then

$$f(tx+(1-t)y) \leq h(t)f(x) + h(1-t)f(y).$$

Definition 1.6. (see [12]) A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y,x) \in K$ for every $x,y \in K$ and $t \in [0,1]$.

Definition 1.7. (see [13]) The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect η, if for every $x,y \in K$ and $t \in [0,1]$,

$$f(x+t\eta(y,x)) \leq (1-t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to mapping $\eta(y,x) = y - x$, but the converse is not true. The Gauss-Jacobi type quadrature formula has the following

$$\int_a^b (x-a)^p (b-x)^q f(x)dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R_m^\star |f|, \tag{1.1}$$

for certain $B_{m,k}, \gamma_k$ and rest $R_m^\star|f|$; see [14] and the references therein. Recently, Liu [15] obtained several integral inequalities for the left-hand side of (1.1) under Definition 1.1 of the P-function. In [16], Özdemir, Set and Alomari established several integral inequalities concerning the left-hand side of (1.1) via some kinds of convexity.

The main purpose of this paper is to introduce some new concepts: generalized (s,m,φ)-preinvex Godunova-Levin functions of the second kind and explicitly generalized (s,m,φ)-preinvex Godunova-Levin functions of the second kind, and give some interesting properties for the newly introduced functions. In Section 3, some new integral inequalities for the left-hand side of (1.1) involving generalized (s,m,φ)-preinvex Godunova-Levin functions of the second kind functions along with beta function are established. Some extensions integral inequalities involving generalized $(h,(m,\varphi))$-preinvex functions are also given. In Section 4, some conclusions are provided and future research is discussed.

2. New definitions and properties

Definition 2.1. (see [17]) A set $K \subseteq \mathbb{R}^n$ is said to be m-invex with respect to mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\eta(y,x,m) \in K$ holds for each $x,y \in K$ and any $t \in [0,1]$.

Remark 2.2. In Definition 2.1, under certain conditions, mapping $\eta(y,x,m)$ could reduce to $\eta(y,x)$. For example, when $m = 1$, the m-invex set degenerates an invex set on K.

Next, we give new definitions, to be referred as generalized (s,m,φ)-preinvex Godunova-Levin functions of the second kind and explicitly generalized (s,m,φ)-preinvex Godunova-Levin functions of the second kind, respectively.

Definition 2.3. Let $K \subseteq \mathbb{R}$ be an open m-invex set with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ and $\varphi : I \longrightarrow K$ be a continuous function. For $f : K \longrightarrow \mathbb{R}$ and any fixed $s \in [0,1]$, $m \in (0,1]$, if

$$f(m\varphi(y) + t\eta(\varphi(x),\varphi(y),m)) \leq \frac{f(\varphi(x))}{t^s} + \frac{mf(\varphi(y))}{(1-t)^s}, \tag{2.1}$$

is valid for all $x,y \in I$, $t \in (0,1)$, then we say that f is a generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind with respect to η or $f \in Q^{\star 2}_{(s,m,\varphi)}$.

The function f is said to be a strictly generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind on K with respect to η, if a strict inequality holds on (2.1) for any $x,y \in I$ and $\varphi(x) \neq \varphi(y)$.

Remark 2.4. In Definition 2.3, it is worth to note that the generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind is an (s,m)-Godunova-Levin functions of the second kind on K with respect to $\eta(\varphi(x),\varphi(y),m)=\varphi(x)-m\varphi(y)$ and $\varphi(x)=x$, for all $x,y\in I$.

Definition 2.5. Let $K\subseteq\mathbb{R}$ be an open m-invex set with respect to $\eta: K\times K\times(0,1]\longrightarrow\mathbb{R}$ and $\varphi: I\longrightarrow K$ be a continuous function. For $f: K\longrightarrow\mathbb{R}$ and any fixed $s\in[0,1]$, $m\in(0,1]$, if

$$f(m\varphi(y)+t\eta(\varphi(x),\varphi(y),m))<\frac{f(\varphi(x))}{t^s}+\frac{mf(\varphi(y))}{(1-t)^s}, \tag{2.2}$$

is valid for all $x,y\in I$, $t\in(0,1)$ and $f(\varphi(x))\neq f(\varphi(y))$, then we say that f is an explicitly generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind with respect to η or $f\in Q^{\diamond 2}_{(s,m,\varphi)}$.

According to the above definitions, we now derive some interesting properties of the generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind and the explicitly generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind as follows. The proof of the following two propositions are straightforward.

Proposition 2.6. Let $\varphi: I\longrightarrow K$ be a continuous function. If $f_i: K\subseteq\mathbb{R}\longrightarrow\mathbb{R}$ $(i=1,2,\ldots,n)$ are generalized (s,m,φ)-preinvex Godunova-Levin functions of the second kind (explicitly generalized (s,m,φ)-preinvex Godunova-Levin functions of the second kind) with respect to the same $\eta: K\times K\times(0,1]\longrightarrow\mathbb{R}$, for any fixed $s\in[0,1]$, $m\in(0,1]$, then

$$f=\sum_{i=1}^{n}a_if_i,\quad a_i\geq 0,\ (i=1,2,\ldots,n),$$

is also a generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind (explicitly generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind) on K with respect to the same η for any fixed $s\in[0,1]$, $m\in(0,1]$.

Proposition 2.7. Let $\varphi: I\longrightarrow K$ be a continuous function. If $f_i: K\subseteq\mathbb{R}\longrightarrow\mathbb{R}$ $(i=1,2,\ldots,n)$ are generalized (s,m,φ)-preinvex Godunova-Levin functions of the second kind (explicitly generalized (s,m,φ)-preinvex Godunova-Levin functions of the second kind) with respect to the same $\eta: K\times K\times(0,1]\longrightarrow\mathbb{R}$, for any fixed $s\in[0,1]$, $m\in(0,1]$, then

$$f=\max\left\{f_i, i=1,2,\ldots,n\right\}$$

is also a generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind (explicitly generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind) on K with respect to the same η for any fixed $s\in[0,1]$, $m\in(0,1]$.

Next, we prove that the combination of a generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind with a positively homogenous and nondecreasing function is a generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind with respect to the same η on K for any fixed $s\in[0,1]$, $m\in(0,1]$.

Proposition 2.8. Let $\varphi: I\longrightarrow K$ be a continuous function. Assume that K be a nonempty m-invex set in $\mathbb{R}$ with respect to $\eta: K\times K\times(0,1]\longrightarrow\mathbb{R}$. Let $f: K\longrightarrow\mathbb{R}$ be a generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind with respect to η for any fixed $s\in[0,1]$, $m\in(0,1]$, and let $g: W\longrightarrow\mathbb{R}$ $(W\subseteq\mathbb{R})$ be a positively homogenous and nondecreasing function, where $\text{rang}(f)\subseteq W$. Then the composite function $g(f)$ is a generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind with respect to the same η on K for any fixed $s\in[0,1]$, $m\in(0,1]$.

Proof. Since f is a generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind, then

$$f(m\varphi(y)+t\eta(\varphi(x),\varphi(y),m))\le\frac{f(\varphi(x))}{t^s}+\frac{mf(\varphi(y))}{(1-t)^s},\quad\forall x,y\in K$$

holds for any $t\in(0,1)$. Since g is a positively homogenous and nondecreasing function, one has

$$g\left(f(m\varphi(y)+t\eta(\varphi(x),\varphi(y),m))\right)\le g\left(\frac{f(\varphi(x))}{t^s}+\frac{mf(\varphi(y))}{(1-t)^s}\right)$$

$$=\frac{g(f(\varphi(x)))}{t^s}+\frac{mg(f(\varphi(y)))}{(1-t)^s},$$

which follows that $g(f)$ is a generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind with respect to the same η on K for any fixed $s\in[0,1]$, $m\in(0,1]$.

Proposition 2.9. Let $\varphi: I\longrightarrow K$ be a continuous function. If $g_i:\mathbb{R}\longrightarrow\mathbb{R}$ $(i=1,2,\dots,n)$ are generalized (s,m,φ)-preinvex Godunova-Levin functions of the second kind with respect to the same η for any fixed $s\in[0,1]$, $m\in(0,1]$, then $M=\max\left\{x\in\mathbb{R}:\, g_i(\varphi(x))\le 0,\, i=1,2,\dots,n\right\}$ is an m-invex set.

Proof. Since $g_i(x),\ (i=1,2,\ldots,n)$ are generalized (s,m,φ)-preinvex Godunova-Levin functions of the second kind, one sees, for all $x,y\in\mathbb{R}$, that

$$g_i(m\varphi(y)+t\eta(\varphi(x),\varphi(y),m))\le\frac{g_i(\varphi(x))}{t^s}+\frac{mg_i(\varphi(y))}{(1-t)^s},\quad i=1,2,\ldots,n,$$

holds for any $t\in(0,1)$. When $x,y\in M$, we know $g_i(\varphi(x))\le 0$ and $g_i(\varphi(y))\le 0$. From the above inequality, one has

$$g_i(m\varphi(y)+t\eta(\varphi(x),\varphi(y),m))\le 0,\quad i=1,2,\ldots,n,$$

that is, $m\varphi(y)+t\eta(\varphi(x),\varphi(y),m)\in M$. Hence, M is an m-invex set related to function φ.

Proposition 2.10. Let $\varphi:\mathbb{R}_+\longrightarrow K$ be a continuous function. Assume that $f:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ is a generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind with respect to $\eta:\mathbb{R}_+\times\mathbb{R}_+\times(0,1]\longrightarrow\mathbb{R}_+$ for any fixed $s\in[0,1]$, $m\in(0,1]$. Also, suppose that f is monotone decreasing, η is monotone increasing regarding m for fixed $x,y\in\mathbb{R}_+$, and $m_1\le m_2$ $(m_1,m_2\in(0,1])$. If f is a generalized (s,m_1,φ)-preinvex Godunova-Levin function of the second kind on $\mathbb{R}_+$ with respect to η, then f is a generalized (s,m_2,φ)-preinvex Godunova-Levin function of the second kind on $\mathbb{R}_+$ with respect to η.

Proof. Since f is a generalized (s,m_1,φ)-preinvex Godunova-Levin function of the second kind, we have, for all $x,y\in\mathbb{R}_+$, that

$$f(m_1\varphi(y)+t\eta(\varphi(x),\varphi(y),m_1))\le\frac{f(\varphi(x))}{t^s}+\frac{m_1f(\varphi(y))}{(1-t)^s}.$$

Combining the conditions that f is monotone decreasing, η is monotone increasing regarding m for fixed $x,y\in\mathbb{R}_+$, and $m_1\le m_2$, we have

$$f(m_2\varphi(y)+t\eta(\varphi(x),\varphi(y),m_2))\le f(m_1\varphi(y)+t\eta(\varphi(x),\varphi(y),m_1))$$

and

$$\frac{f(\varphi(x))}{t^s}+\frac{m_1f(\varphi(y))}{(1-t)^s}\le\frac{f(\varphi(x))}{t^s}+\frac{m_2f(\varphi(y))}{(1-t)^s}.$$

Following the above two inequalities, we have

$$f(m_2\varphi(y)+t\eta(\varphi(x),\varphi(y),m_2))\le\frac{f(\varphi(x))}{t^s}+\frac{m_2f(\varphi(y))}{(1-t)^s}.$$

Hence, f is also a generalized (s,m_2,φ)-preinvex Godunova-Levin function of the second kind on $\mathbb{R}_+$ with respect to η, for any fixed $s\in[0,1]$.

Proposition 2.11. Let $\varphi : I \longrightarrow K$ be a continuous function. Assume that K is a nonempty m-invex set in $\mathbb{R}$ with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$. Let $f_i : K \longrightarrow \mathbb{R}$ $\left(i \in I = \{1,2,\ldots,n\}\right)$ be a family of real-valued functions which are explicitly generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind with respect to the same η for any fixed $s \in [0,1]$, $m \in (0,1]$ and bounded from above on K. Then the function $f(x) = \sup\{f_i(x),\, i \in I\}$ is also an explicitly generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind on K with respect to the same η for any fixed $s \in [0,1]$, $m \in (0,1]$.

Proof. Since each $f_i(x)\,(i \in I)$ is an explicitly generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind with respect to the same η for any fixed $s \in [0,1]$, $m \in (0,1]$, we have for each $i \in I$

$$f_i(m\varphi(y) + t\eta(\varphi(x),\varphi(y),m)) < \frac{f_i(\varphi(x))}{t^s} + \frac{mf_i(\varphi(y))}{(1-t)^s},$$

$\forall x,y \in K$, $t \in (0,1)$. Therefore, for each $i \in I$, we have

$$f_i(m\varphi(y) + t\eta(\varphi(x),\varphi(y),m)) < \frac{\sup_{i\in I} f_i(\varphi(x))}{t^s} + \frac{m\sup_{i\in I} f_i(\varphi(y))}{(1-t)^s},$$

$\forall x,y \in I$, $t \in (0,1)$. Taking sup of the left-hand side of the above equation, we obtain

$$\sup_{i\in I} f_i(m\varphi(y) + t\eta(\varphi(x),\varphi(y),m)) < \frac{\sup_{i\in I} f_i(\varphi(x))}{t^s} + \frac{m\sup_{i\in I} f_i(\varphi(y))}{(1-t)^s},$$

$\forall x,y \in I$, $t \in (0,1)$, that is, $f(x) = \sup\{f_i(x),\, i \in I\}$ is also an explicitly generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind on K with respect to the same η for any fixed $s \in [0,1]$, $m \in (0,1]$.

3. New integral inequalities for generalized (s,m,φ)-preinvex Godunova-Levin functions of the second kind

In this section, in order to prove our main results regarding some new integral inequalities involving generalized (s,m,φ)-preinvex Godunova-Levin functions of the second kind along with the beta function, we need the following new interesting lemma.

Lemma 3.1. *Let $\varphi : I \longrightarrow K$ be a continuous function and $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b),\varphi(a),m)] \longrightarrow \mathbb{R}$ is a continuous function on the interval of real numbers K°, with $a < b$ and $m\varphi(a) < m\varphi(a) + \eta(\varphi(b),\varphi(a),m)$. Then for any fixed*

$m\in(0,1]$ *and* $p,q>0$, *we have*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$=\eta(\varphi(b),\varphi(a),m)^{p+q+1}\int_0^1 t^p(1-t)^q f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt.$$

Proof.

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$=\eta(\varphi(b),\varphi(a),m)\int_0^1(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m)-m\varphi(a))^p$$

$$\times(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-m\varphi(a)-t\eta(\varphi(b),\varphi(a),m))^q$$

$$\times f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt$$

$$=\eta(\varphi(b),\varphi(a),m)^{p+q+1}\int_0^1 t^p(1-t)^q f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt.$$

This completes the proof.

The following definition will be used in the sequel.

Definition 3.2. The Euler beta function is defined for $x,y>0$ as

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Theorem 3.3. *Let* $\varphi: I\longrightarrow K$ *be a continuous function and* $\eta: K\times K\times(0,1]\longrightarrow\mathbb{R}$. *Assume that* $f: K=[m\varphi(a),m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]\longrightarrow\mathbb{R}$ *is a continuous function on* K°, $a<b$ *with* $m\varphi(a)<m\varphi(a)+\eta(\varphi(b),\varphi(a),m)$ *and* $\eta(\varphi(b),\varphi(a),m)\neq 0$. *If* $k>1$ *and* $|f|^{\frac{k}{k-1}}$ *is a generalized* (s,m,φ)*-preinvex Godunova-Levin function of the second kind on* K *for any fixed* $m\in(0,1]$, $s\in[0,1)$, *then for any fixed* $p,q>0$,

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq\frac{|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}}{(1-s)^{\frac{k-1}{k}}}\Big[\beta(kp+1,kq+1)\Big]^{\frac{1}{k}}\left(m|f(\varphi(a))|^{\frac{k}{k-1}}+|f(\varphi(b))|^{\frac{k}{k-1}}\right)^{\frac{k-1}{k}}.$$

Proof. Since $|f|^{\frac{k}{k-1}}$ is a generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind on K, combining with Lemma 3.1, Definition 3.2 and Hölder inequality for all $t\in(0,1)$

and for any fixed $m \in (0,1]$, $s \in [0,1)$, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$
$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\left[\int_0^1 t^{kp}(1-t)^{kq}dt\right]^{\frac{1}{k}}$$
$$\times\left[\int_0^1 \left|f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))\right|^{\frac{k}{k-1}}dt\right]^{\frac{k-1}{k}}$$
$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\Big[\beta(kp+1,kq+1)\Big]^{\frac{1}{k}}$$
$$\times\left[\int_0^1 \left(\frac{m|f(\varphi(a))|^{\frac{k}{k-1}}}{(1-t)^s}+\frac{|f(\varphi(b))|^{\frac{k}{k-1}}}{t^s}\right)dt\right]^{\frac{k-1}{k}}$$
$$=\frac{|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}}{(1-s)^{\frac{k-1}{k}}}\Big[\beta(kp+1,kq+1)\Big]^{\frac{1}{k}}\left(m|f(\varphi(a))|^{\frac{k}{k-1}}+|f(\varphi(b))|^{\frac{k}{k-1}}\right)^{\frac{k-1}{k}}.$$

The proof of Theorem 3.3 is completed.

Theorem 3.4. *Let $\varphi : I \longrightarrow K$ be a continuous function and $\eta : K\times K\times(0,1] \longrightarrow \mathbb{R}$. Assume that $f : K = [m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)] \longrightarrow \mathbb{R}$ is a continuous function on the interval of real numbers K°, $a<b$ with $m\varphi(a) < m\varphi(a)+\eta(\varphi(b),\varphi(a),m)$ and $\eta(\varphi(b),\varphi(a),m)\neq 0$. If $l\geq 1$ and $|f|^l$ is a generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind on K for any fixed $m\in(0,1]$, $s\in[0,1]$, then for any fixed $p,q>0$,*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$
$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\Big[\beta(p+1,q+1)\Big]^{\frac{l-1}{l}}$$
$$\times\Big[m|f(\varphi(a))|^l\beta\left(p+1,q-s+1\right)+|f(\varphi(b))|^l\beta\left(p-s+1,q+1\right)\Big]^{\frac{1}{l}}.$$

Proof. Since $|f|^l$ is a generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind on K, combining with Lemma 3.1, Definition 3.2 and the well-known power mean inequality for all $t\in(0,1)$ and for any fixed $m\in(0,1]$, $s\in[0,1]$, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$
$$=\eta(\varphi(b),\varphi(a),m)^{p+q+1}$$

$$\times \int_0^1 \left[t^p(1-t)^q\right]^{\frac{l-1}{l}} \left[t^p(1-t)^q\right]^{\frac{1}{l}} f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt$$

$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} \left[\int_0^1 t^p(1-t)^q dt\right]^{\frac{l-1}{l}}$$

$$\times \left[\int_0^1 t^p(1-t)^q \left|f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))\right|^l dt\right]^{\frac{1}{l}}$$

$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} \Big[\beta(p+1,q+1)\Big]^{\frac{l-1}{l}}$$

$$\times \left[\int_0^1 t^p(1-t)^q \left(\frac{m|f(\varphi(a))|^l}{(1-t)^s}+\frac{|f(\varphi(b))|^l}{t^s}\right)dt\right]^{\frac{1}{l}}$$

$$= |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} \Big[\beta(p+1,q+1)\Big]^{\frac{l-1}{l}}$$

$$\times \Big[m|f(\varphi(a))|^l \beta\,(p+1,q-s+1)+|f(\varphi(b))|^l \beta\,(p-s+1,q+1)\Big]^{\frac{1}{l}}.$$

The proof of Theorem 3.4 is completed.

Definition 3.5. Let $h : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a positive function and $\varphi : I \longrightarrow K$ be a continuous function. We say that $f : K \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is a generalized $(h,(m,\varphi))$-preinvex function with respect to $\eta : \mathbb{K}\times\mathbb{K}\times(0,1] \longrightarrow \mathbb{R}$, or that f belongs to the class $SX^{\star}(h,\varphi,K)$, if f is nonnegative and for all $x,y \in I$ and $t \in [0,1]$ we have

$$f(m\varphi(y)+t\eta(\varphi(x),\varphi(y),m)) \leq h(t)f(\varphi(x))+mh(1-t)f(\varphi(y)). \tag{3.1}$$

Definition 3.6. Let $h : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a positive function and $\varphi : I \longrightarrow K$ be a continuous function. We say that $f : K \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is an explicitly generalized $(h,(m,\varphi))$-preinvex function with respect to $\eta : \mathbb{K}\times\mathbb{K}\times(0,1] \longrightarrow \mathbb{R}$, or that f belongs to the class $SX^{\diamond}(h,\varphi,K)$, if f is nonnegative and for all $x,y \in I, t \in [0,1]$ and $f(\varphi(x)) \neq f(\varphi(y))$, we have

$$f(m\varphi(y)+t\eta(\varphi(x),\varphi(y),m)) < h(t)f(\varphi(x))+mh(1-t)f(\varphi(y)). \tag{3.2}$$

Remark 3.7. If $h(t)=\dfrac{1}{t^s}$, $t \in (0,1)$, then (3.1) is reduced to the generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind with respect to η, and any fixed $s \in [0,1], m \in (0,1]$ given in Definition 2.3.

Remark 3.8. Propositions 2.6-2.11 can be reformulated for functions f that belongs to the class $SX^{\star}(h,\varphi,K)$ or $SX^{\diamond}(h,\varphi,K)$.

Recently, Liu [15] obtained a new result for P-functions as follows.

Theorem 3.9. *Let $f:[a,b]\longrightarrow\mathbb{R}$ be continuous on $[a,b]$ such that $f\in L([a,b])$, $0\leq a<b<+\infty$. If $|f|$ is a P-function on $[a,b]$, for some fixed $p,q>0$, then*

$$\begin{aligned}&\int_a^b (x-a)^p(b-x)^q f(x)dx\\ &\leq (b-a)^{p+q+1}\beta(p+1,q+1)(|f(a)|+|f(b)|).\end{aligned}\tag{3.3}$$

Now we are in a position to prove the following interesting theorems.

Theorem 3.10. *Let $h:[0,1]\longrightarrow\mathbb{R}$ be continuous on $[0,1]$ such that $h\in L([0,1])$, and let $|h|$ be a P-function on $[0,1]$. Assume that $\varphi:I\longrightarrow K$ is a continuous function. Also, suppose that $f:K=[m\varphi(a),m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]\longrightarrow\mathbb{R}$ is a continuous function on K°, $a<b$ with $m\varphi(a)<m\varphi(a)+\eta(\varphi(b),\varphi(a),m)$. If $|f|$ is a generalized $(h,(m,\varphi))$-preinvex function with respect to $\eta:\mathbb{K}\times\mathbb{K}\times(0,1]\longrightarrow\mathbb{R}$ on K for any fixed $m\in(0,1]$, then for any fixed $p,q>0$,*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\beta(p+1,q+1)(h(0)+h(1))(m|f(\varphi(a))|+|f(\varphi(b))|).$$

Proof. Let $h:[0,1]\longrightarrow\mathbb{R}$ be continuous on $[0,1]$ such that $h\in L([0,1])$, and let $|h|$ be a P-function on $[0,1]$. Note that $|f|$ is a generalized $(h,(m,\varphi))$-preinvex function with respect to η on K for any fixed $m\in(0,1]$. For any fixed $p,q>0$, by Theorem 3.9 and Lemma 3.1, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\int_0^1 t^p(1-t)^q\big|f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))\big|dt$$

$$\leq|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\int_0^1 t^p(1-t)^q\left(mh(1-t)|f(\varphi(a))|+h(t)|f(\varphi(b))|\right)dt$$

$$\leq|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\beta(p+1,q+1)(h(0)+h(1))(m|f(\varphi(a))|+|f(\varphi(b))|).$$

Theorem 3.11. *Let $h:[0,1]\longrightarrow\mathbb{R}$ be continuous on $[0,1]$ such that $h\in L([0,1])$, and let $|h|$ be a P-function on $[0,1]$. Assume that $\varphi:I\longrightarrow K$ be a continuous function. Also, suppose that $f:K=[m\varphi(a),m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]\longrightarrow\mathbb{R}$ is a continuous function on the interval of real numbers K°, $a<b$ with $m\varphi(a)<m\varphi(a)+\eta(\varphi(b),\varphi(a),m)$. Let $l\geq 1$. If $|f|^l$ is a*

generalized $(h,(m,\varphi))$-preinvex function with respect to $\eta : \mathbb{K}\times\mathbb{K}\times(0,1] \longrightarrow \mathbb{R}$ on K for any fixed $m \in (0,1]$, then, for any fixed $p,q > 0$,

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\beta(p+1,q+1)(h(0)+h(1))^{\frac{1}{l}}(m|f(\varphi(a))|^l+|f(\varphi(b))|^l)^{\frac{1}{l}}.$$

Proof. Let $h : [0,1] \longrightarrow \mathbb{R}$ be continuous on $[0,1]$ such that $h \in L([0,1])$, and let $|h|$ be a P-function on $[0,1]$. Note $|f|^l$ is a generalized $(h,(m,\varphi))$-preinvex function with respect to η on K for any fixed $m \in (0,1]$. For any fixed $p,q > 0$, by Theorem 3.9, Lemma 3.1 and the well-known power mean inequality, we get

$$\begin{aligned}
&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx \\
&= |\eta(\varphi(b),\varphi(a),m)|^{p+q+1} \\
&\quad\times\int_0^1 \left[t^p(1-t)^q\right]^{\frac{l-1}{l}}\left[t^p(1-t)^q\right]^{\frac{1}{l}} f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt \\
&\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\left[\int_0^1 t^p(1-t)^q dt\right]^{\frac{l-1}{l}} \\
&\quad\times\left[\int_0^1 t^p(1-t)^q\left|f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))\right|^l dt\right]^{\frac{1}{l}} \\
&\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\left[\beta(p+1,q+1)\right]^{\frac{l-1}{l}} \\
&\quad\times\left[\int_0^1 t^p(1-t)^q\Big(mh(1-t)|f(\varphi(a))|^l+h(t)|f(\varphi(b))|^l\Big)dt\right]^{\frac{1}{l}} \\
&\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\beta(p+1,q+1)(h(0)+h(1))^{\frac{1}{l}}(m|f(\varphi(a))|^l+|f(\varphi(b))|^l)^{\frac{1}{l}}.
\end{aligned}$$

4. Conclusion

In this paper, we introduced a new classes of generalized (s,m,φ)-preinvex Godunova-Levin functions of the second kind. Moreover, some characterization theorems and some new integral inequalities for the left-hand side of the Gauss-Jacobi type quadrature formula involving the generalized (s,m,φ)-preinvex Godunova-Levin functions of the second kind along with the beta function are established. Some extensions integral inequalities involving the generalized $(h,(m,\varphi))$-preinvex functions are also given. We remark here that our methods may be a stimulant for further investigations concerning the Hermite-Hadamard and Ostrowski type integral inequalities for various kinds of preinvex functions involving classical integrals, Riemann-Liouville fractional integrals, k-fractional integrals, local fractional integrals, fractional integral operators, Caputo k-fractional derivatives, q-calculus, (p,q)-calculus, time scale calculus and conformable fractional integrals.

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