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MOUDAFI'S VISCOSITY APPROXIMATION METHOD FOR MONOTONE OPERATORS AND A BIFUNCTION

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Abstract. In this paper, a Moudafi's viscosity approximation algorithm is investigated for solving zero points of the sums of monotone operators and solutions of a generalized equilibrium problem. Strong convergence of the algorithm is obtained in the framework of Hilbert spaces. Applications are also provided to support the main results presented in this article.

Keywords. Moudafi's viscosity approximation; Monotone operator; Strong convergence; Zero point.

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1. Introduction

Variational inclusion problem is a useful and important extension of the variational principles with a wide range of applications in finance, economics, network analysis, transportation, elasticity and optimization. This theory has witnessed an explosive growth in theoretical advances and applications across all disciplines of pure and applied sciences. There are a substantial number of numerical methods including projection method and its variant forms, Wiener-Hopf equations, auxiliary principle and descent for solving various classes of variational inequalities and complementarity problems; see [1, 2, 3, 4, 5, 6] and the references therein. It is well known that the projection methods, Wiener-Hopf equations techniques and auxiliary principle

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techniques cannot be extended and modified for solving variational inclusions. This fact motivated to develop another technique, which involves the use of the resolvent operator associated with maximal monotone operator; see [7, 8, 9, 10, 11, 12] and the references therein. Using the resolvent operator technique, one can show that that the variational inclusions are equivalent to a fixed point problem of a composition operator.

There are several fixed point algorithms for solving the variational inclusions, such as, Mann-type iterative algorithms, Halper-type iterative algorithms, Ishikawa-type iterative algorithms and so on. Among them, Mann-type iterative algorithms and Ishikawa-type iterative algorithms are convergence in infinite dimensional spaces. To obtain the strong convergence, we study a Moudafi's approximation algorithm for solving variational inclusion with maximal monotone operators and strong monotone operators and a generalized equilibrium problem. A strong convergence theorem is established in the framework of Hilbert spaces. The rest of this article is organized as follows. In Section 2, we provide some definitions and lemmas which play an important role in this paper. In Section 3, we give the algorithm and convergence analysis in a Hilbert space. Finally, we give some applications in Section 4.

2. Preliminaries

In what follows, we always assume that H is a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let C be a nonempty, closed, and convex subset of H and let P_C be the metric projection from H onto C.

Let $A : C \to H$ be a mapping. Recall that A is said to be *monotone* iff

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

Recall that A is said to be *strongly monotone* iff there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in C.$$

For such a case, A is also said to be *α-strongly monotone*. Recall that A is said to be *inverse-strongly monotone* iff there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

For such a case, A is also said to be *α-inverse-strongly monotone*.

Recall that a set-valued mapping $M : H \rightrightarrows H$ is said to be *monotone* iff, for all $x, y \in H$, $f \in Mx$ and $g \in My$ imply $\langle x-y, f-g\rangle \geq 0$. M is *maximal* iff the graph $Graph(M)$ of R is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if, for any $(x, f) \in H \times H$, $\langle x-y, f-g\rangle \geq 0$, for all $(y, g) \in Graph(M)$ implies $f \in Rx$. For a maximal monotone operator M on H, and $r > 0$, we may define the single-valued resolvent $J_r : H \to D(M)$, where $D(M)$ denote the domain of M. It is known that $M^{-1}(0) = F(J_r)$, where $F(J_r) := \{x \in D(M) : x = J_r x\}$, and $M^{-1}(0) := \{x \in H : 0 \in Mx\}$.

Let $A : C \to H$ be a inverse-strongly monotone mapping, and let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. We consider the following generalized equilibrium problem.

$$\text{Find } x \in C \text{ such that } F(x,y) + \langle Ax, y-x\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

In this paper, the set of such an $x \in C$ is denoted by $EP(F,A)$. The generalized equilibrium problem has been extensively studied from the viewpoint of numerical computation; see [13, 14, 15, 16, 17] and the references therein.

To study problem (1.1), we may assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\limsup_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x,y)$;

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and weakly lower semi-continuous.

Let $S : C \to C$ be a mapping. $F(S)$ stands for the fixed point set of S; that is, $F(S) := \{x \in C : x = Sx\}$.

Recall that S is said to be *contractive* iff there exists a constant $\alpha \in [0,1)$ such that $\|Sx - Sy\| \leq \alpha\|x-y\|$, $\forall x, y \in C$. S is said to be *nonexpansive* iff $\|Sx - Sy\| \leq \|x-y\|$, $\forall x, y \in C$. S is said to be *firmly nonexpansive* iff $\|Sx - Sy\|^2 \leq \langle x-y, Sx-Sy\rangle$, $\forall x, y \in C$.

The following mathematical tools play an important role in this paper.

Lemma 2.1. [18] *Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1)-(A4)$. Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that $F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0$, $\forall y \in C$. Define a mapping $T_r : H \to C$ as follows:*

$$T_r x = \{z \in C : F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \quad \forall y \in C\}, \quad x \in H,$$

then the following conclusions hold:

(1) T_r *is single-valued firmly nonexpansive.*
(2) $F(T_r) = EP(F)$ *is closed and convex.*

Lemma 2.3. [19] *Assume that* $\{\alpha_n\}$ *is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.4. [20] *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in* H *and let* $\{\beta_n\}$ *be a sequence in* $(0,1)$ *with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Suppose that* $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ *for all* $n \geq 0$ *and* $\limsup_{n\to\infty} \left(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\right) \leq 0$. *Then* $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

3. Main results

Theorem 2.1. *Let* C *be a nonempty closed convex subset of a Hilbert space* H *and* F *a bifunction from* $C \times C$ *to* $\mathbb{R}$ *which satisfies (A1)-(A4). Let* $f : C \to C$ *be a* α*-contraction. Let* $A_1 : C \to H$ *be a* δ_1*-inverse-strongly monotone mapping,* $A_2 : C \to H$ *be a* δ_2*-inverse-strongly monotone mapping,* $A_3 : C \to H$ *be a* δ_3*-inverse-strongly monotone mapping,* $M_1 : H \rightrightarrows H$ *a maximal monotone operator such that* $Dom(M_1) \subset C$ *and* $M_2 : H \rightrightarrows H$ *a maximal monotone operator such that* $Dom(M_2) \subset C$. *Assume that* $\Omega := EP(F, A_3) \cap (A_1 + M_1)^{-1}(0) \cap (A_2 + M_2)^{-1}(0) \neq \emptyset$. *Let* $x_1 \in C$ *and* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} \lambda_n F(u_n, y) + \lambda_n \langle A_3 x_n, y - u_n \rangle + \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ z_n = J_{s_n}(u_n - s_n A_2 u_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n J_{r_n}(z_n - r_n A_1 z_n), \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $(0,1)$ *such that* $\alpha_n + \beta_n + \gamma_n = 1$ *for each* $n \geq 1$ *and* $\{r_n\}$, $\{s_n\}$ *and* $\{\lambda_n\}$ *are positive number sequences. Assume that the above control sequences satisfy the following restrictions:* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} |\lambda_n - \lambda_{n+1}| = \lim_{n\to\infty} |s_n - s_{n+1}| = \lim_{n\to\infty} |r_n - r_{n+1}| = 0$, $0 < a \leq$

$\lambda_n \le b < 2\delta_3$, $0 < a' \le r_n \le b' < 2\delta_1$, $0 < \bar{a} \le s_n \le \bar{b} < 2\delta_2$. Then $\{x_n\}$ converges strongly to $\bar{x} \in \Omega$, where $\bar{x} = P_\Omega f(\bar{x})$, that is, $\bar{x}$ is the unique solution to the following variational inequality

$$\langle f(\bar{x}) - \bar{x}, x - \bar{x}\rangle \le 0, \quad \forall x \in \Omega.$$

Proof. The proof is split into three steps.

Step 1. Show that sequence $\{x_n\}$ is bounded.

Fixing a element x^* in the common solution set, one has

$$\begin{aligned}
\|x_{n+1} - x^*\| &\le \alpha_n \|f(x_n) - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|J_{r_n}(z_n - r_n A_1 z_n) - J_{r_n}(x^* - r_n A_1 x^*)\| \\
&\le \alpha_n \alpha \|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| + \beta_n \|x_n - x^*\| \\
&\quad + \gamma_n \|J_{r_n}(z_n - r_n A_1 z_n) - J_{r_n}(x^* - r_n A_1 x^*)\| \\
&\le \alpha_n \alpha \|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| + \beta_n \|x_n - x^*\| \\
&\quad + \gamma_n \|(z_n - r_n A_1 z_n) - (x^* - r_n A_1 x^*)\|.
\end{aligned} \tag{3.1}$$

Note that $I - r_n A_1$ is nonexpansive. Indeed, one has

$$\begin{aligned}
&\|(I - r_n A_1)x - (I - r_n A_1)y\|^2 \\
&\le \|x - y\|^2 - 2 r_n \delta_1 \|A_1 x - A_1 y\|^2 + r_n^2 \|A_1 x - A_1 y\|^2 \\
&= \|x - y\|^2 + r_n (r_n - 2\delta_1) \|A_1 x - A_1 y\|^2 \\
&\le \|x - y\|^2, \quad \forall x, y \in C,
\end{aligned}$$

which implies that the mapping $I - r_n A_1$ is nonexpansive. In the same way, we find $I - s_n A_2$ and $I - \lambda_n A_3$ are also nonexpansive. It follows from (3.1) that

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&\le (\alpha_n \alpha + \beta_n)\|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| + \gamma_n \|J_{s_n}(u_n - s_n A_2 u_n) - J_{s_n}(x^* - s_n A_2 x^*)\| \\
&\le (\alpha_n \alpha + \beta_n)\|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| + \gamma_n \|(u_n - s_n A_2 u_n) - (x^* - s_n A_2 x^*)\| \\
&\le (\alpha_n \alpha + \beta_n)\|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| + \gamma_n \|u_n - x^*\| \\
&\le (\alpha_n \alpha + \beta_n)\|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| + \gamma_n \|T_{\lambda_n} x_n - T_{\lambda_n} x^*\| \\
&\le (1 - \alpha_n(1-\alpha))\|x_n - x^*\| + \alpha_n (1-\alpha) \frac{\|f(x^*) - x^*\|}{1-\alpha}.
\end{aligned}$$

This implies that $\|x_{n+1} - x^*\| \le \max\{\|x_1 - x^*\|, \frac{\|f(x^*) - x^*\|}{1-\alpha}\}$. This implies that $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{z_n\}$ and $\{u_n\}$. This completes Step 1.

Step 2. Show that

$$\limsup_{n\to\infty}\langle f(\bar{x})-\bar{x},x_n-\bar{x}\rangle\le 0.$$

To see this, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(\bar{x})-\bar{x},x_n-\bar{x}\rangle=\lim_{i\to\infty}\langle f(\bar{x})-\bar{x},x_{n_i}-\bar{x}\rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to w. Without loss of generality, we may assume that $x_{n_i}\rightharpoonup q$. We are in a position to show that q is in Ω.

From the iterative process, one has

$$\langle u_{n+1}-u_n,\frac{u_n-(I-\lambda_nA_3)x_n}{\lambda_n}-\frac{u_{n+1}-(I-\lambda_{n+1}A_3)x_{n+1}}{\lambda_{n+1}}\rangle\ge 0.$$

It follows that

$$\begin{aligned}\|u_{n+1}-u_n\|^2\le\|u_{n+1}-u_n\|\Big(&\|(I-\lambda_{n+1}A_3)x_{n+1}-(I-\lambda_nA_3)x_n\|\\&+\frac{|\lambda_{n+1}-\lambda_n|}{\lambda_{n+1}}\|u_{n+1}-(I-\lambda_{n+1}A_3)x_{n+1})\|\Big).\end{aligned}$$

Hence, one has

$$\begin{aligned}\|u_{n+1}-u_n\|&\le\frac{|\lambda_{n+1}-\lambda_n|}{\lambda_{n+1}}\|u_{n+1}-(I-\lambda_{n+1}A_3)x_{n+1}\|\\&\quad+\|(I-\lambda_{n+1}A_3)x_{n+1}-(I-\lambda_nA_3)x_n\|\\&\le\|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|M_1,\end{aligned}\tag{3.2}$$

where M_1 is an appropriate constant. Since J_{s_n} is firmly nonexpansive, we find that

$$\|z_{n+1}-z_n\|\le\|u_{n+1}-u_n\|+|s_n-s_{n+1}|\|A_2u_n\|.$$

This implies from (3.2) that

$$\|z_{n+1}-z_n\|\le\|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|M_1+|s_n-s_{n+1}|\|A_2u_n\|.\tag{3.3}$$

Put $y_n=J_{r_n}(z_n-r_nA_1z_n)$. Since J_{r_n} is also firmly nonexpansive, we find that

$$\begin{aligned}\|y_{n+1}-y_n\|&\le\|z_{n+1}-z_n\|+|r_n-r_{n+1}|\|A_1z_n\|\\&\le\|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|M_1\\&\quad+|s_n-s_{n+1}|\|A_2u_n\|+|r_n-r_{n+1}|\|A_1z_n\|\end{aligned}\tag{3.4}$$

Setting $x_{n+1}=(1-\beta_n)\Xi_n+\beta_n x_n$, we see that

$$\|\Xi_{n+1}-\Xi_n\|\leq\frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})y_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|f(x_n)-y_n\|$$
$$+\|y_{n+1}-y_n\|.$$

This implies from (3.4) that

$$\|\Xi_{n+1}-\Xi_n\|-\|x_{n+1}-x_n\|\leq\frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u-W_{n+1}y_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|u-W_ny_n\|$$
$$+(|r_{n+1}-r_n|+|\lambda_n-\lambda_{n+1}|+|s_n-s_{n+1}|)M_2,$$

where M_2 is an appropriate constant. It follows from Lemma 2.4 that $\lim_{n\to\infty}\|\Xi_n-x_n\|=0$. This in turn implies that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.5}$$

For any $x^*\in\Omega$, we see that

$$\begin{aligned}\|x_{n+1}-x^*\|^2&\leq\alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|y_n-x^*\|^2\\&\leq\alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|(I-r_nA_1)z_n-(I-r_nA_1)x^*\|^2\\&\leq\alpha_n\|f(x_n)-x^*\|^2+\|x_n-x^*\|^2+r_n(r_n-2\delta_1)\gamma_n\|A_1z_n-A_1x^*\|^2.\end{aligned}$$

Using the restrictions imposed on the control sequences, one has

$$\lim_{n\to\infty}\|A_1x^*-A_1z_n\|=0. \tag{3.6}$$

In the same way, one has

$$\lim_{n\to\infty}\|A_2x^*-A_2u_n\|=0. \tag{3.7}$$

Note that

$$\begin{aligned}\|x_{n+1}-x^*\|^2\leq{}&\alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\big(\|x_n-x^*\|^2\\&+\lambda_n^2\|A_3x_n-A_3x^*\|^2-2\lambda_n\langle A_3x_n-A_3x^*,x_n-x^*\rangle\big)\\\leq{}&\alpha_n\|f(x_n)-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|x_n-x^*\|^2\\&-\lambda_n\gamma_n(2\delta_3-\lambda_n)\|A_3x_n-A_3x^*\|^2.\end{aligned}$$

Using the restrictions imposed on the control sequences, one finds that

$$\lim_{n\to\infty}\|A_3x^*-A_3x_n\|=0. \tag{3.8}$$

Since T_{λ_n} is firmly nonexpansive, we find that

$$\begin{aligned}\|u_n - x^*\|^2 &\leq \langle (I-\lambda_n A_3)x_n - (I-\lambda_n A_3)x^*, u_n - x^*\rangle \\ &\leq \frac{1}{2}\big(\|x_n - x^*\|^2 + \|u_n - x^*\|^2 - \|x_n - u_n\|^2 - \lambda_n^2\|A_3x_n - A_3x^*\|^2 \\ &\quad + 2\lambda_n\langle A_3x_n - A_3x^*, x_n - u_n\rangle\big).\end{aligned}$$

This in turn implies that

$$\begin{aligned}\gamma_n\|x_n - u_n\|^2 &\leq \alpha_n\|f(x_n) - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\| \\ &\quad + 2\lambda_n\|A_3x_n - A_3x^*\|\|x_n - u_n\|.\end{aligned}$$

It follows that

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.9}$$

Since J_{s_n} is also firmly nonexpansive mapping, we see that

$$\begin{aligned}\|z_n - x^*\|^2 &\leq \frac{1}{2}\Big(\|u_n - x^*\|^2 + \|z_n - x^*\|^2 - \|u_n - z_n - s_n(A_2u_n - A_2x^*)\|^2\Big) \\ &\leq \frac{1}{2}\Big(\|u_n - x^*\|^2 + \|z_n - x^*\|^2 - \|u_n - z_n\|^2 \\ &\quad + 2s_n\|u_n - z_n\|\|A_2u_n - A_2x^*\| - s_n^2\|A_2u_n - A_2x^*\|^2\Big).\end{aligned}$$

Hence, one has

$$\begin{aligned}\gamma_n\|u_n - z_n\|^2 &\leq \alpha_n\|f(x_n) - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\| \\ &\quad + 2s_n\|u_n - z_n\|\|A_2u_n - A_2x^*\|.\end{aligned}$$

Using the restrictions imposed on the control sequences, one has

$$\lim_{n\to\infty}\|u_n - z_n\| = 0. \tag{3.10}$$

In the same way, one has

$$\lim_{n\to\infty}\|y_n - z_n\| = 0. \tag{3.11}$$

From the iterative process, one has $(1-\beta_n)\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \alpha_n\|f(x_n) - y_n\|$. This obtains that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.12}$$

Now, we are in a position to prove that $q \in (A_1 + M_1)^{-1}(0)$. Notice that $\frac{z_n - y_n}{r_n} - A_1z_n \in M_1y_n$. Let $\mu \in M_1\nu$. Since M_1 is monotone, we find that

$$\left\langle \frac{z_n - y_n}{r_n} - A_1z_n - \mu, y_n - \nu \right\rangle \geq 0.$$

This implies from (3.11) and (3.12) that $\langle -A_1q-\mu, q-\nu\rangle \geq 0$. This implies that $-A_1q \in M_1q$, that is, $q \in (A_1+M_1)^{-1}(0)$.

Now, we prove that $q \in (A_2+M_2)^{-1}(0)$. Notice that $\frac{u_n-z_n}{s_n} - A_2u_n \in M_2z_n$. Let $\mu' \in M_2\nu'$. Since M_2 is monotone, we find that

$$\left\langle \frac{u_n-z_n}{s_n} - A_2u_n - \mu', z_n - \nu' \right\rangle \geq 0.$$

Hence $\langle -A_2q-\mu', q-\nu'\rangle \geq 0$. This implies from (3.10) that $-A_2q \in M_2q$, that is, $q \in (A_2+M_2)^{-1}(0)$.

Next, we show that $q \in EP(F,A_3)$. Since $u_n = T_{\lambda_n}(I-\lambda_nA_3)x_n$, for any $y \in C$, we have

$$F(u_n,y) + \lambda_n\langle A_3x_n, y-u_n\rangle + \langle y-u_n, u_n-x_n\rangle \geq 0.$$

Replacing n by n_i, we find from $(A2)$ that

$$\langle A_3x_{n_i}, y-u_{n_i}\rangle + \langle y-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{\lambda_{n_i}}\rangle \geq F(y,u_{n_i}), \quad \forall y \in C.$$

Putting $y_t = ty+(1-t)q$ for any $t \in (0,1]$ and $y \in C$, we see that $y_t \in C$. It follows that

$$\begin{aligned}
&\langle y_t - u_{n_i}, A_3y_t\rangle \\
&\geq \langle y_t - u_{n_i}, A_3y_t\rangle - \langle A_3x_{n_i}, y_t-u_{n_i}\rangle - \langle y_t-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{\lambda_{n_i}}\rangle + F(y_t,u_{n_i}) \\
&= \langle y_t - u_{n_i}, A_3y_t - A_3u_{n_i}\rangle + \langle y_t-u_{n_i}, A_3u_{n_i} - A_3x_{n_i}\rangle - \langle y_t-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{\lambda_{n_i}}\rangle \\
&\quad + F(y_t,u_{n_i}).
\end{aligned}$$

Using the monotonicity of A_3, we obtain from $(A4)$ that $\langle y_t-q, A_3y_t\rangle \geq F(y_t,q)$. From $(A1)$ and $(A4)$, we see that

$$\begin{aligned}
0 = F(y_t,y_t) &\leq tF(y_t,y) + (1-t)F(y_t,q) \\
&\leq tF(y_t,y) + (1-t)\langle y_t-q, A_3y_t\rangle \\
&= tF(y_t,y) + (1-t)t\langle y-q, A_3y_t\rangle.
\end{aligned}$$

It follows that $0 \leq F(y_t,y) + (1-t)\langle y-w, A_3y_t\rangle$, $\forall y \in C$. It follows from $(A3)$ that $q \in EP(F,A_3)$. This completes Step 2.

Step 3. Show that $x_n \to \bar{x}$ as $n \to \infty$.

Note that

$$
\begin{aligned}
\|x_{n+1}-\bar{x}\|^2 &\le \alpha_n\langle f(x_n)-\bar{x},x_{n+1}-\bar{x}\rangle+\beta_n\|x_n-\bar{x}\|\|x_{n+1}-\bar{x}\|+\gamma_n\|y_n-\bar{x}\|\|x_{n+1}-\bar{x}\| \\
&\le \alpha_n\langle f(x_n)-f(\bar{x}),x_{n+1}-\bar{x}\rangle+\alpha_n\langle f(\bar{x})-\bar{x},x_{n+1}-\bar{x}\rangle \\
&\quad+\frac{1-\alpha_n}{2}(\|x_n-\bar{x}\|^2+\|x_{n+1}-\bar{x}\|^2) \\
&\le \alpha_n\alpha\|x_n-\bar{x}\|\|x_{n+1}-\bar{x}\|+\alpha_n\langle f(\bar{x})-\bar{x},x_{n+1}-\bar{x}\rangle \\
&\quad+\frac{1-\alpha_n}{2}(\|x_n-\bar{x}\|^2+\|x_{n+1}-\bar{x}\|^2) \\
&\le \alpha_n\langle f(\bar{x})-\bar{x},x_{n+1}-\bar{x}\rangle+\frac{1-\alpha_n(1-\alpha)}{2}(\|x_n-\bar{x}\|^2+\|x_{n+1}-\bar{x}\|^2).
\end{aligned}
$$

This implies that

$$
\|x_{n+1}-\bar{x}\|^2\le(1-\alpha_n(1-\alpha))\|x_n-\bar{x}\|^2+2\alpha_n\langle f(\bar{x})-\bar{x},x_{n+1}-\bar{x}\rangle.
$$

Using Lemma 2.3, we find that $\lim_{n\to\infty}\|x_n-\bar{x}\|=0$. This completes the proof.

4. Applications

In this section, we consider some applications of the main results. Recall that a mapping $T:C\to C$ is said to be a k-strict pseudo-contraction if there exists a constant $k\in[0,1)$ such that

$$
\|Tx-Ty\|^2\le k\|(I-T)x-(I-T)y\|^2+\|x-y\|^2,\quad \forall x,y\in C.
$$

If $k=0$, then the class of strict pseudo-contractions is reduce to the class of nonexpansive mappings. Putting $A=I-T$, where $T:C\to C$ is a k-strict pseudo-contraction, we find that A is $\frac{1-k}{2}$-inverse-strongly monotone.

Next, we consider common fixed points of strict pseudo-contractions.

Theorem 4.1. *Let C be a nonempty closed convex subset of a Hilbert space H and let F be a bifunction from $C\times C$ to $\mathbb{R}$ which satisfies (A1)-(A4). Let f be a contraction on C. Let $T_1:C\to H$ be a k_1-strict pseudo-contraction, $T_2:C\to H$ be a k_2-strict pseudo-contraction, $A_3:C\to H$ be a δ-inverse-strongly monotone mapping, and $\{S_i:C\to C\}$ be a family of infinitely nonexpansive mappings. Assume that $\Omega:=\cap EP(F,A_3)\cap F(T_1)\cap F(T_2)\neq\emptyset$. Let $x_1\in C$ and*

$\{x_n\}$ be a sequence generated by

$$\begin{cases} \lambda_n F(u_n,y)+\lambda_n\langle A_3x_n,y-u_n\rangle+\langle y-u_n,u_n-x_n\rangle\geq 0, \quad \forall y\in C\\ z_n=(1-s_n)u_n+s_nT_2u_n,\\ x_{n+1}=\alpha_n f(x_n)+\beta_n x_n+\gamma_n\big((1-r_n)u_n+r_nT_1u_n\big), \quad \forall n\geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\alpha_n+\beta_n+\gamma_n=1$ for each $n\geq 1$ and $\{r_n\}$, $\{s_n\}$ and $\{\lambda_n\}$ are positive number sequences. Assume that the above control sequences satisfy the restrictions as in Theorem 3.1. Then $\{x_n\}$ converges strongly to $\bar{x}=P_\Omega f(\bar{x})$.

Proof. Taking $A_i=I-T_i$, wee see that $A_i:C\to H$ is a δ_i-strict pseudo-contraction with $\delta_i=\frac{1-k_i}{2}$ and $F(T_i)=VI(C,A_i)$ for $i=1,2$. In view of Theorem 3.1, we find the desired conclusion immediately.

Recall that the classical variational inequality is to find an $x\in C$ such that

$$\langle Ax,y-x\rangle\geq 0, \quad \forall y\in C.$$

In this paper, we use $VI(C,A)$ to denote the solution set of the inequality. It is known that $x\in C$ is a solution of the inequality iff x is a fixed point of the mapping $P_C(I-rA)$, where $r>0$ is a constant, I stands for the identity mapping. If A is α-inverse-strongly monotone and $r\in(0,2\alpha]$, then the mapping $I-rA$ is nonexpansive. It follows that $VI(C,A)$ is closed, and convex.

Let $g:H\to(-\infty,+\infty]$ be a proper convex lower semicontinuous function. Then the subdifferential ∂g of g is defined as follows:

$$\partial fg(x)=\{y\in H: g(z)\geq g(x)+\langle z-x,y\rangle, \quad z\in H\}, \quad \forall x\in H.$$

From Rockafellar [21], we know that ∂g is maximal monotone. It is not hard to verify that $0\in\partial g(x)$ if and only if $g(x)=\min_{y\in H}g(y)$.

Let I_C be the indicator function of C, i.e., $I_C(x)=0$, if $x\in C$ and $I_C(x)=+\infty$, if $x\notin C$. Since I_C is a proper lower semicontinuous convex function on H, we see that the subdifferential ∂I_C of I_C is a maximal monotone operator. It is clearly that $J_rx=P_Cx$, $\forall x\in H$, $(A_1+\partial I_C)^{-1}(0)=VI(C,A_1)$ and $(A_2+\partial I_C)^{-1}(0)=VI(C,A_2)$.

Theorem 4.2. *Let C be a nonempty closed convex subset of a Hilbert space H and let f be a contraction on C. Let F be a bifunction from $C\times C$ to $\mathbb{R}$ which satisfies $(A1)$-$(A4)$. Let $A_1:C\to H$ be a δ_1-inverse-strongly monotone mapping, $A_2:C\to H$ be a δ_2-inverse-strongly monotone* *mapping, $A_3 : C \to H$ be a δ_3-inverse-strongly monotone mapping, and $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings. Assume that $\Omega := EP(F,A_3) \cap VI(C,A_1) \cap VI(C,A_2) \neq \emptyset$. Let $x_1 \in C$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} \lambda_n F(u_n,y) + \lambda_n \langle A_3 x_n, y-u_n \rangle + \langle y-u_n, u_n-x_n \rangle \geq 0, \quad \forall y \in C, \\ z_n = P_C(u_n - s_n A_2 u_n), \\ y_n = P_C(z_n - r_n A_1 z_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$ for each $n \geq 1$ and $\{r_n\}$, $\{s_n\}$ and $\{\lambda_n\}$ are positive number sequences. Assume that the above control sequences satisfy the restrictions as in Theorem 3.1. Then $\{x_n\}$ converges strongly to $\bar{x} = P_\Omega f(\bar{x})$.

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REFERENCES

[1] X.P. Ding, Perturbed proximal point algorithms for generalized quasi variational in clusions, J. Math. Anal. Appl. 210 (1997), 88-101.

[2] M.A. Noor, Two-step approximation schemes for multivalued quasi variational inclusions, Nonlinear Funct. Anal. Appl. 7 (2002), 1-14.

[3] M.A. Noor, Auxiliary principle for generalized mixed variational-like inequalities, J. Math. Anal. Appl. 215 (1997), 75-85.

[4] M.A. Noor, A predictor-corrector algorithm for general variational inequalities, Appl. Math. Lett. 14 (2001), 53-58.

[5] P. Shi, Equivalence of variational inequalities with WienerHopf equations, Proc. Amer. Math. Soc. 111 (1991), 339-346.

[6] D.V. Hieu, A. Moudafi, A barycentric projected-subgradient algorithm for equilibrium problems, J. Nonlinear Var. Anal. 1 (2017), 43-59.

[7] S.Y. Cho, X. Qin, L. Wang, Strong convergence of a splitting algorithm for treating monotone operators, Fixed Point Theory Appl. 2014 (2014), Article ID 94.

[8] S.Y. Cho, S.M. Kang, Approximation of common solutions of variational inequalities via strict pseudocontractions, Acta Math. Sci. 32 (2012), 1607-1618.

[9] M. Zhang, Iterative algorithms for common elements in fixed point sets and zero point sets with applications Fixed Point Theory Appl. 2012 (2012), Article ID 21.

[10] X. Qin, S.Y. Cho, L. Wang, A regularization method for treating zero points of the sum of two monotone operators, Fixed Point Theory Appl. 2014 (2014), Article ID 75.

[11] X. Qin, S.Y. Cho, L. Wang, Convergence of splitting algorithms for the sum of two accretive operators with applications, Fixed Point Theory Appl. 2014 (2014), Article ID 166.

[12] J.U. Jeong, Fixed point solutions of generalized mixed equilibrium problems and variational inclusion problems for nonexpansive semigroups, Fixed Point Theory Appl. 2014 (2014), Article ID 57.

[13] S.A. Khan, Auxiliary problem and algorithm for a generalized mixed equilibrium problem, J. Egyptian Math. Soc. 23 (2015), 155-160.

[14] L. Sun, Hybrid methods for common solutions in Hilbert spaces with applications, J. Inequal. Appl. 2014 (2014), Article ID 183.

[15] I. Inchan, Viscosity iteration method for generalized equilibrium problems and fixed point problems of finite family of nonexpansive mappings, Appl. Math. Comput. 219 (2012), 2949-2959.

[16] Y. Shehu, An iterative method for nonexpansive semigroups, variational inclusions and generalized equilibrium problems, Math. Comput. Modelling 55 (2012), 1301-1314.

[17] X. Qin, S.Y. Cho, S.M. Kang, Some results on generalized equilibrium problems involving a family of nonexpansive mappings, Appl. Math. Comput. 217 (2010), 3113-3126.

[18] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Stud. 63 (1994), 123-145.

[19] L.S. Liu, Ishikawa and Mann iteration process with errors for nonlinear strongly accretive mappings in Banach spaces, J. Math. Anal. Appl. 194 (1995), 114-125.

[20] T. Suzuki, Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochne integrals, J. Math. Anal. Appl. 305 (2005), 227-239.

[21] R.T. Rockafellar, On the maximality of sums of nonlinear monotone operators, Trans. Amer. Math. Soc. 149 (1970), 75-88.