Commun. Optim. Theory 2017 (2017), Article ID 3 https://doi.org/10.23952/cot.2017.3

Communications in Optimization Theory

Available online at http://cot.mathres.org

WEAK CONVERGENCE OF AN ITERATIVE ALGORITHMS FOR ZERO POINT PROBLEMS IN A BANACH SPACE

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Abstract. Common solutions of two convex optimization problems are investigated based on an iterative algorithm. A weak convergence theorem is obtained in a q-uniformly smooth and uniformly convex Banach space.

Keywords. Algorithm; Accretive operator; Convex optimization; Fixed point; Monotone operator.

2010 Mathematics Subject Classification. 47H05, 90C33.

1. Introduction

Let H be a Hilbert space and let C be a nonempty closed and convex subset of H. Let $T : H \to 2^H$ be a maximal monotone operator. The corresponding zero problem of operator T is to find $\bar{x} \in C$ such that $0 \in T\bar{x}$. A classical method for solving the problem is the proximal point algorithm, proposed by Martinet [1,2] and generalized by Rockafellar [3,4]. In the case of $T = A + B$, where A and B are monotone operators, the problem is reduced to the following inclusion problem: find $\bar{x} \in C$ such that $0 \in (A+B)\bar{x}$. The solution set of the inclusion problem is denoted by $(A+B)^{-1}(0)$.

A splitting method for the inclusion problem means an iterative algorithm for which each iteration involves only with the individual operators A and B, but not the sum $A+B$. Splitting methods have recently received much attention due to the fact that many nonlinear problems

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Received July 30, 2016; Accepted November 24, 2016.

arising in applied areas such as image recovery, signal processing, and machine learning are mathematically modeled as a nonlinear operator equation and this operator is decomposed as the sum of two possibly simpler nonlinear operators. Splitting methods for linear equations were introduced by Peaceman and Rachford [5] and Douglas and Rachford [6]. Extensions to nonlinear equations in Hilbert spaces were carried out by Kellogg [7] and Lions and Mercier [8]. The central problem is to iteratively find a zero of the sum of two monotone operators A and B in a Hilbert space H. Many problems can be formulated as the inclusion problem. For instance, a stationary solution to the initial value problem of the evolution equation $0 \in \frac{\partial u}{\partial t} + Ku$, $u(0) = u_0$ can be recast as the inclusion problem when the governing maximal monotone K is of the form $K = A + B$. To solve the inclusion problem Lions and Mercier [8] introduced the nonlinear Peaceman-Rachford and Douglas-Rachford splitting iterative algorithms which generate a sequence $\{x_n\}$ by the recursion

$$x_{n+1} = (2(I + r_n A)^{-1} - I)(2(I + r_n B)^{-1} - I)x_n \tag{1.1}$$

and respectively, a sequence $\{y_n\}$ by the recursion

$$y_{n+1} = (I + r_n A)^{-1}(2(I + r_n B)^{-1} - I)y_n + (I - (I + r_n B)^{-1})y_n. \tag{1.2}$$

The nonlinear Peaceman-Rachford algorithm (1.1) fails, in general, to converge (even in the weak topology in the infinite-dimensional setting). This is due to the fact that the generating operator $(2(I + r_n A)^{-1} - I)(2(I + r_n B)^{-1} - I)$ for algorithm (1.1) is merely nonexpansive. However, the mean averages of $\{y_n\}$ can be weakly convergent [9]. The nonlinear Douglas-Rachford algorithm (1.2) always converges in the weak topology, since the generating operator $(I + r_n A)^{-1}(2(I + r_n B)^{-1} - I) + (I - (I + r_n B)^{-1})$ for this algorithm is firmly nonexpansive, namely, the operator is of the form $\frac{I+T}{2}$, where T is a nonexpansive mapping.

The aim of this paper is to present a forward-backward splitting method for solving zero point problems of two accretive operators in a q-uniformly smooth and uniformly convex Banach space. The main results mainly improve the corresponding results in [10].

2. Preliminaries

Let E be a real Banach space with the dual E^*. Given of continuous strictly increasing function: $\varphi : R^+ \to R^+$, where R^+ denotes the set of nonnegative real numbers, such that $\varphi(0) = 0$ and $\lim_{r\to\infty} \varphi(r) = \infty$, we associate with it a (possibly mutivalued) generalized duality map $\mathfrak{J}_\varphi(x) : E \to 2^{E^*}$, defined as $\mathfrak{J}_\varphi(x) := \{x^* \in E^* : x^*(x) = \varphi(\|x\|)\|x\|, \varphi(\|x\|) = \|x^*\|\}, \forall x \in E$. In this paper, we use the generalized duality map associated with the gauge function $\varphi(t) = t^{q-1}$ for $q > 1$,

$$\mathfrak{J}_q(x) := \{x^* \in E^* : \langle x^*, x\rangle = \|x\|^q, \|x\|^{q-1} = \|x^*\|\}, \quad \forall x \in E.$$

Let $B_E = \{x \in E : \|x\| = 1\}$. Let $\rho_E : [0,\infty) \to [0,\infty)$ be the modulus of smoothness of E by

$$\rho_E(t) = \sup\{\frac{\|x+y\| - \|y-x\|}{2} - 1 : \|y\| \le t, x \in B_E\}.$$

A Banach space E is said to be uniformly smooth if $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$. Let $q > 1$. E is said to be q-uniformly smooth if there exists a fixed constant $k > 0$ such that $\rho_E(t) \le kt^q$. The modulus of convexity of E is the function $\delta_E(\varepsilon) : (0,2] \to [0,1]$ defined by $\delta_E(\varepsilon) = \inf\{1 - \frac{\|x+t\|}{2} : \|y\| = \|x\| = 1, \|y-x\| \ge \varepsilon\}$. Recall that E is said to be uniformly convex if $\delta_E(\varepsilon) > 0$ for any $\varepsilon \in (0,2]$. Let $p > 1$. We say that E is p-uniformly convex if there exists a constant $k_q > 0$ such that $\delta_E(\varepsilon) \ge k_p\varepsilon^p$ for any $\varepsilon \in (0,2]$. It is known that E is p-uniformly convex if and only if E^* is q-uniformly smooth, where $p+q = pq$.

Let $T : C \to C$ be a mapping. The fixed point set of T is denoted by $F(T)$. Recall that T is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x, y\|, \quad \forall x, y \in C.$$

Let I denote the identity operator on E. An operator $A \subset E \times E$ with domain $D(A) = \{z \in E : Az \ne \emptyset\}$ and range $R(A) = \cup\{Az : z \in D(A)\}$ is said to be accretive if, for $t > 0$ and $x, y \in D(A)$,

$$\|x - y\| \le \|t(u-v) + x - y\|, \quad \forall v \in Ay, u \in Ax.$$

In this paper, we use $A^{-1}(0)$ to denote the set of zeros of A. It follows from Kato [11] that A is accretive if and only if, for $x, y \in D(A)$, there exists $\mathrm{j}_q(x_1 - x_2)$ such that

$$\langle u - v, \mathrm{j}_q(x-y)\rangle \ge 0.$$

An accretive operator A is said to be m-accretive if $R(I+rA)=E$ for all $r>0$. For an accretive operator A, we can define a single valued mapping $J_r^A : R(I+rA) \to D(A)$ by $J_r^A = (I+rA)^{-1}$ for each $r>0$.

Recall that a single valued operator $A : C \to E$ is said to be α-inverse strongly accretive if there exists a constant $\alpha > 0$ and some $\mathrm{j}_q(x-y) \in \mathfrak{J}_q(x-y)$ such that

$$\langle Ax - Ay, \mathrm{j}_q(x-y)\rangle \geq \alpha \|Ax - Ay\|^q, \quad \forall x, y \in C.$$

Zero points of accretive operators have been extensively investigated by iterative methods; see [12-23] and the references therein. In this article, common zero points of two accretive operators are investigated based on a splitting iterative algorithm. A weak convergence theorem is obtained in a q-uniformly smooth and uniformly convex Banach space. Some applications are also provided in Hilbert spaces. In order to obtain our main results, we also need the following lemmas.

Lemma 2.1. [24] *Let E be a real uniformly convex Banach space such that its dual E^* has the Kadec-Klee property. Suppose that $\{x_n\}$ is a bounded sequence such that $\lim_{n\to\infty}\|ax_n + (1-a)p_1 - p_2\|$ exists for all $a \in [0,1]$ and $p_1, p_2 \in \omega_w(x_n)$, where $\omega_w(x_n) : \{x : \exists x_{n_i} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$ Then $\omega_w(x_n)$ is a singleton.*

Lemma 2.2. [25] *Let E be a real uniformly convex Banach space, C a nonempty closed, and convex subset of E and $T : C \to C$ a nonexpansive mapping. Then $I-T$ is demiclosed at zero.*

Lemma 2.3. [26] *Let E be a real q-uniformly smooth Banach space. Then the following inequality holds:*

$$\|x+y\|^q \leq q\langle y, \mathfrak{J}_q(x+y)\rangle + \|x\|^q$$

and

$$\|x+y\|^q \leq q\langle y, \mathfrak{J}_q(x)\rangle + K_q\|y\|^q + \|x\|^q, \quad \forall x, y \in E,$$

where K_q is some fixed positive constant.

Lemma 2.4. [27] *Let E be a real uniformly convex Banach space and let C be a nonempty closed convex and bounded subset of E. Then there is a strictly increasing and continuous* *convex function $\psi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ such that, for every Lipschitzian continuous mapping $T : C \to C$ and, for all $x, y \in C$ and $t \in [0,1]$, the following inequality holds:*

$$\|T(tx+(1-t)y)-(tTx+(1-t)Ty)\| \le L\psi^{-1}\big(\|x-y\|-L^{-1}\|Tx-Ty\|\big),$$

where $L \ge 1$ is the Lipschitz constant of T.

Lemma 2.5. [26] *Let $p > 1$ and $r > 0$ be two fixed real numbers. Then a Banach space E is uniformly convex if and only if there exists a continuous strictly increasing convex function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ such that*

$$\|ax+(1-a)y\|^p \le a\|x\|^p+(1-a)\|y\|^p-\big(a^p(1-a)+(1-a)^pa\big)\varphi(\|x-y\|),$$

for all $x, y \in B_r(0) := \{x \in E : \|x\| \le r\}$ and $a \in [0,1]$.

3. Main results

Theorem 3.1. *Let E be a real uniformly convex and q-uniformly smooth Banach space with the constant K_q and let C be a closed convex subset of E. Let $A : C \to E$ be an α-inverse strongly accretive operator and let $B : Dom(B) \subset E \to 2^E$ be an m-accretive operator such that $Dom(B) \subset C$. Let $\{x_n\}$ be a sequence generated in the following manner: $x_0 \in C$ and*

$$x_{n+1} = (1-\alpha_n)(I+r_nB)^{-1}(x_n-r_nAx_n)+\alpha_nx_n, \quad \forall n \ge 0,$$

where $\{\alpha_n\}$ and $\{r_n\}$ are real sequences satisfying the following restrictions: $0 \le \alpha_n \le \alpha$ and $0 < r \le r_n \le r' < (\frac{q\alpha}{K_q})^{\frac{1}{q-1}}$. Assume that $(A+B)^{-1}(0) \neq \emptyset$. Then $\{x_n\}$ converges weakly to some zero of $A+B$.

Proof. From Lemma 2.3 and the restriction imposed on $\{r_n\}$, one has

$$\begin{aligned}
&\|(I-r_nA)x-(I-r_nA)y\|^q\\
&\le \|x-y\|^q - qr_n\langle Ax-Ay, \mathfrak{J}_q(x-y)\rangle + K_q r_n^q\|Ax-Ay\|^q\\
&\le \|x-y\|^q - qr_n\alpha\|Ax-Ay\|^q + K_q r_n^q\|Ax-Ay\|^q\\
&= \|x-y\|^q - (\alpha q - K_q r_n^{q-1})r_n\|Ax-Ay\|^q\\
&\le \|x-y\|^q.
\end{aligned}$$

Fixing $p\in(A+B)^{-1}(0)$, one has

$$\begin{aligned}
\|x_{n+1}-p\| &\le \alpha_n\|x_n-p\| + (1-\alpha_n)\|J_{r_n}(x_n-r_nAx_n)-p\|\\
&\le \alpha_n\|x_n-p\| + (1-\alpha_n)\|(x_n-r_nAx_n)-(p-r_nA)p\|\\
&\le \|x_n-p\|.
\end{aligned}$$

This shows that $\lim_{n\to\infty}\|x_n-p\|$ exists, in particular, $\{x_n\}$ is bounded. Using Lemma 2.3, we find that

$$\begin{aligned}
&\|(I-r_nA)x_n-(I-r_nA)p\|^q\\
&\le \|x_n-p\|^q - qr_n\langle Ax_n-Ap, \mathfrak{J}_q(x_n-p)\rangle + K_q r_n^q\|Ax_n-Ap\|^q \qquad (3.1)\\
&\le \|x_n-p\|^q - (\alpha q - K_q r_n^{q-1})r_n\|Ax_n-Ap\|^q.
\end{aligned}$$

Putting $y_n = J_{r_n}(x_n-r_nAx_n)$, we find from Lemma 2.5 that

$$\begin{aligned}
&\Big\|\frac{1}{2}(y_n-p)+\frac{1}{2}\big((I-r_nA)x_n-(I-r_nA)p\big)\Big\|^q\\
&\le \frac{1}{2}\|y_n-p\|^q + \frac{1}{2}\|(I-r_nA)x_n-(I-r_nA)p\|^q\\
&\quad - \frac{1}{2^q}\varphi\Big(\|(y_n-p)-\big((I-r_nA)x_n-(I-r_nA)p\big)\|\Big) \qquad (3.2)\\
&\le \|(I-r_nA)x_n-(I-r_nA)p\|^q - \frac{1}{2^q}\varphi\Big(\|(y_n-p)-\big((I-r_nA)x_n-(I-r_nA)p\big)\|\Big).
\end{aligned}$$

Substituting (3.1) into (3.2), we arrive at

$$\begin{aligned}
&\Big\|\frac{1}{2}(y_n-p)+\frac{1}{2}\big((I-r_nA)x_n-(I-r_nA)p\big)\Big\|^q\\
&\le \|x_n-p\|^q - (\alpha q - K_q r_n^{q-1})r_n\|Ax_n-Ap\|^q \qquad (3.3)\\
&\quad - \frac{1}{2^q}\varphi\Big(\|(y_n-p)-\big((I-r_nA)x_n-(I-r_nA)p\big)\|\Big).
\end{aligned}$$

Note that

$$\begin{aligned}\|y_n-p\| &\le \Big\|y_n-p+\frac{r_n}{2}\Big(\frac{x_n-r_nAx_n-y_n}{r_n}-\frac{(I-r_nA)p-p}{r_n}\Big)\Big\| \\ &= \Big\|\frac{1}{2}(y_n-p)+\frac{1}{2}\big((I-r_nA)x_n-(I-r_nA)p\big)\Big\|.\end{aligned} \tag{3.4}$$

Combining (3.3) with (3.4), we see that

$$\begin{aligned}\|y_n-p\|^q \le \|x_n-p\|^q &- (\alpha q-K_q r_n^{q-1})r_n\|Ax_n-Ap\|^q \\ &- \frac{1}{2^q}\varphi\Big(\|(y_n-p)-\big((I-r_nA)x_n-(I-r_nA)p\big)\|\Big).\end{aligned} \tag{3.5}$$

Since $\|\cdot\|^q$ is convex, we find that

$$\begin{aligned}\|x_{n+1}-p\|^q &\le \alpha_n\|x_n-p\|^q+(1-\alpha_n)\|y_n-p\|^q \\ &\le \|x_n-p\|^q-(\alpha q-K_q r_n^{q-1})r_n(1-\alpha_n)\|Ax_n-Ap\|^q \\ &\quad -(1-\alpha_n)\frac{1}{2^q}\varphi\Big(\|(y_n-p)-\big((I-r_nA)x_n-(I-r_nA)p\big)\|\Big).\end{aligned}$$

It follows that

$$\lim_{n\to\infty}\|Ax_n-Ap\|=0 \tag{3.6}$$

and

$$\lim_{n\to\infty}\|y_n-x_n+r_nAx_n-r_nAp\|=0. \tag{3.7}$$

Since

$$\|y_n-x_n\|\le\|y_n-x_n+r_nAx_n-r_nAp\|+r_n\|Ax_n-Ap\|,$$

we find from (3.6) and (3.7) that

$$\lim_{n\to\infty}\|J_{r_n}(x_n-r_nAx_n)-x_n\|=0. \tag{3.8}$$

Notice that

$$\Big\langle\frac{x_n-J_r(I-rA)x_n}{r}-\frac{x_n-J_{r_n}(I-r_nA)x_n}{r_n},\mathfrak{J}_q\big(J_r(I-rA)x_n-J_{r_n}(I-r_nA)x_n\big)\Big\rangle\ge 0.$$

Hence, we find that

$$\begin{aligned}&\|J_r(I-rA)x_n-J_{r_n}(I-r_nA)x_n\|^q \\ &\le \frac{r_n-r}{r_n}\langle x_n-J_{r_n}(I-r_nA)x_n,\mathfrak{J}_q\big(J_r(I-rA)x_n-J_{r_n}(I-r_nA)x_n\big)\rangle \\ &\le \|x_n-J_{r_n}(I-r_nA)x_n\|\|J_r(I-rA)x_n-J_{r_n}(I-r_nA)x_n\|^{q-1}.\end{aligned}$$

This implies that $\|J_r(I-rA)x_n - J_{r_n}(I-r_nA)y_n\| \le \|x_n - J_{r_n}(I-r_nA)x_n\|$. It follows that

$$\begin{aligned}\|J_r(I-rA)x_n - x_n\| &\le \|J_r(I-rA)x_n - J_{r_n}(I-r_nA)x_n\| \\ &\quad + \|J_{r_n}(I-r_nA)x_n - x_n\| \\ &\le 2\|J_{r_n}(I-r_nA)x_n - x_n\|.\end{aligned}$$

From (3.8), we arrive at

$$\lim_{n\to\infty} \|J_r(x_n - rAx_n) - x_n\| = 0. \tag{3.9}$$

Define mappings $T_n : C \to C$ by

$$T_n x := \alpha_n x + (1-\alpha_n) J_{r_n}((I-r_nA)x), \quad \forall x \in C.$$

Set

$$T_{n+m-1}T_{n+m-2}\cdots T_n = S_{n,m}, \quad \forall n,m \ge 1.$$

Then $S_{n,m}x_n = x_{n+m}$ and $S_{n,m}$ is nonexpansive. For all $t \in [0,1]$ and $n,m \ge 1$, put

$$a_n(t) = \|tx_n + (1-t)p_1 - p_2\|,$$

and

$$b_{n,m} = \|S_{n,m}(tx_n + (1-t)p_1) - (tx_{n+m} + (1-t)p_1)\|,$$

where p_1 and p_2 are zeros of $A+B$. Using Lemma 2.4, we find that

$$\begin{aligned}b_{n,m} &\le \psi^{-1}\big(\|x_n - p_1\| - \|S_{n,m}x_n - S_{n,m}p_1\|\big) \\ &= \psi^{-1}\big(\|x_n - p_1\| - \|x_{n+m} - p_1 + p_1 - S_{n,m}p_1\|\big) \\ &\le \psi^{-1}\big(\|x_n - p_1\| - (\|x_{n+m} - p_1\| - \|p_1 - S_{n,m}p_1\|)\big) \\ &\le \psi^{-1}\big(\|x_n - p_1\| - \|x_{n+m} - p_1\|\big).\end{aligned}$$

It follows that $\{b_{n,m}\}$ converges uniformly to zero as $n \to \infty$ for all $m \ge 1$. Hence,

$$\begin{aligned}a_{n+m}(t) &\le b_{n,m} + \|S_{n,m}(tx_n + (1-t)p_1) - p_2\| \\ &\le b_{n,m} + \|S_{n,m}(tx_n + (1-t)p_1) - S_{n,m}p_2\| + \|S_{n,m}p_2 - p_2\| \\ &\le b_{n,m} + a_n(t) + \|S_{n,m}p_2 - p_2\| \\ &\le b_{n,m} + a_n(t).\end{aligned}$$

Taking $\limsup$ as $m \to \infty$ and then the $\liminf$ as $n \to \infty$, we find that $\lim_{n\to\infty} a_n(t)$ for any $t \in [0,1]$. In view of Lemma 2.2, we see that $\omega_w(x_n) \subset (A+B)^{-1}(0)$. This implies from Lemma 2.1 that $\omega_w(x_n)$ is just one point. This proves the proof.

Remark 3.2. The framework of the space in Theorem 3.1 is applicable to L_p, where $p > 1$.

Corollary 3.3. *Let E be a real Hilbert space and let C be a closed convex subset of E. Let $A : C \to E$ be an α-inverse strongly monotone operator and let $B : Dom(B) \subset E \to 2^E$ be a maximal monotone operator such that $Dom(B) \subset C$. Let $\{x_n\}$ be a sequence generated in the following manner: $x_0 \in C$ and $x_{n+1} = Prog_C(x_n - r_n A x_n), \quad \forall n \geq 0$, where $\{r_n\}$ is a real sequence satisfying the following restrictions: $0 < r \leq r_n \leq r' < 2\alpha$. Assume that $VI(C,A) \neq \emptyset$. Then $\{x_n\}$ converges weakly to some zero of $VI(C,A)$.*

Acknowledgments

The author thanks the reviewer for useful suggestions which improved the contents of this article. This article was supported by the National Natural Science Foundation of China under Grant No.11601348.

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