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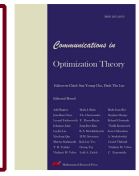

GENERAL VARIATIONAL INCLUSIONS AND DYNAMICAL SYSTEMS

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Abstract. In this paper, we consider and investigate a dynamical system with general variational inclusions involving the difference of monotone operators. We study the existence, uniqueness and the global stability of the resolvent dynamical system and resolvent equations dynamical system under some suitable conditions. Results obtained in this paper can be viewed as significant contribution in this field and may motivate further research.

Keywords. Monotone operator; Nonexpansive mapping; Variational inequality; Dynamical system.

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1. Introduction

Variational inclusions are the natural generalization of variational inequalities having applications to many fields, for example, mechanics, physics, optimization and control theory, nonlinear programming, economics and engineering sciences. For details see [1-7] and references therein. Variational inclusions involving the sum of two (more) monotone operators have been studied widely in recent years. It is known that the sum of two (more) monotone operators is again a monotone operator. However the difference of two monotone operators is not. Due to this fact, the problem of finding a zero of the difference of two monotone operators is very difficult as compared to finding the zeros of monotone operators. The problem of finding the zeros

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of the variational inclusions involving the difference of two monotone operators has recently investigated by many authors; see [8, 9, 10] and the references therein.

These results have been for the general variational inclusions involving the difference of two monotone operators with respect to an arbitrary operators. For recent activities in this direction, see [10, 11, 12, 13] and the references therien.

In recent years, several dynamical systems associated with variational inclusions and inequalities are being studied using projection operator and resolvent operator methods. Xia and Wang [14] considered the projected dynamical system associated with variational inequality using the equivalent fixed point formulation. For the numerical method and other aspects of the dynamical systems associated with the variational inequalities, see [15, 16, 17, 18] and the references therein. The novel feature of the resolvent dynamical system is that the equilibrium point of the dynamical system is the solution of the corresponding variational inequalities. Consequently, all the problems, which can be studied in the general framework of variational inequalities can be studied using the dynamical systems approach. Dynamical systems enable us to describe the trajectories of real world and physical process before reaching to steady state. It is well known that variational inclusions are equivalent to the fixed point problem using the resolvent operator technique [10]. We use this equivalent formulation to consider the dynamical system associated with general variational inclusions involving the difference of two monotone operators. This shows that variational inclusions involving the difference of monotone operator can be reformulated as an initial value first order problem. This equivalent form is used to study the existence of a solution of the variational inclusion involving Lipschitz continuity of the operator. This approach also enables us to investigate the asymptotic stability of the solution.

2. Formulations and basic results

Let $\mathscr{H}$ be a real Hilbert space, whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ respectively. For given monotone operators $T, A, g : \mathscr{H} \to \mathscr{H}$, consider a problem of finding $u \in \mathscr{H} : g(u) \in \mathscr{H}$, such that

$$0 \in A(g(u)) - Tu. \tag{2.1}$$

The problem of type (2.1) is called general variational inclusion involving difference of monotone operators. This type of problem is considered by Noor, Noor and Kamal [10]. We now discuss some applications of the general variational inclusion (2.1).

I. If $g \equiv I$, the identity operator, then problem (2.1) is equivalent to finding $u \in \mathscr{H}$ such that

$$0 \in A(u) - Tu, \tag{2.2}$$

a problem considered by Noor *et al.* [8, 9] and Moudafi [19] separately using two different techniques.

II. If $A(\cdot) \equiv \partial\phi(\cdot)$, the subdifferential of a proper, convex and lower-semicontinuous function $\phi : \mathscr{H} \to R \cup \infty$, then problem (2.1) is equivalent to finding $u \in \mathscr{H}$ such that

$$0 \in \partial\phi(g(u)) - Tu, \tag{2.3}$$

a problem considered and studied by Adly and Oettli [20].
We note that problem (2.3) can be written as: find $u \in \mathscr{H} : g(u) \in \mathscr{H}$ such that

$$\langle -Tu, g(v) - g(u)\rangle + \phi(g(v)) - \phi(g(u)) \geq 0, \forall v \in \mathscr{H} \tag{2.4}$$

which is known as the general mixed variational inequality or the variational inequality of the second kind. For the applications, numerical methods and other aspects of these mixed variational inequalities, see [20, 21] and the references therein.

III. If $A = \partial f$ and $T = \partial h$, then problem (2.2) is equivalent to finding $u \in \mathscr{H}$, such that

$$0 \in \partial f(u) - \partial h(u), \tag{2.5}$$

where ∂f and ∂h are the subdifferentials of the convex functions f and h, which is the necessary optimality condition for finding the minimum of the difference of two convex function $(f(u) - h(u))$.

IV. If $A = \partial f$ and $T = \partial h$, then problem (2.1) is equivalent to finding $u \in \mathscr{H}$, such that

$$0 \in \partial f(g(u)) - \partial h(g(u)), \tag{2.6}$$

where ∂f and ∂h are the subdifferentials of the g-convex functions f and h.

V. If φ is an indicator function of a closed and convex set K in a real Hilbert space H, then problem (2.3) is equivalent to finding $u \in \mathscr{H} : g(u) \in K$, such that

$$\langle Tu, g(v) - g(u)\rangle \leq 0, \forall v \in \mathscr{H} : g(v) \in K, \tag{2.7}$$

which is called the general variational inequality, see [22] and the references therein.

VI. If $g = I$, the identity operator, then problem (2.4) is equivalent to finding $u \in K$, such that

$$\langle Tu, v-u \rangle \leq 0, \forall v \in K, \tag{2.8}$$

which is known as the classical general variational inequalities, introduced and studied by Stampacchia [13] in 1964. See also [10, 15, 17, 21, 23] for more details.

Definition 2.1. An operator $T : \mathscr{H} \to \mathscr{H}$ is said to be:
(i) strongly antimonotone, if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, u-v \rangle \leq -\alpha \|u-v\|^2, \qquad \forall u, v \in \mathscr{H};$$

(ii) Lipschitz continuous, if there exists a constant $\beta > 0$ such that

$$\| Tu - Tv \| \leq \beta \| u-v \|, \qquad \forall u, v \in \mathscr{H},$$

(iii) strongly monotone, if there exists a constant $\alpha_1 > 0$ such that

$$\langle Tu - Tv, u-v \rangle \geq \alpha_1 \|u-v\|^2, \qquad \forall u, v \in \mathscr{H};$$

Definition 2.2. [24] If A is a maximal monotone operator on $\mathscr{H}$, then, for a constant $\rho > 0$, the resolvent operator associated with A is defined by

$$J_A(u) = (I + \rho A)^{-1}(u), \forall u \in \mathscr{H},$$

where I is the identity operator.

It is known that a monotone operator is maximal, if and only if, its resolvent operator is defined everywhere. Furthermore, the resolvent operator is a single valued and nonexpansive, that is,

$$\|J_A(u) - J_A(v)\| \leq \|u-v\|, \forall u, v \in \mathscr{H}.$$

Remark 2.3. Since $\partial\varphi$ be a subdifferential of a proper, convex, and lower semicontinuous function $\varphi : \mathscr{H} \to R \cup \{+\infty\}$is a maximal monotone operator, we define by

$$J_\varphi = (I + \rho\partial\varphi)^{-1},$$

the resolvent operator associated with $\partial\varphi$ and $\rho > 0$ is a constant. Furthermore,

$$\left\|J_\varphi(u) - J_\varphi(v)\right\| \leq \|u-v\|, \forall u, v \in \mathscr{H}$$

Definition 2.5. [16] The dynamical system is said to converge to the solution set $\mathscr{H}^*$ of problem (2.1) if, irrespective of the initial point, the trajectory of the dynamical system satisfies

$$\lim_{t\to\infty} \operatorname{dist}(u(t), \mathscr{H}^*) = 0,$$

where

$$\operatorname{dist}(u, \mathscr{H}^*) = \inf_{v\in\mathscr{H}^*} \|u - v\|.$$

Clearly, if the set $\mathscr{H}^*$ has a unique point u^*, then we have

$$\lim_{t\to\infty} u(t) = u^*.$$

Definition 2.6. [16] The dynamical system is said to be globally exponentially stable with degree η_1 at $u^* \in \mathscr{H}$, if, irrespective of the initial point, the trajectory of the system $u \in H$, satisfies

$$\|u(t) - u^*\| \leq \mu_1 \|u(t_0) - u^*\| e^{-\eta_1(t-t_0)}, \quad \forall t \geq t_0,$$

where $\mu_1 > 0$ and $\eta_1 > 0$ are positive constants independent of the initial point.

It is clear that globally exponential stability is necessarily globally asymptotically stable and the dynamical system converges arbitrarily fast.

Lemma 2.7. [23] (Gronwall's Lemma) *Let u and v be real valued non-negative continuous functions with domain $\{t : t \geq t_0\}$ and let $\alpha(t) = \alpha_0 |t - t_0|$, where α_0 is a monotone increasing function. If, for $t \geq t_0$,*

$$u(t) \leq \alpha(t) + \int_{t_0}^{t} u(s) v(s) ds,$$

then

$$u(t) \leq \alpha(t) \cdot \exp\left(\int_{t_0}^{t} v(s) ds\right).$$

3. Main results

In this section, we consider the dynamical systems associated with general variational inclusions involving the difference of two monotone operators and studied the qualitative properties of the solution. For this end, we need the following result, which is mainly due to Noor *et al.* [10].

Lemma 3.1. [10] *Let A be a maximal monotone operator. Then function $u \in \mathscr{H} : g(u) \in \mathscr{H}$, is a solution of the general variational inclusion (2.1), if and only if, $u \in \mathscr{H} : g(u) \in \mathscr{H}$, satisfies the relation*

$$g(u) = J_A[g(u) + \rho Tu], \tag{3.1}$$

where $J_A = (I + \rho A)^{-1}$ is the resolvent operator and $\rho > 0$ is a constant.

Lemma 3.1 implies that the general variational inclusion (2.1) is equivalent to the fixed point problem (3.1). This alternative equivalent formulation is very useful from the numerical and theoretical points of view. We now define the residue vector $\mathscr{R}(u)$ as:

$$\mathscr{R}(u) = g(u) - J_A[g(u) + \rho Tu]. \tag{3.2}$$

It is clear from Lemma 3.1 that the variational inclusion (2.1) has a solution $u \in \mathscr{H} : g(u) \in \mathscr{H}$, if and only if, $u \in \mathscr{H} : g(u) \in \mathscr{H}$, is a zero of the equation

$$\mathscr{R}(u) = 0. \tag{3.3}$$

We now propose a dynamical system using residue vector $\mathscr{R}(u)$, defined by the relation (3.3), related to the problem (2.1) as:

$$\begin{aligned} \frac{du}{dt} &= -\gamma \mathscr{R}(u) \\ &= \gamma\{J_A[g(u) + \rho Tu] - g(u)\}, \qquad u(t_o) = u_0 \in \mathscr{H}, \end{aligned} \tag{3.4}$$

where γ is any constant. The problem (3.4) is called resolvent dynamical system related to the problem (3.1).

Definition 3.2. [16] The point $u^* \in \mathscr{H}$ is said to be an equilibrium point of the resolvent dynamical system, if,

$$\frac{du^*}{dt} = 0.$$

If u^* is the equilibrium point of the dynamical system (3.4), that is,, $\frac{du^*}{dt} = 0$. Then, from (3.4), we have

$$\mathscr{R}(u^*) = 0.$$

This shows that $u^* \in \mathscr{H}$ satisfies (2.1).

We now discuss some special cases of problem (3.4)

I. If $A(\cdot) \equiv \partial\varphi(\cdot)$, the subdifferential of a proper, convex, and lower semicontinuous function φ then problem (3.4) is equivalent to

$$\frac{du}{dt} = \gamma\{J_{\varphi}[g(u)+\rho T(u)] - g(u)\}, \qquad u(t_0) = u_0 \in \mathscr{H}, \tag{3.5}$$

where γ is any constant and $J_{\varphi} = (I+\rho\partial\varphi)^{-1}$ is the resolvent operator and $\rho > 0$ is a constant.

II. If $g = I$, then problem (3.5) can be written as

$$\frac{du}{dt} = \gamma\{J_{\varphi}[u+\rho T(u)] - u\}, \qquad u(t_0) = u_0 \in \mathscr{H}. \tag{3.6}$$

The problem (3.6) is called resolvent dynamical system related to the problem (2.5).

III. If $\varphi(\cdot)$ is the indicator function of a closed convex set K, then problem (3.5) is equivalent to

$$\frac{du}{dt} = \gamma\{P_K[g(u)+\rho T(u)] - g(u)\}, \qquad u(t_0) = u_0 \in K, \tag{3.7}$$

where γ is any constant and P_K is the projection operator. The problem (3.7) is called projected dynamical system related to the problem (2.4). For more details, see [25, 26, 27] and the references therein.

IV. If $g = I$, then problem (3.7) is equivalent to

$$\frac{du}{dt} = \gamma\{P_K[u+\rho T(u)] - u\}, \qquad u(t_0) = u_0 \in K.$$

This dynamical system is related to problem (2.8).

Now, we are in a position to prove that the residue vector is strongly monotone and Lipschitz continuous.

Theorem 3.3. *Let the operator T be strongly anti-monotone and Lipschitz continuous with constants $\alpha_T > 0$, $\beta_T > 0$, respectively. Let the operator g be strongly monotone and Lipschitz continuous with respect to the constants $\sigma_g > 0$ and $\delta_g > 0$, respectively. If the resolvent operator is non-expensive then the residue vector $\mathscr{R}(u)$, defined by (3.) is strongly monotone and Lipschitz continuous for*

$$\sigma_g > \frac{\kappa}{c} + \tau(\rho),$$

where $\kappa = \sqrt{1-2\sigma_g+\delta_g^2}$ and $\tau(\rho) = \sqrt{1-2\rho\alpha_T+\rho^2\beta_T^2}$.

Proof. Since the residue vector defined by (3.2) is

$$\mathscr{R}(u) = g(u) - J_A[g(u)+\rho T(u)].$$

To prove that $\mathscr{R}(u)$ is strongly monotone for all $u \neq v \in \mathscr{H} : g(u) \neq g(v) \in \mathscr{H}$, we consider

$$\begin{aligned}\langle \mathscr{R}(u) - \mathscr{R}(v), u - v\rangle &= \langle g(u) - J_A[g(u) + \rho T(u)] - g(v) \\ &\quad + J_A[g(v) + \rho T(v)], u - v\rangle \\ &= \langle g(u) - g(v), u - v\rangle - \langle J_A[g(u) + \rho T(u)] \\ &\quad - J_A[g(v) + \rho T(v)], u - v\rangle.\end{aligned} \tag{3.8}$$

Since g is strongly monotone with respect to $\sigma_g > 0$,

$$\langle g(u) - g(v), u - v\rangle \geq \sigma_g \parallel u - v \parallel^2 . \tag{3.9}$$

Consider

$$\begin{aligned}&\langle J_A[g(u) + \rho T(u)] - J_A[g(v) + \rho T(v)], u - v\rangle \\ &\leq \parallel g(u) - g(v) + \rho(T(u) - T(v)) \parallel \parallel u - v \parallel \\ &\leq \{\parallel u - v - (g(u) - g(v)) \parallel + \parallel u - v + \rho(T(u) - T(v)) \parallel\} \parallel u - v \parallel .\end{aligned} \tag{3.10}$$

Since g is strongly monotone with respect to $\sigma_g > 0$ and Lipschitz continuous with respect to $\delta_g > 0$, one has

$$\begin{aligned}\parallel u - v - (g(u) - g(v)) \parallel &\leq \sqrt{1 - 2\sigma_g + \delta_g^2} \parallel u - v \parallel \\ &= \frac{\kappa}{2} \parallel u - v \parallel .\end{aligned} \tag{3.11}$$

Also T is strongly anti-monotne and Lipschitz continuous with constants $\alpha_T > 0$ and $\beta_T > 0$ respectively, i.e.,

$$\begin{aligned}\parallel u - v + \rho(T(u) - T(v)) \parallel &\leq \sqrt{1 - 2\rho\alpha_T + \rho^2\beta_T^2} \parallel u - v \parallel \\ &= \tau(\rho) \parallel u - v \parallel .\end{aligned} \tag{3.12}$$

Using (3.11) and (3.12) in (3.10), we have

$$\langle J_A[g(u) + \rho T(u)] - J_A[g(v) + \rho T(v)], u - v\rangle \leq (\frac{\kappa}{2} + \tau(\rho)) \parallel u - v \parallel^2 . \tag{3.13}$$

After using (3.9) and (3.13), (3.8) taking the form

$$\begin{aligned}\langle \mathscr{R}(u) - \mathscr{R}(v), u - v\rangle \quad &\geq \{\sigma_g - \frac{\kappa}{2} - \tau(\rho)\} \parallel u - v \parallel^2 \\ &= \kappa_2 \parallel u - v \parallel^2,\end{aligned}$$

where $\kappa_2 = \{\sigma_g - \frac{\kappa}{2} - \tau(\rho)\} > 0$, we have the required result.

To prove that $\mathscr{R}(u)$ is Lipschitz continuous for all $u \neq v \in \mathscr{H} : g(u) \neq g(v) \in \mathscr{H}$, we consider

$$\begin{aligned}
\| \mathscr{R}(u) - \mathscr{R}(v) \| &\leq \| g(u) - g(v) \| + \| J_A[g(u) + \rho T(u)] - J_A[g(v) + \rho T(v)] \| \\
&\leq 2 \| g(u) - g(v) \| + \rho \|T(u) - T(v)\| \\
&\leq (2\delta_g + \rho\beta_T) \| u - v \|,
\end{aligned}$$

where we have utilized the Lipschitz continuity of T and g with constants $\beta_T > 0$ and $\delta_g > 0$, respectively. This shows that operator $\mathscr{R}(u)$ is a Lipschitz continuous in $\mathscr{H}$. This completes the proof.

We will now discuss the existence and uniqueness of the dynamical system (2.8) using the technique of Noor [15]. We include the proof for the sake of completion and to convey the main idea.

Theorem 3.4. *Let the operator T and g be Lipschitz continuous with constants $\beta_T > 0$ and $\delta_g > 0$, respectively. If $\gamma > 0$, then for each $u_0 \in \mathscr{H}$, there exists a unique continuous solution $u(t)$ of the dynamical system (3.4) with $u(t_0) = u_0$ and over $[t_0, \infty)$.*

Proof. Let

$$\mathscr{G}_1(u) = \gamma\{J_A[g(u) + \rho T(u)] - g(u)\}.$$

To prove that $\mathscr{G}_1(u)$ is Lipschitz continuous for all $u \neq v \in \mathscr{H} : g(u) \neq g(v) \in \mathscr{H}$, we consider

$$\begin{aligned}
\| \mathscr{G}_1(u) - \mathscr{G}_1(v) \| &\leq \gamma \| J_A[g(u) + \rho T(u)] - J_A[g(v) + \rho T(v)] \| + \gamma \| g(u) - g(v) \| \\
&\leq 2\gamma \| g(u) - g(v) \| + \gamma\rho \|T(u) - T(v)\| \\
&\leq \gamma(2\delta_g + \rho\beta_T) \| u - v \|,
\end{aligned}$$

where we have used Lipschitz continuity of T and g with constants $\beta_T > 0$ and $\delta_g > 0$, respectively. This shows that operator $\mathscr{G}_1(u)$ is a Lipschitz continuous in $\mathscr{H}$. Since operator $\mathscr{G}_1(u)$ is Lipschitz continuous so for each $u_0 \in \mathscr{H} : g(u_0) \in \mathscr{H}$, there exists a unique and continuous solution $u \in \mathscr{H} : g(u) \in \mathscr{H}$, of the dynamical system (3.4), defined in an interval $t_0 \leq t < T_1$ with the initial condition $u(t_0) = u_0$. Let the maximal interval of existence be $[t_0, T_1)$. Now we have to show that $T_1 = \infty$. Consider

$$\begin{aligned}
\| \frac{du}{dt} \| &= \| \mathscr{G}_1(u) \| \\
&= \gamma \| J_A[g(u)+\rho T(u)] - g(u) \| \\
&\leq \gamma \| J_A[g(u)+\rho T(u)] - J_A[g(u)] \| + \gamma \| J_A[g(u)] - J_A[g(u^*)] \| \\
&\quad + \gamma \| J_A[g(u^*)] \| + \gamma \| g(u) \| \\
&\leq \gamma \|g(u)+\rho T(u) - g(u)\| + \gamma \|g(u) - g(u^*)\| \\
&+ \gamma \|J_A[g(u^*)]\| + \gamma \|g(u)\| \\
&\leq \gamma\rho\beta_T \| u \| + \gamma\delta_g \| u - u^* \| + \gamma \|J_A[g(u^*)]\| + \gamma\delta_g \|u\| \\
&\leq \gamma(\delta_g \| u^* \| + \| J_A[g(u^*)] \|) + \gamma(2\delta_g + \rho\beta_T) \| u \|,
\end{aligned} \tag{3.14}$$

where we have utilized Lipschitz continuity of operators T and g with constants $\beta_T > 0$ and $\delta_g > 0$, respectively.

Now, taking the integral of (3.14) over the interval $[t_0, t]$, we have

$$\|u(t)\| - \|u(t_0)\| \leq \lambda_1 (t - t_0) + \lambda_2 \int_{t_0}^{t} \|u(s)\| \, dynamicalsystem.$$

Using Lemma (2.7), we have

$$\|u(t)\| \leq \{\|u(t_0)\| + \lambda_1 (t - t_0)\} \exp\{\lambda_2 (t - t_0)\},$$

where

$$\begin{aligned}
\lambda_1 &= \gamma\{\delta_g \|u^*\| + \|J_A[g(u^*)]\|\}, \\
\lambda_2 &= \gamma(2\delta_g + \rho\beta_T).
\end{aligned}$$

This presents that the solution is bounded on $[t_0, \infty)$. This completes the proof.

We now show that the trajectory of the solution of the dynamical system (3.4) converges globally exponentially to the unique solution of problem (2.1).

Theorem 3.5. *Let the operator T be Lipschitz continuous with constant $\beta_T > 0$. Also suppose that operator g be strongly monotone and Lipschitz continuous with respect to the constants $\sigma_g > 0$ and $\delta_g > 0$ respectively. If the resolvent operator is non-expensive and*

$$\sigma_g > \delta_g + \rho\beta_T,$$

then dynamical system (3.4) converges globally exponentially to the unique solution of problem (2.1).

Proof. Theorem (3.4) states that the dynamical system (3.4) has a unique solution $u(t)$ over $[t_0, T_1)$ for any fixed $u_0 \in \mathscr{H}$ whenever the operators T and g are Lipschitz continuous. Let $u(t) = u(t, t_0; u_0)$ be a solution of problem (3.4). For a given $u^* \in \mathscr{H} : g(u^*) \in \mathscr{H}$, satisfying problem (2.1), consider the following Lyapunov function:

$$\mathfrak{L}(u) = \frac{1}{2}\|u(t) - u^*\|^2, \quad u \in \mathscr{H}.$$

It follows that

$$\begin{aligned} \frac{d\mathfrak{L}}{dt} &= \langle u(t) - u^*, \frac{du}{dt}\rangle \\ &= -\gamma\langle u - u^*, g(u) - J_A[g(u) + \rho T(u)]\rangle \\ &= -\gamma\langle u - u^*, g(u) - g(u^*)\rangle - \gamma\langle u - u^*, g(u^*) - J_A[g(u) + \rho T(u)]\rangle \\ &= -\gamma\langle u - u^*, g(u) - g(u^*)\rangle + \gamma\langle u - u^*, J_A[g(u) + \rho T(u)] - g(u^*)\rangle \\ &\le -\gamma\sigma_g \parallel u - u^* \parallel^2 + \gamma \parallel J_A[g(u) + \rho T(u)] - g(u^*) \parallel\parallel u - u^* \parallel, \end{aligned} \tag{3.15}$$

where we have used the fact that g is strongly monotone with respect to the constant $\sigma_g > 0$. Since $u^* \in \mathscr{H}$ is the solution of problem (2.1), therefore using the Lemma (3.1), we have

$$g(u^*) = J_A[g(u^*) + \rho T(u^*)].$$

Using the above relation in (3.15), we have

$$\begin{aligned} \frac{d\mathfrak{L}}{dt} &\le -\gamma\sigma_g \parallel u - u^* \parallel^2 + \gamma \parallel J_A[g(u) + \rho T(u)] - J_A[g(u^*) + \rho T(u^*)] \parallel\parallel u - u^* \parallel \\ &\le -\gamma\sigma_g \parallel u - u^* \parallel^2 + \{\gamma \parallel g(u) - g(u^*) \parallel \\ &\quad + \gamma\rho \parallel T(u) - T(u^*) \parallel\} \parallel u - u^* \parallel \\ &\le (-\gamma\sigma_g + \gamma\delta_g + \gamma\rho\beta_T) \parallel u - u^* \parallel^2 \\ &= -\gamma(\sigma_g - \delta_g - \rho\beta_T) \parallel u - u^* \parallel^2 \\ &= -\lambda_3 \parallel u - u^* \parallel^2, \end{aligned}$$

where $\lambda_3 = \gamma(\sigma_g - \delta_g - \rho\beta_T) > 0$, which implies that

$$\parallel u(t) - u^* \parallel \le \parallel u(t_0) - u^* \parallel \exp(-\lambda_3(t - t_0)).$$

This proves that the trajectory of the solution of the dynamical system (3.4) converges globally exponentially to the unique solution of problem (2.1).

4. Resolvent equations dynamical System

In this section, we introduce and study a new dynamical system related to (2.1).

This dynamical system is called general resolvent equations dynamical system.

Lemma 4.1. *The GVI (2.1) has a solution $u \in \mathscr{H}$, if and only if, $z \in \mathscr{H}$ satisfies GRE*

$$Tg^{-1}J_A z - \rho^{-1}R_A z = 0, \tag{4.1}$$

provided

$$g(u) = J_A z, \tag{4.2}$$

$$z = g(u) + \rho T(u), \tag{4.3}$$

where $\rho > 0$, a constant.

Using Lemma 4.1, the resolvent equations (4.1) can be written as

$$g(u) = \rho Tg^{-1}J_A[g(u)+\rho T(u)] + J_A[g(u)+\rho T(u)] - \rho T(u),$$

from which we have

$$\mathscr{R}_2(u) = g(u) - \rho Tg^{-1}J_A[g(u)+\rho T(u)] - J_A[g(u)+\rho T(u)] + \rho T(u) = 0. \tag{4.4}$$

Lemma 4.1 clearly tells us that the GVI (2.1) has a solution $u \in \mathscr{H} : g(u) \in \mathscr{H}$, if and only if, $u \in \mathscr{H} : g(u) \in \mathscr{H}$, satisfies (4.4). For constant $\gamma > 0$, we suggest a dynamical system using (4.4):

$$\frac{du}{dt} = \gamma\{\rho Tg^{-1}J_A[g(u)+\rho T(u)] + J_A[g(u)+\rho T(u)] - \rho T(u) - g(u)\}, \quad u(t_o) = u_0 \in \mathscr{H}. \tag{4.5}$$

This is called general resolvent equations dynamical system related to the GVI (2.1). Since the solution of the dynamical system (4.5) belongs to the constraint set, therefore we can study the existence, uniqueness and continuous dependence of the solution.

We now study some important properties of the suggested resolvent equations dynamical system and analyze the global stability of the system. Firstly, we discuss the existence and uniqueness of the dynamical system (4.5) and this is the foremost inspiration of our following result.

Theorem 4.1. *Let the operators T, g and g^{-1} be Lipschitz continuous with constants $\beta_T > 0$, $\delta_g > 0$ and $\beta_1 > 0$ respectively. If $\gamma > 0$, then for each $u_0 \in \mathscr{H}$, there exists a unique continuous solution $u(t)$ of the dynamical system (4.5) with $u(t_0) = u_0$ and over $[t_0, \infty)$.*

Proof. Let

$$\mathscr{G}_2(u) = \gamma\{\rho T g^{-1} J_A[g(u) + \rho T(u)] + J_A[g(u) + \rho T(u)] - \rho T(u) - g(u)\}.$$

To prove $\mathscr{G}_2(u)$ is Lipschitz continuous for all $u \neq v \in \mathscr{H}$, we have

$$\begin{aligned}
&\| \mathscr{G}_2(u) - \mathscr{G}_2(v) \| \\
&= \gamma \| \rho T g^{-1} J_A[g(u) + \rho T(u)] + J_A[g(u) + \rho T(u)] \\
&\quad - \rho T(u) - g(u) - \rho T g^{-1} J_A[g(v) + \rho T(v)] \\
&\quad - J_A[g(v) + \rho T(v)] + \rho T(v) + g(v) \| \\
&\leq \gamma\rho \| T g^{-1} J_A[g(u) + \rho T(u)] - T g^{-1} J_A[g(v) + \rho T(v)] \| \\
&\quad + \gamma \| J_A[g(u) + \rho T(u)] - J_A[g(v) + \rho T(v)] \| \\
&\quad + \gamma \| g(u) - g(v) \| + \gamma\rho \| T(u) - T(v) \| \qquad (4.6) \\
&\leq 2\gamma \| g(u) - g(v) \| + 2\gamma\rho \| T(u) - T(v) \| \\
&\quad + \gamma\rho\beta_T\beta_1 \| J_A[g(u) + \rho T(u)] - J_A[g(v) + \rho T(v)] \| \\
&\leq \gamma(2 + \rho\beta_T\beta_1) \| g(u) - g(v) \| + \gamma(2\rho + \rho^2\beta_T\beta_1) \| T(u) - T(v) \| \\
&\leq \gamma\delta_g(2 + \rho\beta_T\beta_1) \| u - v \| + \gamma\beta_T(2\rho + \rho^2\beta_T\beta_1) \| u - v \| \\
&= \gamma(2\delta_g + \rho\beta_T\beta_1\delta_g + 2\rho\beta_T + \rho^2\beta_T^2\beta_1) \| u - v \| \\
&= \gamma(\delta_g + \rho\beta_T)(2 + \rho\beta_T\beta_1) \| u - v \|,
\end{aligned}$$

where we have used the fact that the resolvent operator is nonexpensive and the operators T, g and g^{-1} are Lipschitz continuous with constants $\beta_T > 0$, $\delta_g > 0$ and $\beta_1 > 0$ respectively.

Since (4.6) shows that the operator $\mathscr{G}_2(u)$ is Lipschitz continuous so for each $u_0 \in \mathscr{H} : g(u_0) \in \mathscr{H}$, one sees that there exists a unique and continuous solution $u \in \mathscr{H} : g(u) \in \mathscr{H}$, of dynamical system (4.5), defined in an interval $t_0 \leq t < T_1$ with the initial condition $u(t_0) = u_0$. Let $[t_0, T_1)$ be its maximal interval of existence.

Now we have to show that $T_1 = \infty$. Note that

$$
\begin{aligned}
\| \frac{du}{dt} \| =& \| \mathscr{G}_2(u) \| \\
&= \gamma \| \rho T g^{-1} J_A[g(u)+\rho T(u)] + J_A[g(u)+\rho T(u)] - \rho T(u) - g(u) \| \\
&\leq \gamma\rho \| T g^{-1} J_A[g(u)+\rho T(u)] - T(u) \| + \gamma \| J_A[g(u)+\rho T(u)] - J_A[g(u^*)] \| \\
&\quad + \gamma \| J_A[g(u^*)] \| + \gamma \| g(u) \| \\
&\leq \gamma\rho\beta_T \| g^{-1} J_A[g(u)+\rho T(u)] - g^{-1} J_A[g(u^*)] \| + \gamma\rho\beta_T \| g^{-1} J_A[g(u^*)] \| \\
&\quad + \gamma \| g(u) - g(u^*) \| + \gamma\rho \| T(u) \| + \gamma \| J_A[g(u^*)] \| \\
&\quad + \gamma\rho\beta_T \| u \| + \gamma \| g(u) \| \\
&\leq \gamma\rho\beta_T\beta_1 \| J_A[g(u)+\rho T(u)] - J_A[g(u^*)] \| + \gamma\rho\beta_T\beta_1 \| J_A[g(u^*)] \| \\
&\quad + \gamma\delta_g \| u - u^* \| + \gamma\rho\beta_T \| u \| + \gamma \| J_A[g(u^*)] \| + \gamma\rho\beta_T \| u \| + \gamma\delta_g \| u \| \\
&\leq \gamma\rho\beta_T\beta_1 \| g(u) - g(u^*) \| + \gamma\rho^2\beta_T^2\beta_1 \| u \| + \gamma\rho\beta_T\beta_1 \| J_A[g(u^*)] \| \\
&\quad + 2\gamma\delta_g \| u \| + 2\gamma\rho\beta_T \| u \| + \gamma\delta_g \| u^* \| + \gamma \| J_A[g(u^*)] \| \\
&\leq \gamma(2\delta_g + 2\rho\beta_T + \gamma\delta_g\beta_T\beta_1 + \rho^2\beta_T^2\beta_1) \| u \| + \gamma\delta_g(1+\rho\beta_T\beta_1) \| u^* \| \\
&\quad + \gamma(1+\rho\beta_T\beta_1) \| J_A[g(u^*)] \| \\
&= \gamma(\rho\beta_T + \delta_g)(2+\rho\beta_T\beta_1) \| u \| + \gamma(1+\rho\beta_T\beta_1)\{\delta_g \| u^* \| + \| J_A[g(u^*)] \|\},
\end{aligned}
\tag{4.7}
$$

where we have used Lipschitz continuity of operators T, g and g^{-1} with constants $\beta_T > 0$, $\delta_g > 0$ and $\beta_1 > 0$ respectively. Now, taking the integral of (4.7) over the interval $[t_0, t]$, we have

$$
\|u(t)\| - \|u(t_0)\| \leq \lambda_4 (t - t_0) + \lambda_5 \int_{t_0}^{t} \|u(s)\| ds.
$$

Using Lemma 2.7, we have $\|u(t)\| \leq \{\|u(t_0)\| + \lambda_4(t-t_0)\} \exp\{\lambda_5(t-t_0)\}$, where

$$
\begin{aligned}
\lambda_4 &= \gamma(1+\rho\beta_T\beta_1)\{\delta_g \| u^* \| + \| J_A[g(u^*)] \|\}, \\
\lambda_5 &= \gamma(\rho\beta_T + \delta_g)(2+\rho\beta_T\beta_1).
\end{aligned}
$$

This shows that the solution is bounded on $[t_0, \infty)$.

We now show that the trajectory of the solution of the (4.4) converges globally exponentially to the unique solution of problem (2.1).

Theorem 4.2. *Let the operators T, g and g^{-1} be Lipschitz continuous with constants $\beta_T > 0$, $\delta_g > 0$ and $\beta_1 > 0$, respectively. Also suppose that the operator g be strongly monotone with constant $\sigma_g > 0$. If $\sigma_g > \rho\beta_T + (\delta_g + \rho\beta_T)(1 + \rho\beta_T\beta_1)$, then (4.5) converges globally exponentially to the unique solution of problem (2.1).*

Proof. Theorem 4.1 refers that (4.4) has a unique solution $u(t)$ over $[t_0, T_1)$ for any fixed point $u_0 \in \mathscr{H}$, whenever the operators T, g and g^{-1} satisfies the conditions of Theorem 4.1. Letting $u(t) = u(t, t_0; u_0)$ be a solution of (4.5). For a given $u^* \in \mathscr{H} : g(u^*) \in \mathscr{H}$, satisfying problem (2.1), we consider the following Lyapunov function:$\mathfrak{L}(u) = \frac{1}{2}\|u(t) - u^*\|^2$, $u \in \mathscr{H}$. Using (4.4), we have

$$
\begin{aligned}
\frac{d\mathfrak{L}}{dt} &= \langle u(t) - u^*, \frac{du}{dt}\rangle \\
&= -\gamma\langle u(t) - u^*, g(u) + \rho T(u) - J_A[g(u) + \rho T(u)] - \rho T g^{-1} J_A[g(u) + \rho T(u)]\rangle \\
&= -\gamma\langle u(t) - u^*, g(u) - g(u^*)\rangle \\
&\quad + \gamma\langle u(t) - u^*, \rho T g^{-1} J_A[g(u) + \rho T(u)] + J_A[g(u) + \rho T(u)] - \rho T(u) - g(u^*)\rangle \\
&\leq -\gamma\sigma_g \| u - u^* \|^2 + \gamma \| u - u^* \| \| \rho T g^{-1} J_A[g(u) + \rho T(u)] \\
&\quad + J_A[g(u) + \rho T(u)] - \rho T(u) - g(u^*) \|,
\end{aligned} \tag{4.8}
$$

where we have used the strongly monotonicity of the operator g with constant $\sigma_g > 0$. Since $u^* \in \mathscr{H} : g(u^*) \in \mathscr{H}$ is the solution of problem (2.1), therefore using (4.4), we have

$$
\begin{aligned}
&\| \rho T g^{-1} J_A[g(u) + \rho T(u)] + J_A[g(u) + \rho T(u)] - \rho T(u) - g(u^*) \| \\
&= \| \rho T g^{-1} J_A[g(u) + \rho T(u)] + J_A[g(u) + \rho T(u)] - \rho T(u) \\
&\quad - \rho T g^{-1} J_A[g(u^*) + \rho T(u^*)] - J_A[g(u^*) + \rho T(u^*)] + \rho T(u^*) \| \\
&\leq \rho \| T g^{-1} J_A[g(u) + \rho T(u)] - T g^{-1} J_A[g(u^*) + \rho T(u^*)] \| + \| J_A[g(u) + \rho T(u)] \\
&\quad - J_A[g(u^*) + \rho T(u^*)] \| + \rho \| T(u) - T(u^*) \| \\
&\leq \rho\beta_T\beta_1 \| g(u) - g(u^*) + \rho(T(u) - T(u^*)) \| + \| g(u) - g(u^*) + \rho(T(u) - T(u^*)) \| \\
&\quad + \rho \| T(u) - T(u^*) \| \\
&\leq (\rho\beta_T\beta_1 + 1) \| g(u) - g(u^*) \| + (\rho\beta_T\beta_1 + 2)\rho \| T(u) - T(u^*) \| \\
&\leq \{(\delta_g + \rho\beta_T)(1 + \rho\beta_T\beta_1) + \rho\beta_T\} \| u - u^* \|,
\end{aligned}
$$

where we have used the Lipschitz continuity of the operators T, g and g^{-1} with constants $\beta_T > 0$, $\delta_g > 0$ and $\beta_1 > 0$ respectively. Substituting above inequality in (4.8) , we have

$$\begin{aligned} \frac{d\mathfrak{L}}{dt} &\leq -\gamma\sigma_g \parallel u-u^* \parallel^2 + \gamma\{(\delta_g+\rho\beta_T)(1+\rho\beta_T\beta_1)+\rho\beta_T\} \parallel u-u^* \parallel^2 \\ &= -\gamma\{\sigma_g-\rho\beta_T-(\delta_g+\rho\beta_T)(1+\rho\beta_T\beta_1)\} \parallel u-u^* \parallel^2 \\ &= -\lambda_6 \parallel u-u^* \parallel^2, \end{aligned} \tag{4.9}$$

where $\lambda_6 = \gamma\{\sigma_g-\rho\beta_T-(\delta_g+\rho\beta_T)(1+\rho\beta_T\beta_1)\} > 0$.
Integrating (4.9), we have

$$\parallel u(t)-u^* \parallel \leq \parallel u(t_0)-u^* \parallel \exp(-\lambda_6(t-t_0)).$$

This shows that the trajectory of the solution of (4.4) converges globally exponentially to the unique solution of problem (2.1).

Conclusion

In this paper, we introduced resolvent dynamical systems and resolvent equations dynamical system related to the general variational inclusion. By using these dynamical systems, we studied the existence, uniqueness and continuity of the solution of variational inclusion (2.1) involving the difference of two monotone operators using the Lipschitz continuity of operator T. Results proved int his paper may stimulate further research. These dynamical systems may be used to suggest some iterative methods for solving the variational inclusions. This is challenging problem for future research.

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