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SOME RESULTS ON GENERALIZED NONLINEAR CONTRACTIVE MAPPINGS

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Abstract. Sufficient conditions for the existence and uniqueness of a fixed point for generalized nonlinear contractive mappings in metric spaces and ordered metric spaces are obtained. The proved results generalize and extend various known results in the literature. Some examples are also provided to illustrate the main results in the paper.

Keywords. Metric space; Order set; Generalized nonlinear contractive mapping.

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1. Introduction-Preliminaries

Fixed point theory is one of the traditional branch of nonlinear analysis. The importance of fixed point theory has been increasing rapidly over the time as this theory provide useful tools for proving the existence and uniqueness of the solutions to various mathematical models (integral and partial differential equations, variational inequalities etc). It is well known that the contractive-type conditions are very indispensable in the study of fixed point theory and Banach's fixed point theorem [1] for contraction mappings is one of the pivotal result in analysis. This theorem that has been extended and generalized by various authors in metric spaces and

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ordered metric spaces (see, for example, [2], [3]-[21]). Bhaskar and Lakshmikantham [3], Chandok [7]-[8], Nieto *et al.* [14, 15], O'Regan *et al.* [16], Ran and Reurings [17] extended the Banach's contraction principle to partial ordered metric spaces and provided many applications to matrix equations and solution of differential equations. Khan et al. [13] introduced concept of altering distance function and many researchers generalize the contraction mapping using altering distance function and proved many interesting results in metric spaces, complex valued metric spaces (see, for example, [5], [9], [10], [21] and references cited therein).

In this paper, using the idea of set valued mappings, we introduced the notions of C-class and A-class mappings. These classes of mappings are used to obtain some fixed point results for generalized contraction mappings with some auxiliary functions in the framework of metric spaces and ordered metric spaces. The proved results generalize and extend some well known results of the lilterature. Some examples are also presented to illustrate our obtained results.

To begin with, first we give some definitions and notations which will be used in the sequel.

Definition 1.1. $\mathfrak{C}$ a family functions $f:[0,\infty)^2\to\mathbb{R}$ is called *C-class* if it is continuous and satisfies following axioms:

(1) $f(s,t)\le s$;

(2) $f(s,t)=s$ implies that either $s=0$ or $t=0$;

for all $s,t\in[0,\infty)$.

Note that for some f, we have $f(0,0)=0$.

Example 1.2. *The following functions $f:[0,\infty)^2\to\mathbb{R}$ are elements of $\mathfrak{C}$. For each $s,t\in[0,\infty)$,*

(1) $f(s,t)=s-t$, $f(s,t)=s\Rightarrow t=0$;

(2) $f(s,t)=\frac{s-t}{1+t}$, $f(s,t)=s\Rightarrow t=0$;

(3) $f(s,t)=\frac{s}{1+t}$, $f(s,t)=s\Rightarrow s=0$ or $t=0$;

(4) $f(s,t)=\log\frac{t+a^s}{1+t}$, $a>1$, $f(s,t)=s\Rightarrow s=0$ or $t=0$;

(5) $f(s,1)=\ln\frac{1+a^s}{2}$, $a>e$, $f(s,1)=s\Rightarrow s=0$;

(6) $f(s,t)=(s+l)^{\frac{1}{1+t}}-l$, $l>1$, $f(s,t)=s\Rightarrow t=0$;

(7) $f(s,t)=s\log_{a+t}a$, $a>1$, $f(s,t)=s\Rightarrow s=0$ or $t=0$.

Definition 1.3. $\mathfrak{A}$ a family of functions $h:\mathbb{R}^+\to\mathbb{R}^+$ is called a *A-class* if it is a continuous function such that $h(t)\geq t$, for all $t\in\mathbb{R}^+$.

Example 1.4. *The following functions $h:\mathbb{R}^+\to\mathbb{R}^+$ are elements of $\mathfrak{A}$.*

(1) $h(t)=a^t-1$, $a>1$, $t\in\mathbb{R}^+$;

(2) $h(t)=mt$, $m\geq 1$, $t\in\mathbb{R}^+$.

Definition 1.5. Let (X,d) be a metric space and $T:X\to X$. F a subset of X is invariant under T if and only if $x\in F\Rightarrow T(x)\in F$

Let Ψ denote the set of all monotone non-decreasing continuous functions $\psi:[0,\infty)\to[0,\infty)$, with $\psi^{-1}(\{0\})=0$.

Let Φ denote the set of all continuous functions $\phi:[0,\infty)\to[0,\infty)$, with $\phi(0)\geq 0$.

2. Main results

2.1. Fixed points in metric spaces

Theorem 2.1. *Let T be a self-mapping defined on a complete metric space (X,d). Suppose that F is a closed subset of X and invariant under T and T satisfies*

$$h(\psi(d(Tx,Ty)))\leq f(\psi(d(x,y)),\phi(d(x,y))), \tag{1}$$

for all $x,y\in F$, $\psi\in\Psi,\phi\in\Phi$, f a function of C-class, h a function of A-class. Then T has a unique fixed point in F.

Proof. For any $x_0\in F$, we can construct the sequence $\{x_n\}$ such that $x_n=Tx_{n-1}\in F$, $n\geq 1$. Substituting $x=x_{n-1}$ and $y=x_n$ in (1), we obtain $\{x_n\}$ which is invariant under T, and

$$\psi(d(x_n,x_{n+1}))\leq h(\psi(d(x_n,x_{n+1})))\leq f(\psi(d(x_{n-1},x_n)),\phi(d(x_{n-1},x_n))), \tag{2}$$

for any $n\geq 1$. Since f is a function of C-class, we obtain

$$\psi(d(x_n,x_{n+1}))\leq\psi(d(x_{n-1},x_n)).$$

Now, as $\psi\in\Psi$, we find that

$$d(x_n,x_{n+1})\leq d(x_{n-1},x_n) \tag{3}$$

for every $n \geq 1$. Hence the sequence $\{d(x_n, x_{n+1})\}$ is a decreasing sequence. So for the non-negative decreasing sequence $\{d(x_n, x_{n+1})\}$, there exists some $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r. \tag{4}$$

Assume that $r > 0$. On letting $n \to \infty$ in (2) and using (4), we obtain

$$\psi(r) \leq f(\psi(r), \phi(r)). \tag{5}$$

Since f is a C-class, we have $\phi(r) = 0$ or $\psi(r) = 0$. It further imples that $r = 0$, which is a contradiction. Hence

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{6}$$

Now, we prove that $\{x_n\}$ is a Cauchy sequence. If possible, let $\{x_n\}$ be not a Cauchy sequence. Then there exists $\delta > 0$ for which we can find subsequences $\{x_{n_k})\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ with $n_k > m_k > k$ such that

$$d(x_{n_k}, x_{m_k}) \geq \delta. \tag{7}$$

Further, corresponding to m_k, we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ and satisfying (7). Therefore, we have

$$d(x_{n_k-1}, x_{m_k}) < \delta. \tag{8}$$

Using (7) and (8), we have

$$0 < \delta \leq d(x_{n_k}, x_{m_k}) \leq d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k}) < \delta + d(x_{n_k}, x_{n_k-1}). \tag{9}$$

On letting $k \to \infty$ and using (6), in (9), we have

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \delta. \tag{10}$$

Consider

$$\begin{aligned} d(x_{n_k}, x_{m_k}) &\leq d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k-1}) + d(x_{m_k-1}, x_{m_k}) \\ &\leq 2d(x_{n_k}, x_{n_k-1}) + d(x_{n_k}, x_{m_k}) + 2d(x_{m_k-1}, x_{m_k}). \end{aligned} \tag{11}$$

Letting $k \to \infty$ in the above inequality (11) and using (6), (10), we get

$$\lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) = \delta. \tag{12}$$

Now, by setting $x = x_{m_k-1}$ and $y = x_{n_k-1}$ in (1), we obtain

$$\psi(d(x_{n_k}, x_{m_k})) \le h(\psi(d(x_{n_k}, x_{m_k})) \le f(\psi(d(x_{n_k-1}, x_{m_k-1})), \phi(d(x_{n_k-1}, x_{m_k-1}))). \tag{13}$$

Letting $k \to \infty$, using (10) and (12), we obtain

$$\psi(\delta) \le f(\psi(\delta), \phi(\delta)), \tag{14}$$

which is a contradiction if $\delta > 0$. This shows that $\{x_n\}$ is a Cauchy sequence. As (X,d) is complete metric space and F is closed subset of X and invariant under T, $\{x_n\}$ is convergent in F. Let $x_n \to z \in F$. Substituting $x = x_{n-1}$ and $y = z$ in (1), we obtain

$$\begin{aligned} \psi(d(x_n, Tz)) &\le h(\psi(d(x_n, Tz))) \\ &\le f(\psi(d(x_{n-1}, z)), \phi(d(x_{n-1}, z))). \end{aligned} \tag{15}$$

Letting $n \to \infty$, using continuity of ψ,ϕ and f, we have

$$\psi(d(z, Tz)) \le f(\psi(0), \phi(0)) = 0,$$

which implies $d(z, Tz) = 0$ and hence $z = Tz$.

Now, let $z_1, z_2 \in F$ such that $z_1 \ne z_2$ and $z_1 = Tz_1$, $z_2 = Tz_2$. Then

$$\begin{aligned} \psi(d(z_1, z_2)) &= \psi(d(Tz_1, Tz_2)) \\ &\le h(\psi(d(Tz_1, Tz_2))) \\ &\le f(\psi(d(z_1, z_2)), \phi(d(z_1, z_2))), \end{aligned}$$

which implies that $d(z_1, z_2) = 0$, i.e. $z_1 = z_2$. This completes the proof.

Example 2.2. *Let $X = [0,1]$, $d(x,y) = |x-y|$ and $F = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$. Define $T : X \to X$ by*

$$T(x) = \begin{cases} \frac{1}{4}, x = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \\ 1 - x, x \ne \frac{1}{2}, \frac{1}{3}, \frac{1}{4}. \end{cases}$$

Take $\psi(t) = t$, $\phi(t) = \frac{t}{5}$, $f(s,t) = s - t$, $h(t) = t$, for each $s,t \geq 0$. It is easy to to verify that

$$h(\psi(d(Tx,Ty))) \leq f(\psi(d(x,y)),\phi(d(x,y))),$$

holds for all $x,y \in F$. Therefore, Theorem 2.1 implies that T has a fixed point and $\frac{1}{4}$ is a fixed point of T. Note that in this example without F, there do not exist $\phi,\psi : [0,\infty) \to [0,\infty)$, such that inequality (1) holds.

2.2. Fixed points in ordered metric spaces

Let $(X,\preceq)$ be a partially ordered set endowed with a metric d.

$(X,\preceq)$ is called directed if for every pair $(a,b) \in X \times X$, there exists $c \in X$ such that $a \preceq c$ and $b \preceq c$.

A mapping $T : X \to X$ is said to be nondecreasing if $x,y \in X$, $x \preceq y \Rightarrow Tx \preceq Ty$.

Theorem 2.3. *Let $(X,\preceq)$ be a partially ordered set endowed with a metric d such that (X,d) is complete. Suppose that the mapping $T : X \to X$ is nondecreasing and satisfies the following condition for all $x,y \in X$, such that $x \succeq y$,*

$$h(\psi(d(Tx,Ty))) \leq f(\psi(d(x,y)),\phi(d(x,y))),$$

where $\psi \in \Psi$,$\phi \in \Phi$, f a function of C-class, h a function of A-class. Suppose also that there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$. Further assume that either

(a) T is continuous;

(b) if $\{x_n\} \subset X$ is a nondecreasing sequence with $x_n \to z$ then $x_n \preceq z$ for every n.

Then T has a fixed point.

Proof. For any $x_0 \in X$, we can construct the sequence $\{x_n\}$ such that $x_{n+1} = Tx_n \in X$, $n \geq 0$. Since T is nondecreasing, we have

$$x_0 \preceq Tx_0 = x_1 \Rightarrow x_1 \preceq Tx_1 \ldots \Rightarrow Fx_{n-1} = x_n \preceq Tx_n \preceq x_{n+1},$$

that is,

$$x_0 \preceq x_1 \preceq x_2 \ldots \preceq x_n \preceq x_{n+1} \preceq \ldots$$

Taking $x = x_{n-1} \succeq y = x_n$ in (1), we obtain

$$\psi(d(x_n,x_{n+1})) \leq h(\psi(d(x_n,x_{n+1}))) \leq f(\psi(d(x_{n-1},x_n)),\phi(d(x_{n-1},x_n))), \tag{16}$$

for any $n \geq 1$. Since f is a function of C-class, we obtain

$$\psi(d(x_n,x_{n+1})) \leq \psi(d(x_{n-1},x_n)).$$

Now, as $\psi \in \Psi$, we have

$$d(x_n,x_{n+1}) \leq d(x_{n-1},x_n) \tag{17}$$

for every $n \geq 1$. Hence the sequence $\{d(x_n,x_{n+1})\}$ is a decreasing sequence. So for the non-negative decreasing sequence $\{d(x_n,x_{n+1})\}$, there exists some $r \geq 0$, such that

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = r. \tag{18}$$

Assume that $r > 0$. Letting $n \to \infty$ in (16) and using (18), we obtain

$$\psi(r) \leq f(\psi(r),\phi(r)). \tag{19}$$

Since f is a C-class, we find that $\phi(r) = 0$ or $\psi(r) = 0$. It further implies that $r = 0$, which is a contradiction. Hence

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{20}$$

Now, we prove that $\{x_n\}$ is a Cauchy sequence. If possible, let $\{x_n\}$ be not a Cauchy sequence. Then there exists $\delta > 0$ for which we can find subsequences $\{x_{n_k})\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ with $n_k > m_k > k$ such that

$$d(x_{n_k},x_{m_k}) \geq \delta. \tag{21}$$

Further, corresponding to m_k, we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ and satisfying (21). Therefore, we have

$$d(x_{n_k-1},x_{m_k}) < \delta. \tag{22}$$

Using (21) and (22), we have

$$0 < \delta \leq d(x_{n_k},x_{m_k}) \leq d(x_{n_k},x_{n_k-1}) + d(x_{n_k-1},x_{m_k}) < \delta + d(x_{n_k},x_{n_k-1}). \tag{23}$$

Letting $k \to \infty$ and using (20), in (23), we have

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \delta. \tag{24}$$

Consider

$$\begin{aligned} d(x_{n_k}, x_{m_k}) &\leq d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k-1}) + d(x_{m_k-1}, x_{m_k}) \\ &\leq 2d(x_{n_k}, x_{n_k-1}) + d(x_{n_k}, x_{m_k}) + 2d(x_{m_k-1}, x_{m_k}). \end{aligned} \tag{25}$$

Letting $k \to \infty$ in the above inequality (25) and using (20), (24), we get

$$\lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) = \delta. \tag{26}$$

Setting $x = x_{m_k-1}$ and $y = x_{n_k-1}$ in (1), we obtain

$$\psi(d(x_{n_k}, x_{m_k})) \leq h(\psi(d(x_{n_k}, x_{m_k})) \leq f(\psi(d(x_{n_k-1}, x_{m_k-1})), \phi(d(x_{n_k-1}, x_{m_k-1}))). \tag{27}$$

Letting $k \to \infty$, using (24) and (26), we obtain

$$\psi(\delta) \leq f(\psi(\delta), \phi(\delta)), \tag{28}$$

which is a contradiction if $\delta > 0$ This shows that $\{x_n\}$ is a Cauchy sequence in a complete metric space (X,d). This implies that there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. First, we set that T is continuous. It follows that $\lim_{n\to\infty} Tx_n = Tx^*$. Since $x_{n+1} = Tx_n$, we have also $\lim_{n\to\infty} Tx_n = x^*$. By the uniqueness of the limit, we get $Tx^* = x^*$, that is, x^* is a fixed point of T.

Now, we consider second part, that is, if $\{x_n\} \subset X$ is a nondecreasing sequence with $x_n \to z$ then $x_n \preceq z$ for every n. Since $x_0 \preceq x_1 \preceq \ldots \preceq x_n \preceq \ldots$ and $\lim_{n\to\infty} x_n = x^*$. So $x^* \succeq x_n$ for all n. Substituting $x = x^*$ and $y = x_n$ in (1), we obtain

$$\begin{aligned} \psi(d(x_n, Tx^*)) &\leq h(\psi(d(x_n, Tx^*))) \\ &\leq f(\psi(d(x_{n-1}, x^*)), \phi(d(x_{n-1}, x^*))). \end{aligned} \tag{29}$$

Letting $n \to \infty$, using continuity of ψ,ϕ and f, we have

$$\psi(d(x^*, Tx^*)) \leq f(\psi(0), \phi(0)) = 0,$$

which implies $d(x^*, Tx^*) = 0$ and hence $x^* = Tx^*$. This completes the proof.

Theorem 2.4. *In addition to the hypotheses of Theorem 2.3, suppose that $(X,\preceq)$ is directed. Then, T has a unique fixed point $x^* \in X$.*

Proof. Following the proof of Theorem 2.3, we know that T admits a fixed point $x^* \in X$ satisfying $\lim_{n\to\infty} x_n = x^*$. Suppose now that $y^* \in X$ is also a fixed point of T. We prove that $x^* = y^*$. Since $(X,\preceq)$ is directed, there exists $u \in X$ such that $x^* \preceq u$ and $y^* \preceq u$. We define the sequence $\{u_n\}$ as follows: $u_0 = u$, $u_{n+1} = Tu_n$, $n \geq 0$. Since T is nondecreasing, x^* is a fixed point of T and $x \preceq u$, we obtain $x^* \preceq u_n$, $n \geq 0$.

Now, consider (1) with $x = u_n$ and $y = x^*$. Then

$$\begin{aligned} \psi(d(u_{n+1},x^*)) &\leq h(\psi(d(Tu_n,Tx^*))) \\ &\leq f(\psi(d(u_n,x^*)),\phi(d(u_n,x^*))) \\ &\leq \psi(d(u_n,x^*)), \end{aligned} \tag{30}$$

which implies that $\psi(d(u_{n+1},x^*)) \leq \psi(d(u_n,x^*))$, $n \geq 0$. Thus, by the monotonicity of ψ, we obtain that the sequence $\{\Delta\}$ defined by $\{\Delta\} = d(u_n,x^*)$, $n \geq 0$, is nonincreasing. Hence, there exists $\Delta \geq 0$ such that

$$\lim_{n\to\infty} \Delta_n = \lim_{n\to\infty} d(u_n,x^*) = \Delta. \tag{31}$$

Now, we prove that $\Delta = 0$. Suppose, to the contrary, that $\Delta > 0$. Letting $n \to \infty$ in (30) and using (31), we get

$$\begin{aligned} \psi(\Delta)) &\leq f(\psi(\Delta),\phi(\Delta))) \\ &\leq \psi(\Delta), \end{aligned}$$

which is a contradiction. Thus $\Delta = 0$, that is, $\lim_{n\to\infty} d(u_n,x^*) = 0$. Similarly, we obtain that $\lim_{n\to\infty} d(u_n,y^*) = 0$, and hence $x^* = y^*$. This completes the proof.

Example 2.5. *Let $X = [0,1] \cup \{2,3,4,...\}$ and define the metric on X by*

$$d(x,y) = \begin{cases} |x-y|, \ \text{if } x,y \in [0,1] \text{ and } x \neq y, \\ x+y, \ \text{if } x \text{ or } y \notin [0,1] \text{ and } x \neq y, \\ 0, \ \text{if } x = y. \end{cases}$$

We see that (X,d) is a complete metric space (see [4]). Define $T : X \to X$ as

$$T(x) = \begin{cases} x - \frac{x^2}{2}, x \in [0,1], \\ x - 1, x \in \{2,3,4,...\}. \end{cases}$$

Now, define $f \in \mathfrak{C}$, $h \in \mathfrak{A}$, $\psi \in \Psi$ and $\phi \in \Phi$, for each $s,t \geq 0$, as $f(s,t) = \frac{s}{1+t}$, $h(t) = t$,

$$\psi = \begin{cases} x, x \in [0,1], \\ x^2, x > 1; \end{cases} \phi = \begin{cases} \frac{x}{2-x}, x \in [0,1], \\ \frac{1}{2x^2-1}, x > 1. \end{cases}$$

Now, consider inequality (1),

$$\begin{aligned} \psi(d(Tx,Ty)) &= h(\psi(d(Tx,Ty))) \\ &\leq f(\psi(d(x,y)), \phi(d(x,y))) \\ &= \frac{\psi(d(x,y))}{1+\phi(d(x,y))}. \end{aligned}$$

Let $x \geq y$.

Case 1. When $x, y \in [0,1]$, we have

$$\begin{aligned} \frac{\psi(d(x,y))}{1+\phi(d(x,y))} &= \frac{\psi(|x-y|)}{1+\psi(|x-y|)} = |x-y| - \frac{|x-y|^2}{2} \\ &\geq (x-y) - \frac{(x-y)(x+y)}{2} \\ &= (x - \frac{x^2}{2}) - (y - \frac{y^2}{2}) = Tx - Ty \\ &= \psi(Tx - Ty) = \psi(d(Tx,Ty)). \end{aligned}$$

Case 2. When $x \in \{3,4,...\}$, there are two sub-cases:

When $y \in [0,1]$, we have

$$d(Tx,Ty) = d(x-1, y - \frac{y^2}{2}) = (x-1) + (y - \frac{y^2}{2}) \leq x + y - 1.$$

When $y = \{2\}$, we have

$$d(Tx,Ty) = d(x-1, y-1) = (x-1) + (y-1) \leq x + y - 1.$$

Here in both the sub-cases, $d(Tx,Ty) \leq x+y-1$. Consider

$$\begin{aligned}
\frac{\psi(d(x,y))}{1+\phi(d(x,y))} &= \frac{\psi(x+y)}{1+\psi(x+y)} \\
&= \frac{(x+y)^2}{1+\frac{1}{2(x-y)^2-1}} = (x+y)^2 - \frac{1}{2} \\
&\geq (x+y)^2 - 1 \geq (d(Tx,Ty))^2 \\
&= \psi(d(Tx,Ty)).
\end{aligned}$$

Case 3. When $x = \{2\}$ and $y \in [0,1]$, we have

$$d(Tx,Ty) = d(2-1, y-\frac{y^2}{2}) = 1+(y-\frac{y^2}{2}) \leq y+1,$$

$$\begin{aligned}
\frac{\psi(d(x,y))}{1+\phi(d(x,y))} &= \frac{\psi(2+y)}{1+\psi(2+y)} = \frac{(2+y)^2}{1+\frac{1}{2(2-y)^2-1}} \\
&= (2+y)^2 - \frac{1}{2} \geq (2+y)^2 - 1 \geq (1+y)^2 \\
&= (d(Tx,Ty))^2 = \psi(d(Tx,Ty)).
\end{aligned}$$

Hence in all the above cases, we conclude that inequality (1) holds. Hence by the application of our result T has a fixed point and 0 is a unique fixed point of T.

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