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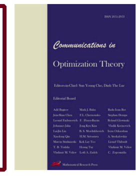

GLOBAL POSITIVE SOLUTIONS FOR A SYSTEM OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH VARIABLE DELAYS

ASMA BOUAZIZ*, LAMINE NISSE

Laboratory of Applied Mathematics, Badji Mokhtar University - Annaba, Algeria

Abstract. In this paper, by employing some fixed point theorems, we establish the existence and uniqueness of the global positive solution for a system of nonlinear fractional differential equations with variable delays, involving the Caputo fractional derivative. Some examples are provided to illustrate our main results.

Keywords. Fixed point theorem; Global positive solution; Fractional differential system; Variable delay; Caputo fractional derivative.

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1. Introduction

Fractional differential equations theory has emerged as an interesting area to explore in recent years due to its applications in engineering and other related fields. A very interesting account of the study of fractional differential equations can be found in [2, 3, 4, 5, 6, 7]. Furthermore, fractional differential equations with delays have gained much attention of research; see [1, 8] and references cited therein.

*Corresponding author.

E-mail address: a_bouaziz@hotmail.com.

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In [9], Ye, Ding and Gao have studied the existence of a positive solution for a delay fractional differential equation

$$\begin{cases} D^{\alpha}\left[x(t)-x(0)\right]=x(t)\,f(t,x_t), & t\in(0,T], \\ x(t)=\phi(t)\geq 0, & t\in[-r,0], \end{cases}$$

where $\alpha\in(0,1)$, D^{α} is the Riemann-Liouville fractional derivative, and ϕ and f are continuous. By using the sub- and super-solution method, they gave some sufficient conditions for the existence of positive solutions.

In this paper, motivated by the work in [9], we are interested in studying the existence and uniqueness of a positive solution for the fractional differential system with variable delays

$$^{c}D^{\alpha}x_i(t)=f_i\left(t,x_i(t),x_1(t-\tau_1(t)),\ldots,x_n(t-\tau_n(t))\right),\quad i\in[\![1,n]\!],\ t\geq 0, \tag{1.1}$$

$$x(t)=\Phi(t)\geq 0,\quad t\in[-\tau,0], \tag{1.2}$$

where $^{c}D^{\alpha}$ is the Caputo fractional derivative of order $\alpha\in(0,1)$, $x(t)=(x_1(t),\ldots,x_n(t))'$, where $'$ denote the transpose of the vector and $f_i:\mathbb{R}^+\times C^{n+1}\to\mathbb{R}$ are continuous such that $C=C([-\tau,\infty],\mathbb{R}^+)$ is the space of continuous function from $[-\tau,\infty]$ to $\mathbb{R}^+$. $\Phi(t)=(\phi_1(t),\ldots,\phi_n(t))'$ are given vector, such that $\phi_i(t)\in C([-\tau,0],\mathbb{R}^+)$. τ_i are continuous real-valued functions defined on $\mathbb{R}^+$ such that $\tau=\max\{\sup_{t\in\mathbb{R}^+}\tau_i(t),\, i\in[\![1,n]\!]\}>0$.

First, by using the sub- and super-solution method (on a cone), we establish the existence of a global positive solution to the problem (1.1)-(1.2). Second, we show the uniqueness of a positive solution of the problem(1.1)-(1.2) by using the Banach fixed point theorem.

The remainder of this paper is structured as follows. In Section 2, we list some preliminaries to make the paper self-contained. In Section 3, we present and prove our main results on the global existence of a positive solution. Furthermore, the uniqueness of the solution is shown in Section 4. Finally, we provide two examples to illustrate our results.

2. Preliminaries

Let E be a real Banach space. A cone K introduces a partial order $\leq$ in E in the following manner [7]

$$x\leq y\ \text{ if } y-x\in K.$$

Definition 2.1. [7] For $x, y \in E$, the order interval $\langle x, y\rangle$ is defined as

$$\langle x, y\rangle = \{z \in E : \ x \leq z \leq y\}.$$

Definition 2.2. [7] The functional $h(t, x, x_1, ..., x_n)$ is nondecreasing with respect to arguments (starting from the second argument) on $I \times E^{n+1}$, if for any $(t, \phi, \phi_1, ..., \phi_n) \in I \times E^{n+1}$ and $(t, \psi, \psi_1, ..., \psi_n) \in I \times E^{n+1}$, such that

$$\phi(\theta) \leq \psi(\theta) \text{ and } \phi_i(\theta) \leq \psi_i(\theta), \ \ i = 1, 2, ..., n, \ \theta \in [-\tau, 0],$$

$h(t, \phi, \phi_1, ..., \phi_n) \leq h(t, \psi, \psi_1, ..., \psi_n)$ holds.

Theorem 2.3. [7] *Let D be a subset of the cone K of partially ordered space E, $F : D \to E$ be nondecreasing. If there exist $x_0, y_0 \in D$ such that $x_0 \leq y_0$, $\langle x_0, y_0\rangle \subset D$ and x_0, y_0 are respectively lower and upper solutions of equation $x - F(x) = 0$, then the equation $x - F(x) = 0$ has minimum solution and maximum solution x^*, y^* in $\langle x_0, y_0\rangle$ such that $x^* \leq y^*$, when one of the following conditions holds*

(1) K is normal and F is completely continuous;

(2) K is regular and F is continuous;

(3) E is reflexive, K is normal, and F is continuous or weak continuous.

In this paper, the functions we are going to manipulate are defined on infinite (unbounded) intervals. Therefore, in our case, the theorem of Arzela-Ascoli does not work. So, we need the following modification [10].

Proposition 2.4. *Let $\Omega \subset E$. If the functions $x \in \Omega$ are almost equicontinuous on I and uniformly bounded in the sens of the norm*

$$\|x\|_q = \sup_I \{|x(t)| q(t)\},$$

where function q is positive and continuous on I and

$$\lim_{t\to\infty} \frac{p(t)}{q(t)} = 0,$$

then Ω is relatively compact in E, where $p : I := [0, \infty) \to (0, \infty)$ be a continuous function such that

$$\sup_I \{|x(t)| p(t)\} < \infty.$$

Recall that the functions $x \in \Omega$ are said to be almost equicontinuous on I if they are equicontinuous in each interval $[0,T]$, $0 < T < \infty$.

Definition 2.5. [3] For all $T > 0$, the Riemann–Liouville fractional integral of order $\alpha \in \mathbb{R}$ of a function $f \in L^1[0,T]$ is given by:

$$I^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds,\ t \in [0,T].$$

For $\alpha = 0$, we set $I^0 := Id$, the identity operator.

Definition 2.6. [3] The Caputo fractional derivative of order $\alpha \in \mathbb{R}$ of the function f with $D^n f \in L^1[0,T]$ is defined by

$${}^cD^{\alpha} f(t) = I^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,ds,\ t \geq 0,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α and $D = \frac{d}{dt}$.

Obviously, Caputo's derivative of a constant is equal to zero. From the definition of Caputo's derivative, we can acquire the following properties; see [3] and the references therein.

Lemma 2.7. *Let* $\alpha > 0$ $(n-1 < \alpha \leq n), n = [\alpha]+1$ *and* $f \in L^1[0,\infty)$. *Then*

$$ {}^cD^{\alpha} f(t) = I^{n-\alpha} D^n f(t). \tag{2.1}$$

Lemma 2.8. *Let* $\alpha > 0$. *Then*

$$ {}^cD^{\alpha} I^{\alpha} f(t) = f(t). \tag{2.2}$$

3. Existence of positive solutions

In this section, we prove the existence of global positive solutions.

Lemma 3.1. *The vector function* $x(t) := (x_1(t), \ldots, x_n(t))$ *is a solution of the problem* $(1.1)-(1.2)$ *if and only if*

$$x_i(t) = \begin{cases} \phi_i(0) + I^{\alpha} f_i(t, x_i(t), x_1(t-\tau_1(t)), \ldots, x_n(t-\tau_n(t))), & t > 0, \\ \phi_i(0), & t \in [-\tau, 0],\ \ i \in [\![1,n]\!]. \end{cases}$$

Proof. For $t>0$, equation (1.1) can be written as

$$I^{1-\alpha}Dx_i(t)=f_i(t,x_i(t),x_1(t-\tau_1(t)),...,x_n(t-\tau_n(t))).$$

Applying the operator I^α on both sides, we have

$$IDx_i(t)=I^\alpha f_i(t,x_i(t),x_1(t-\tau_1(t)),...,x_n(t-\tau_n(t))),$$
$$x_i(t)-x_i(0)=I^\alpha f_i(t,x_i(t),x_1(t-\tau_1(t)),...,x_n(t-\tau_n(t))).$$

Then

$$x_i(t)=\phi_i(0)+I^\alpha f_i(t,x_i(t),x_1(t-\tau_1(t)),...,x_n(t-\tau_n(t))). \tag{3.1}$$

Conversely, by operating ${}^cD^\alpha$ on both sides of (3.1), we get

$${}^cD^\alpha x_i(t)={}^cD^\alpha\phi_i(0)+{}^cD^\alpha I^\alpha f_i(t,x_i(t),x_1(t-\tau_1(t)),...,x_n(t-\tau_n(t))).$$

Using (2.2) and the fact that Caputo's derivative of a constant is equal to zero, we have

$${}^cD^\alpha x_i(t)=f_i(t,x_i(t),x_1(t-\tau_1(t)),...,x_n(t-\tau_n(t))).$$

This completes the proof.

Next, we consider space $E=[C([-\tau,\infty),\mathbb{R}^+)]^n$, where $[C([-\tau,\infty),\mathbb{R}^+)]^n$ is the class of all continuous column $n-$vectors function with the norm

$$\|x\|_N=\sum_{i=1}^{n}\sup_{t\in\mathbb{R}^+}\left\{e^{-Nt}|x_i(t)|\right\},\ x\in E,$$

where $N\in\mathbb{R}^+$ will be chosen later.

Next, we define the cone $K=\{x\in E: x_i(t)\geq 0,\ t\geq-\tau,\ \ i\in[\![1,n]\!]\}$, and the subset $D=\{x\in K: x(t)=\Phi(t),\ -\tau\leq t\leq 0\}\subset K$. We define the integral operator F by

$$Fx_i=\begin{cases}\phi_i(t), & t\in[-\tau,0],\\ \phi_i(0)+I^\alpha f_i(t,x_i(t),x_1(t-\tau_1(t)),...,x_n(t-\tau_n(t))), & t>0,\end{cases}\qquad i\in[\![1,n]\!]. \tag{3.2}$$

Assume that:

(H_1)**:** there exists $g_i,\psi_{ij}:\mathbb{R}^+\to\mathbb{R}^+$ continuous such that

$$|f_i(t,u_i,v_1,...,v_n)|\leq g_i(u_i)+\sum_{j=1}^{n}\psi_{ij}(v_j)\quad \text{for } i\in[\![1,n]\!].$$

(H_2)**:** $\forall\, A,\ \exists\, B_i,B_i^{'}: g_i(A)\subset B_i$ and $\psi_{ij}(A)\subset B_i^{'}$, where A,B_i and $B_i^{'}$ are bounded subset in D for $i\in[\![1,n]\!]$.

Lemma 3.2. *Assume that* $(H_1)-(H_2)$ *holds. Then operator* $F:D\to D$ *is completely continuous.*

Proof. From the assumption of continuity of f, we see that $F:D\to D$ is continuous. Let $G\subset D$ be bounded, i.e. there exists a positive constant l such that $\|x\|\le l$, $\forall x\in G$. So, for each $x\in G$, we have for $i\in[\![1,n]\!]$:

$$\begin{aligned}|Fx_i(t)| &\le |\phi_i(0)|+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f_i(s,x_i(s),x_1(s-\tau_1(s)),\ldots,x_n(s-\tau_n(s)))|ds\\ &\le |\phi_i(0)|+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g_i(x_i(s))ds+\sum_{j=1}^n\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\psi_{ij}(x_j(s-\tau_j(s)))ds.\end{aligned}$$

It follows that

$$\begin{aligned}e^{-Nt}|Fx_i(t)| &\le e^{-Nt}|\phi_i(0)|+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}e^{-N(t-s)}e^{-Ns}g_i(x_i(s))ds+\\ &+\sum_{j=1}^n\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}e^{-N(t-s+\tau_j(s))}e^{-N(s-\tau_j(s))}\psi_{ij}(x_j(s-\tau_j(s)))ds\\ &\le \|\phi\|_N+\sup_{\xi\in\mathbb{R}^+}\{e^{-N\xi}g_i(x_i(\xi))\}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}e^{-N(t-s)}ds+\\ &+\sum_{j=1}^n\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}e^{-N(t-s+\tau_j(s))}e^{-Nr_j(s)}\psi_{ij}(x_j(r_j(s)))ds\ /\ r_j(s)=s-\tau_j(s)\\ &\le \|\phi\|_N+\sup_{\xi\in\mathbb{R}^+}\{e^{-N\xi}g_i(x_i(\xi))\}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}e^{-N(t-s)}ds+\\ &+\sum_{j=1}^n\sup_{\xi\in\mathbb{R}^+}\{e^{-N\xi}\psi_{ij}(x_j(\xi))\}\int_0^{Nt}\frac{u^{\alpha-1}}{N^\alpha\Gamma(\alpha)}e^{-u}e^{-N\tau_j(t-\frac{u}{N})}du\\ &\le \|\phi\|_N+\sup_{\xi\in\mathbb{R}^+}\{e^{-N\xi}g_i(x_i(\xi))\}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}e^{-N(t-s)}ds+\\ &+\sum_{j=1}^n\sup_{\xi\in\mathbb{R}^+}\{e^{-N\xi}\psi_{ij}(x_j(\xi))\}\int_0^{Nt}\frac{u^{\alpha-1}}{N^\alpha\Gamma(\alpha)}e^{-u}du.\end{aligned}$$

From hypothesis (H_2), there exists $L_i,L_j^{'}$ such that $L_i=\sup_{t\in\mathbb{R}^+}\{e^{-Nt}|g_i(x_i(t))|\}$, $L_j^{'}=\sup_{t\in\mathbb{R}^+}\{e^{-Nt}|\psi_{ij}(x_j(t))|\}$, $\forall j$. Then

$$\begin{aligned}e^{-Nt}|Fx_i(t)| &\le \|\phi\|_N+L_i\int_0^{Nt}\frac{u^{\alpha-1}}{N^\alpha\Gamma(\alpha)}e^{-u}du+\sum_{j=1}^nL_i^{'}\int_0^{Nt}\frac{u^{\alpha-1}}{N^\alpha\Gamma(\alpha)}e^{-u}du\\ &\le l+\frac{L_i+\sum_{j=1}^nL_j^{'}}{N^\alpha}.\end{aligned}$$

Hence FG is bounded.

Next, we show that FG is locally (almost) equicontinuous. There are three possible cases for $i \in [\![1,n]\!]$:

Case 1. For each $x \in G$, $\varepsilon_i > 0, \forall T \in]0,\infty)$, $t_1,t_2 \in [0,T]$, $t_1 < t_2$. Let $\delta_i = \left(\frac{\varepsilon_i \Gamma(\alpha+1)}{2\left(c_i + \sum_{j=1}^n c'_j\right)}\right)^{\frac{1}{\alpha}}$, when $t_2 - t_1 < \delta_i$, we have:

$$
\begin{aligned}
& \left|Fx_i(t_1) - Fx_i(t_2)\right| \\
\leq\ & \tfrac{1}{\Gamma(\alpha)} \int_0^{t_1} \left((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\right) \left|g_i(x_i(s)) + \textstyle\sum_{j=1}^n \psi_{ij}\left(x_j\left(s-\tau_j(s)\right)\right)\right| ds + \\
+\ & \int_{t_1}^{t_2} \tfrac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} \left|g_i(x_i(s)) + \textstyle\sum_{j=1}^n \psi_{ij}\left(x_j\left(s-\tau_j(s)\right)\right)\right| ds \\
\leq\ & \tfrac{1}{\Gamma(\alpha)} \int_0^{t_1} \left((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\right) \left\{g_i(x_i(s)) + \textstyle\sum_{j=1}^n \psi_{ij}\left(x_j\left(s-\tau_j(s)\right)\right)\right\} ds + \\
+\ & \int_{t_1}^{t_2} \tfrac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} \left\{g_i(x_i(s)) + \textstyle\sum_{j=1}^n \psi_{ij}\left(x_j\left(s-\tau_j(s)\right)\right)\right\} ds,
\end{aligned}
$$

$\exists l > 0$ such that for $x \in G$, $\|x\| \leq l$ then $|x_i(t)| \leq le^{Nt} \leq le^{NT}$. Indeed, the subset $X = \{x(t),\ t \in [0,T], x \in G\}$ is a closed bounded subset. Then g_i, ψ_{ij} have a maximum on X. Therefore, $\exists$ $c_i, c'_i : c_i = \sup_{t\in[0,T]} g_i(x_i(t)),\ c'_i = \sup_{t\in[0,T]} \psi_{ij}(x_i(t-\tau_i)),\ \forall i,j = 1,2,...,n$. Then

$$
\begin{aligned}
& \left|Fx_i(t_1) - Fx_i(t_2)\right| \\
\leq\ & \tfrac{1}{\Gamma(\alpha)} \int_0^{t_1} \left((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\right) \left(c_i + \textstyle\sum_{j=1}^n c'_j\right) ds + \\
+\ & \int_{t_1}^{t_2} \tfrac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} \left(c_i + \textstyle\sum_{j=1}^n c'_j\right) ds \\
\leq\ & \tfrac{c_i + \sum_{j=1}^n c'_j}{\Gamma(\alpha)} \left\{\int_0^{t_1} \left((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\right) ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} ds\right\} \\
\leq\ & \tfrac{c_i + \sum_{j=1}^n c'_j}{\alpha\Gamma(\alpha)} \left\{(t_2-t_1)^{\alpha} + t_1^{\alpha} - t_2^{\alpha} + (t_2-t_1)^{\alpha}\right\} \\
<\ & 2\tfrac{c_i + \sum_{j=1}^n c'_j}{\Gamma(\alpha+1)} (t_2-t_1)^{\alpha} < 2\tfrac{c_i + \sum_{j=1}^n c'_j}{\Gamma(\alpha+1)} \delta_i^{\alpha} = \varepsilon_i.
\end{aligned}
$$

Case 2. For each $x \in G$, $\varepsilon_i > 0$, $t_1 \in [-\tau, 0]$, $t_2 \in [0, T], \forall T \in [0, \infty)$. Since $\phi_i \in C[-\tau, 0]$, we see that $\exists \delta' : |\phi_i(t_1) - \phi_i(0)| < \frac{\varepsilon_i}{2}$ when $0 - t_1 < \delta'$. If $t_2 - t_1 < \delta_i$, $\delta_i = \min\left(\delta', \left(\frac{\varepsilon_i \Gamma(\alpha+1)}{2\left(c_i + \sum_{j=1}^{n} c'_j\right)}\right)^{\frac{1}{\alpha}}\right)$, we have

$$
\begin{aligned}
& |Fx_i(t_1) - Fx_i(t_2)| \\
\leq\ & |\phi_i(t_1) - \phi_i(0)| + \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} |f_i(s, x_i(s), x_1(s - \tau_1(s)), \ldots, x_n(s - \tau_n(s)))| \, ds \\
\leq\ & \frac{\varepsilon_i}{2} + \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} \left\{ g_i(x_i(s)) + \sum_{j=1}^{n} \psi_{ij}\left(x_j\left(s - \tau_j(s)\right)\right) \right\} ds \\
<\ & \frac{\varepsilon_i}{2} + \frac{c_i + \sum_{j=1}^{n} c'_j}{\Gamma(\alpha+1)} \delta_i^{\alpha} \\
<\ & \frac{\varepsilon_i}{2} + \frac{\varepsilon_i}{2} = \varepsilon_i.
\end{aligned}
$$

Case 3. For each $x \in G$, $\varepsilon_i > 0$, $t_1, t_2 \in [-\tau, 0]$, by continuity of ϕ_i, when $t_2 - t_1 < \delta_i$, we have

$$|Fx_i(t_1) - Fx_i(t_2)| = |\phi_i(t_1) - \phi_i(t_2)| < \varepsilon_i.$$

Therefore, FG is equicontinuous in each bounded interval. We now appeal proposition (2.4) to conclude that FG is relatively compact. Hence operator F is completely continuous. This completes the proof.

Definition 3.3. The function $u \in E$ is called a lower solution of problem (1.1)-(1.2) if

$${}^{c}D^{\alpha} u_i(t) \leq f_i(t, u_i(t), u_1(t - \tau_1(t)) \ldots, u_n(t - \tau_n(t))), \quad t \geq 0, \ \ i \in [\![1, n]\!],$$

and

$$u(t) \leq \Phi(t), \quad t \in [-\tau, 0].$$

Similarly, function $v \in E$ is called an upper solution of problem (1.1)-(1.2) if

$${}^{c}D^{\alpha} v_i(t) \geq f_i(t, v_i(t), v_1(t - \tau_1(t)) \ldots, v_n(t - \tau_n(t))), \quad t \geq 0, \ \ i \in [\![1, n]\!],$$

and

$$v(t) \geq \Phi(t), \quad t \in [-\tau, 0].$$

If the strict inequalities hold, then $u(t), v(t)$ are called strict lower and upper solutions.

Theorem 3.4. *Assume that* $(H_1)-(H_2)$ *hold, and* $(H_3:)$ *for* $i \in [\![1,n]\!]$: $f_i : \mathbb{R}^+ \times E^{n+1} \to \mathbb{R}$ *is continuous and nondecreasing function for each* $t \in [0,\infty)$. (H_4) $u_0 = \left(u_0^1,...,u_0^n\right)', v_0 = \left(v_0^1,...,v_0^n\right)'$ *are respectively lower and upper solutions of* (1.1)-(1.2) *satisfying* $u_0(t) \leq v_0(t)$, $t \in [0,\infty)$, $u_0, v_0 \in D$. *Then* (1.1)-(1.2) *has at least a global positive solution.*

Proof. By Lemma (3.2), we have $F : D \to D$ is completely continuous. And by (3.2), u_0^i, v_0^i are lower and upper solutions of F respectively. By (H_3), $x, y \in D$, $x \leq y$, we have for $i \in [\![1,n]\!]$:

$$\begin{aligned} Fx_i(t) &= x_i(0) + I^\alpha f_i(t, x_i(t), x_1(t-\tau_1(t)), ..., x_n(t-\tau_n(t))) \\ &\leq y_j(0) + I^\alpha f_i(t, y_i(t), y_1(t-\tau_1(t)), ..., y_n(t-\tau_n(t))) \\ &\leq Fy_i(t). \end{aligned}$$

Hence F is a nondecreasing operator. Clearly, for $i \in [\![1,n]\!]$: $Fu_0^i \geq u_0^i$, $Fv_0^i \leq v_0^i$ by definition of lower and upper solution of F. Hence, $F : \langle u_0, v_0 \rangle \to \langle u_0, v_0 \rangle$ is a compact continuous operator. As K is a normal cone, by Theorem 2.3, F has a fixed point $x \in \langle u_0, v_0 \rangle$. This completes the proof.

Remark 3.5. The condition (H_4) in Theorem 3.4 can be replaced by (H_5) : there exists a positive function $H(t) = (h_1(t), ..., h_n(t))'$, $t \geq 0$, such that

$$f_i(t, x_i(t), x_1(t-\tau_1(t)), ..., x_n(t-\tau_n(t))) \leq h_i(t), \quad t \geq 0,$$

where $\forall\ t \geq 0 : \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h_i(s)\, ds < \infty$. We can easily find lower and upper solutions of the problem as follow. Consider the problem

$$\begin{cases} {}^cD^\alpha u_0^i(t) = 0, & t \geq 0 \\ u_0^i(t) = \phi_i(t), & -\tau \leq t \leq 0 \end{cases}, \ i = 1, 2, ..., n. \tag{3.3}$$

Obviously, the equation ${}^cD^\alpha u_0^i(t) = 0$ has a solution $u_0^i(t) = \phi_i(0)$, $t \geq 0$, $i = 1, 2, ..., n$, which is a lower solution of the problem $(1.1)-(1.2)$. Similarly, consider the problem

$$\begin{cases} {}^cD^\alpha v_0^i(t) = h_i(t) \geq f_i(t, x_i(t), x_1(t-\tau_1(t)), ..., x_n(t-\tau_n(t))), & t \geq 0 \\ v_0^i(t) = \phi_i(t), \quad -\tau \leq t \leq 0 \end{cases}, \ i \in [\![1,n]\!].$$

We have

$$v_0^i(t) = \phi_i(0) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h_i(s)\, ds, \ t \geq 0,$$

which is an upper solution of the problem $(1.1)-(1.2)$ and $u_0^i(t) \leq v_0^i(t)$. By Theorem 3.4, $(1.1)-(1.2)$ has at least a positive solution.

4. Uniqueness of solutions

In this section, we discuss the uniqueness of solutions.

Theorem 4.1. *Let* $f_i : \mathbb{R}^+ \times [C([-\tau,\infty),\mathbb{R}^+)]^{n+1} \to \mathbb{R}$ *be continuous and satisfy the Lipschitz condition for* $i \in [\![1,n]\!]$*:* $|f_i(t,u_i,u_1,...,u_n) - f_i(t,v_i,v_1,...,v_n)| \le l_i|u_i - v_i| + \sum_{j=1}^n k_{ij}\left|u_j - v_j\right|,\ i \in [\![1,n]\!]$. *Then* $(1.1)-(1.2)$ *has a unique global positive solution.*

Proof. Letting $u,v \in D$, we have

$$\begin{aligned}
& |Fu_i(t) - Fv_i(t)| \\
\le\ & \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left\{ l_i|u_i(s) - v_i(s)| + \sum_{j=1}^n k_{ij}\left|u_j\left(s-\tau_j(s)\right) - v_j\left(s-\tau_j(s)\right)\right|\right\} ds \\
\le\ & l_i \int_0^t \tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|u_i(s) - v_i(s)|\,ds + \sum_{j=1}^n k_{ij}\int_0^t \tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|u_j\left(s-\tau_j(s)\right) - v_j\left(s-\tau_j(s)\right)\right| ds.
\end{aligned}$$

Let $l = \sum_{i=1}^n |l_i|,\ k = \sum_{i=1}^n |k_i| = \sum_{j=1}^n \max_{\forall i}\left|k_{ij}\right|$. Then

$$\begin{aligned}
& e^{-Nt}\left|Fu_j(t) - Fv_i(t)\right| \\
\le\ & l_i \int_0^t \tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s)}e^{-Ns}|u_i(s) - v_i(s)|\,ds + \\
+\ & k_i \sum_{j=1}^n \int_0^t \tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s+\tau j)}e^{-N\left(s-\tau_j\right)}\left|u_j\left(s-\tau_j(s)\right) - v_j\left(s-\tau_j(s)\right)\right| ds. \\
\le\ & \|u-v\|_N \tfrac{l_i}{N^\alpha}\int_0^{Nt} \tfrac{u^{\alpha-1}e^{-u}}{\Gamma(\alpha)}du + nk_i\|u-v\|_N \int_0^{Nt} \tfrac{u^{\alpha-1}e^{-u}}{N^\alpha\Gamma(\alpha)}du \\
\le\ & \tfrac{l_i}{N^\alpha}\|u-v\|_N + \tfrac{k_i}{N^\alpha}\|u-v\|_N .
\end{aligned}$$

Hence, we have

$$\begin{aligned}\|Fu-Fv\|_N &= \sum_{i=1}^{n}\sup_{t\in\mathbb{R}^+} e^{-Nt}\,|Fu_i(t)-Fv_i(t)| \\ &\le \sum_{i=1}^{n}\tfrac{l_i+nk_i}{N^\alpha}\,\|u-v\|_N \\ &\le \tfrac{l+nk}{N^\alpha}\,\|u-v\|_N\,.\end{aligned}$$

It follows that

$$\|Fu-Fv\|_N \le \frac{l+nk}{N^\alpha}\,\|u-v\|_N\,.$$

We choose N large enough such that $\frac{l+nk}{N^\alpha}<1$. By Banach fixed point theorem, F has a unique fixed point in D, which is the unique positive solution. This completes the proof.

5. Examples

Example 5.1. Consider the problem

$$\begin{cases} {}^cD^\alpha x_i(t)=\sum_{j=1}^{n}a_{ij}x_j\left(t-\tau_j(t)\right), & t\ge 0\\ x(t)=\Phi(t)\ge 0, & -\tau\le t\le 0\end{cases}\quad,\ i\in[\![1,n]\!], \tag{5.1}$$

where $A=\left(a_{ij}\right)_{n\times n}$ is given matrix. The hypothesis (H_1)-(H_4) are verified. By Theorem 3.4, problem (5.1) has at least a global positive solution. In addition, $\sum_{j=1}^{n}a_{ij}x_j\left(t-\tau_j(t)\right)$ satisfies the Lipschitz condition. By Theorem 4.1, the problem (5.1) has unique global positive solution.

Example 5.2. Consider the problem

$$\begin{cases} {}^cD^\alpha x_i(t)=\sum_{k=1}^{n}\frac{x_i^2(t-\tau_k(t))}{1+x_i^2(t-\tau_k(t))}, & t\ge 0\\ x(t)=\Phi(t)\ge 0, & -\tau\le t\le 0\end{cases}\quad,i\in[\![1,n]\!]. \tag{5.2}$$

The hypothesis $(H_1)-(H_4)$ are verified. Using Theorem 3.4, we find that problem (5.2) has at least a global positive solution.

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