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COMMON FIXED POINT THEOREMS FOR EXPANSIVE MAPPINGS VIA AN IMPLICIT RELATION

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Abstract. In this paper, we prove some common fixed point theorems for two weakly subsequentially continuous pairs of mappings which are compatible of type (E) and satisfy implicit relations in metric spaces. The results presented in this article generalize and extend some previous known results.

Keywords. Integral expansive condition; Weakly subsequentially continuous; Compatible of type (E); Implicit function.

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1. Introduction

Fixed point problems of integral type contractions was first initiated by Branciari [1]. In [1], Branciari obtain a generalized version of the Banach contraction principle. Suziki [2] showed that integral contractions are Meer-keeler type contractions in a integral form. Recently, Wang [3] proved some results on the existence of fixed points for expansive mappings. Subsequently, many authors obtained some common fixed point results via the expansive condition, see [4, 5, 6] and the references therein. In [7], Jungck introduced the notion of compatible maps. In [8], Jungck and Rhoades further weakened the concept of compatibility to the weak compatibility. Recently Al-Thagafi and Shahzad [9] gave a generalization, which is called the occasional weak

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compatibility property. We remark here that this notion is weaker than the weak compatibility; see Jungck and Rhoades [8] and the references therein. Recently, Doric *et al.* [10] showed that the condition of occasionally weak compatibility reduces to weak compatibility in the case that the two mappings have a unique point of coincidence (or a unique common fixed point). In 2009, Bouhadjera and Tobie [11] introduced the concepts of subcompatibility and subsequential continuity which are more general than the occasional weak compatibility and the reciprocal continuity. Later, Imdad *et al.* [12] improved the results of Bouhadjera and Godet Thobie [11] by using the subcompatibility with reciprocal continuity or subsequential continuity with compatibility.

In this paper, we prove some common fixed point theorems for two weakly subsequentially continuous pairs of mappings which are compatible of type (E) and satisfy implicit relations in metric spaces. The results presented in this article generalize and extend some previous known results.

2. Preliminaries

Definition 2.1. Let (X,d) be a metric space. Two self mappings $A,S:X\to X$ are said to be

(1) Compatible [7] iff

$$\lim_{n\to\infty} d(ASx_n, SAx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t\in X$.

(2) Compatible mappings of type (A) [8] iff

$$\lim_{n\to\infty} d(ASx_n, S^2x_n) = 0 \text{ and } \lim_{n\to+\infty} d(SAx_n, A^2x_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to+\infty} Sx_n = t$, for some $t\in X$.

(3) Compatible mappings of type (B) [13] iff

$$\lim_{n\to\infty} d(SAx_n, A^2x_n) \le \frac{1}{2}\Big[\lim_{n\to\infty} d(SAx_n, St) + \lim_{n\to\infty} d(St, S^2x_n)\Big] \text{ and}$$

$$\lim_{n\to\infty} d(ASx_n, S^2x_n) \le \frac{1}{2}\Big[\lim_{n\to\infty} d(ASx_n, At) + \lim_{n\to\infty} d(At, A^2x_n)\Big],$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t\in X$.

(4) Compatible mappings of type (P) [14] iff

$$\lim_{n\to\infty} d(A^2x_n, S^2x_n) = 0,$$

where $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$.

(5) Compatible mappings of type (C) [15] iff

$$\lim_{n\to\infty} d(ASx_n, S^2x_n) \le \frac{1}{3}[\lim_{n\to\infty} d(ASx_n, At) + \lim_{n\to\infty} d(At, S^2x_n) + \lim_{n\to\infty} d(At, A^2x_n)]$$

and

$$\lim_{n\to\infty} d(SAx_n, A^2x_n) \le \frac{1}{3}[\lim_{n\to\infty} d(SAx_n, St) + \lim_{n\to\infty} d(St, S^2x_n) + \lim_{n\to\infty} d(St, S^2x_n)],$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$.

(6) Weakly compatible [16] iff they commute in their coincidence points, i.e., if $Au = Su$, then $ASu = SAu$, for some $u \in X$.

If A and S are compatible or compatible of type (A) or compatible of type (B) or compatible of type (C) or compatible of type (P), then they are weakly compatible. Singh *et al.* [17] introduced the notion of compatibility of type (E) as follows.

Definition 2.2. Two self maps A and S of a metric space (X,d) are said to be compatible of type (E) iff $\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} SAx_n = At$ and $\lim_{n\to\infty} A^2x_n = \lim_{n\to+\infty} ASx_n = St$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$.

Remark 2.3. If $At = St$, then compatible of type (E) implies compatible (compatible of type (A), compatible of type (B), compatible of type (C), compatible of type (P)), however the converse may not be true. In general case, compatibility of type (E) implies compatibility of type (B).

In 2011, Singh *et al.* [18] used the two following definitions and the concept of reciprocal continuity to prove the existence of fixed point for two pairs of self mappings in metric spaces.

Definition 2.4. Two self maps A and S of a metric space (X,d) are A-compatible of type (E) iff $\lim_{n\to\infty} A^2x_n = \lim_{n\to\infty} ASx_n = St$, for some $t \in X$.

Definition 2.5. Two self maps A and S of a metric space (X,d) are S-compatible of type (E) iff $\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} SAx_n = At$, for some $t \in X$.

Notice that if A and S are compatible of type (E), then they are A-compatible and S-compatible of type (E), but the converse may not be true.

Example 2.6. Let $X=[0,2], d(x,y)=|x-y|$. We define A and S as follows:

$$Ax=\begin{cases} 1, & 0\le x<1,\\ 2-x, & 1\le x\le 2,\end{cases} \quad Sx=\begin{cases} 0, & 0\le x<1,\\ \frac{x+1}{2}, & 1\le x\le 2.\end{cases}$$

Consider $\{x_n\}$ defined by $x_n=1+\dfrac{1}{n}$, $\forall n\ge 1$. It is clear that, for all $n\ge 1$, $1<x_n\le 2$,

$$\lim_{n\to\infty} Ax_n=\lim_{n\to\infty} Sx_n=1,$$

$$A^2x_n=A(2-x_n)=1\longrightarrow 1=S(1) \text{ and}$$

$$ASx_n=A(x_n)=2-x_n\longrightarrow 1=S(1).$$

Then (A,S) is A-compatible of type (E), but

$$SAx_n=S(2-x_n)=0\longrightarrow 0\neq A(1).$$

So (A,S) is not compatible of type (E).

More recently, Al-Thagafi and Shahzad [9] introduced the notion of occasionally weakly compatible maps in metric spaces.

Two self mappings f and g of a metric space (X,d) are to be occasionally weakly compatible (owc) iff there exists a point $u\in X$ such that $fu=gu$ and $fgu=gfu$ for some $u\in X$. The last concept generalizes the notion of weak compatibility, i.e., weak compatibility implies occasional weakly compatibility. In 2009, Bouhadjera and Thobie [11] introduced the concepts of sub-compatibility and subsequential continuity as follows.

Two self-mappings A and S on a metric space (X,d) are said to be sub-compatible iff there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} Ax_n=\lim_{n\to\infty} Sx_n=z \text{ and } \lim_{n\to\infty} d(ASx_n,SAx_n)=0,$$

for some $z\in X$. The pair (A,S) is called to be subsequentially continuous iff there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Ax_n=\lim_{n\to\infty} Sx_n=z$, and $\lim_{n\to\infty} ASx_n=Az$, $\lim_{n\to\infty} SAx_n)=Sz$, for some $z\in X$. Notice that two occasionally weakly compatible mappings are subcompatible , however the converse may not be true.

Definition 2.7. [19] Let A and S two self mappings of a metric space (X,d), the pair (A,S) is to be weakly subsequentially continuous if there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

for some $z \in X$ and

$$\lim_{n\to\infty} ASx_n = Az, \text{ or } \lim_{n\to\infty} Sf(x_n) = Sz.$$

It is easy to show that the subsequential continuity implies weak subsequential continuity, but the converse is not true in general.

Example 2.8. Let $X = [0,\infty)$ and d be the euclidian metric. We define A,S by

$$Ax = \begin{cases} x+3, & 0 \le x \le 2, \\ 0, & x > 2, \end{cases} \qquad Sx = \begin{cases} 2x+1, & 0 \le x \le 2, \\ \frac{x+5}{2}, & x > 2. \end{cases}$$

It is clear that A and S are not continuous at 2. We consider $\{x_n\}$ defined by $x_n = 2 - \frac{1}{n}$, $\forall n \ge 1$. Hence, we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 5,$$

$$\lim_{n\to\infty} SAx_n = \lim_{n\to\infty} S(5 - \frac{1}{n}) \to 5 = S(2).$$

Then $\{A,S\}$ is weakly subsequentially continuous, but

$$\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} A(5 - \frac{1}{n}) = 0 \ne A(2),$$

which implies that A and S are not reciprocally continuous.

Definition 2.9. [19] Let A and S two self mappings of a metric space (X,d). (A,S) is said to be A-subsequentially continuous iff there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$$

for some $z \in X$ and

$$\lim_{n\to\infty} ASx_n = Az.$$

Definition 2.10. [19] Let A and S two self mappings of a metric space (X,d). (A,S) is said to be S-subsequentially continuous iff there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

for some $z\in X$ and

$$\lim_{n\to\infty} SAx_n = Sz.$$

Example 2.11. Let $X=[0,\infty)$ and d is the euclidian metric. We define f,S by

$$Ax=\begin{cases} 2x+1, & 0\le x\le 1,\\ x+4, & x>1,\end{cases} \qquad Sx=\begin{cases} x+2, & 0\le x\le 1,\\ 0, & x>1.\end{cases}$$

Consider $\{x_n\}$ defined by $x_n = 1-\frac{1}{n}$, $\forall n\ge 1$. Hence, we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 3,$$

$$\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} A(3-\frac{1}{n}) \to 7 = A(3).$$

Then $\{A,S\}$ is A-subsequentially continuous, but

$$\lim_{n\to\infty} SAx_n = \lim_{n\to\infty} S(3-\frac{2}{n}) = 0 \neq S(3).$$

So the pair $\{A,S\}$ is not reciprocally continuous.

Notice that two subsequentially continuous mappings A and S are A-subsequentially continuous and S-subsequentially continuous, the converse may not be true. Also if the pair $\{A,S\}$ is A or S-subsequentially continuous, then it is weakly subsequentially continuous.

Let $\mathscr{G}$ is the set of of all upper semicontinuous functions $G:\mathbb{R}_+^6\to\mathbb{R}^+$ such that

$$G(u,u,0,0,u,u)<0, \quad \forall u>0.$$

Example 2.12.

$$G(t_1,t_2,t_3,t_4,t_5,t_6) = at_1^2 - bt_3^2 - \frac{ct_5t_6}{dt_3^2+et_4^2+1},$$

where $c,d,e\ge 0$, $b>a>0, c>a>0$.

Example 2.13.

$$G(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - [at_2^2 + \frac{bt_3^2+ct_4^2}{t_5t_6+1}]^{\frac{1}{2}},$$

where $b\ge 0$, $c<1, a>1$.

Example 2.14.

$$G(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - [at_2^p + bt_3^p + ct_4^p]^{\frac{1}{p}} - dt_5t_6,$$

where $a>(1+d)^p$, $d\ge 0, c\ge 0, b<1, p\in\mathbb{N}$.

The aim of this paper is to prove some common fixed point theorems for two weakly subsequentially continuous pairs of mappings which are compatible of type (E) and satisfy implicit relations in metric spaces. The results presented in this article generalize and extend the results in Bouhadjera and Thobie [11] and some other corresponding results.

3. Main results

Theorem 3.1. *Let A,B,S and T be self maps on a metric space (X,d) into itself such for all $x,y in X$. Then*

$$G(\int_0^{d(Ax,By)}\varphi(t)dt,\int_0^{d(Sx,Ty)}\varphi(t)dt,\int_0^{d(Ax,Sx)}\varphi(t)dt,$$
$$\int_0^{d(By,Ty)}\varphi(t)dt,\int_0^{d(Ax,Ty)}\varphi(t)dt,\int_0^{d(By,Sx)}\varphi(t)dt)\geq 0. \tag{3.1}$$

If (A,S) and (B,T) are weakly subsequentially continuous and compatible of type (E) as well as B and T, then A,B,S and T have a unique common fixed point in X.

Proof. Since the pair (A,S) is weakly subsequentially continuous, we see that there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty}Ax_n=\lim_{n\to\infty}Sx_n=z$ and $\lim_{n\to\infty}ASx_n=Az$, $\lim_{n\to\infty}SAx_n=Sz$. The pair (A,S) is compatible of type (E) implies

$$\lim_{n\to\infty}A^2x_n=\lim_{n\to\infty}ASx_n=Sz,\ \lim_{n\to\infty}S^2x_n\lim_{n\to\infty}SAx_n=Az.$$

Then $Az=Sz$. Similarly, for (B,T), there is a sequence $\{y_n\}$ in X such

$$\lim_{n\to\infty}By_n=\lim_{n\to\infty}Ty_n=t$$

and

$$\lim_{n\to\infty}BTy_n=Bt,\ \lim_{n\to\infty}TBy_n=Tt.$$

The pair (B,T) is compatible of type (E) implies

$$\lim_{n\to\infty}B^2x_n=\lim_{n\to\infty}BTx_n=Tt$$

and $\lim_{n\to\infty}T^2x_n=\lim_{n\to\infty}TBy_n=Bt$. Consequently $Bt=Tt$. We are in a position to claim $Az=Bt$. If not, we get

$$G\Big(\int_0^{d(Az,Bt)}\varphi(t)dt,\int_0^{d(Sz,Tt)}\varphi(t)dt,\int_0^{d(Az,Sz)}\varphi(t)dt,$$
$$\int_0^{d(By,Tt)}\varphi(t)dt,\int_0^{d(Az,Tt)}\varphi(t)dt,\int_0^{d(Bt,Sz)}\varphi(t)dt\Big)\geq 0,$$

$$G\Big(\int_0^{d(Az,Bt)}\varphi(t)dt,\int_0^{d(Az,Bt)}\varphi(t)dt,0,0,\int_0^{d(Az,Bt)}\varphi(t)dt,\int_0^{d(Bt,Az)}\varphi(t)dt\Big)\geq 0,$$

which is a contradiction, then $Az = Bt$. Now, we show $z = t$. If not, we get

$$G\Big(\int_0^{d(Ax_n,By_n}\varphi(t)dt),\int_0^{d(Sx_n,Ty_n}\varphi(t)dt),\int_0^{d(Ax_n,Sx_n}\varphi(t)dt,$$
$$\int_0^{d(By_n,Ty_n)}\varphi(t)dt),\int_0^{d(Ax_n,Ty_n)}\varphi(t)dt,\int_0^{d(By_n,Sx_n)}\varphi(t)dt\Big)\geq 0.$$

Letting $n\to\infty$, we get

$$G\Big(\int_0^{d(z,t)}\varphi(t)dt,\int_0^{d(z,t)}\varphi(t)dt,0,0,\int_0^{d(z,t)}\varphi(t)dt,\int_0^{d(z,t)}\varphi(t)dt\Big)\geq 0,$$

which is a contradiction. This proves that $z = t$. Next, we prove $z = Az$. If not, we find that

$$G\Big(\int_0^{d(Az,By_n}\varphi(t)dt),\int_0^{d(Sz,Ty_n})\varphi(t)dt,\int_0^{d(Az,Sz)}\varphi(t)dt,$$
$$\int_0^{d(By_n,Ty_n)}\varphi(t)dt,\int_0^{d(Az,Ty_n)}\varphi(t)dt,\int_0^{d(By_n,Sz)}\varphi(t)dt\Big)\geq 0.$$

Letting $n\to\infty$, we get

$$G\Big(\int_0^{d(Az,z)}\varphi(t)dt,\int_0^{d(Az,z)}\varphi(t)dt,0,0,\int_0^{d(Az,z)}\varphi(t)dt,\int_0^{d(Az,z)}\varphi(t)dt\Big)\geq 0,$$

which is a contradiction. Hence, $z = Az = Bz = Sz = Tz$. For the uniqueness, suppose there exists an other common point $w\in X$. It follows that

$$G\Big(\int_0^{d(Az,Bw)}\varphi(t)dt,\int_0^{d(Sz,Tw)}\varphi(t)dt,0,0,\int_0^{d(Az,Tw)}\varphi(t)dt,\int_0^{d(Sz,Bw)}\varphi(t)dt\Big)$$
$$G\Big(\int_0^{d(z,w)}\varphi(t)dt,\int_0^{d(z,w)}\varphi(t)dt,0,0,\int_0^{d(z,w)}\varphi(t)dt,\int_0^{d(z,w)}\varphi(t)dt\Big)\geq 0,$$

which is a contradiction. This obtains $w = z$. This completes the proof.

If $A = B$ and $S = T$, we get the following result.

Corollary 3.2. *Let (X,d) be a metric space and let $S,A : X\to X$ be two self mappings such that, $\forall x,y\in X$,*

$$G\Big(\int_0^{d(Ax,By)}\varphi(t)dt,\int_0^{(Sx,Sy)}\varphi(t)dt,\int_0^{d(Sx,Ax)}\varphi(t)dt,$$
$$\int_0^{d(Ay,Sy)}\varphi(t)dt,\int_0^{d(Ax,Sy)}\varphi(t)dt,\int_0^{d(Ay,Sx)}\varphi(t)dt\Big)\geq 0,$$

where $G\in\mathscr{G}$. Assume that the pair (A,S) is wsc compatible of type (E). Then A and S have a unique common fixed point in X.

Putting $d = 0$ in Example 2.12, we find from Theorem 3.1 the following result.

Corollary 3.3. *Let A,B,S and T be self maps in a metric space (X,d) such that*

$$\Big(\int_0^{d(Ax,By)}\varphi(t)dt\Big)^p \geq a\Big(\int_0^{d^p(Sx,Ty)}\varphi(t)dt\Big)^p + b\Big(\int_0^{d(Sx,Ty)}\varphi(t)dt\Big)^p + c\Big(\int_0^{d(Sx,Ty)}\varphi(t)dt\Big)^p,$$

where a,b,c are nonnegative real numbers such $a>1$, $0\leq b,c<1$ and $p\in\mathbb{N}^*$. If $\{A,S\}$ is weakly subsequentially continuous and compatible of type (E) as well as B and T, then A,B,S and T have a unique common fixed point.

We remark here that Corollary 3.3 generalizes the corollary 3.3 in Djoudi [20].

Theorem 3.4. *Let S,T and $\{f_n\}_{n\in\mathbb{N}^*}$ be self maps on metric space (Xd) into itself such S and $\{f_n\}$ are weakly subsequentially continuous and compatible of type (E) as well as T and $\{f_{n+1}\}$. Assume that*

$$G(\int_0^{d(f_nx,f_{n+1}y)}\varphi(t)dt, \int_0^{d(Sx,Ty)}\varphi(t)dt, \int_0^{d(f_nx,Sx)}\varphi(t)dt,$$
$$\int_0^{d(f_{n+1}y,Ty)}\varphi(t)dt, \int_0^{d(f_nx,Ty)}\varphi(t)dt, \int_0^{d(f_{n+1}y,Sx)}\varphi(t)dt)\geq 0,$$

for all $x,y\in X$ and $n\in\mathbb{N}^*$, where $G\in\mathscr{G}$. Then S,T and $\{f_n\}$ have a unique common fixed point in X.

Proof. For $n=1$, the four mappings f_1,f_2,S and T satisfy all the hypothesis of Theorem 3.1. Then they have a unique common fixed point z. So z is a common fixed point for S,T and f_1. For S,T and f_2, if t another fixed point for S,T and f_1 such $t\neq z$, we get

$$G(\int_0^{d(f_1z,f_2t)}\varphi(t)dt, 0, 0, \int_0^{d(f_2z,Tt)}\varphi(t)dt, \int_0^{d(f_1z,Tt)}\varphi(t)dt), \int_0^{d(f_2t,Sz)}\varphi(t)dt) =$$

$$G(\int_0^{d(z,t)}, 0, 0, \int_0^{d(t,t)}\varphi(t)dt, \int_0^{d(z,t)}\varphi(t)dt, \int_0^{d(t,z)}\varphi(t)dt)\geq 0,$$

which is a contradiction. Hence $z=t$. Similarly, by using the same method, we get that z is the unique fixed point for S,T and f_2. For $n=2$, the mappings f_2,f_3,S and T satisfy all hypothesis of Theorem 3.1, they have a unique common fixed point. Using the same method, we find the the uniqueness. By continuing in this manner, we find that z is the required point.

We remark here that Theorem 3.4 extends the theorem 3.4 in Djoudi [20].

Corollary 3.5. *Let (X,d) be a metric space and let A,B and S be three self mappings such that $\forall x,y\in X$,*

$$G\Big(\int_0^{d(Sx,Sy)}\varphi(t)dt, \int_0^{d(Ax,By)}\varphi(t)dt, \int_0^{d(Ax,Sx)}\varphi(t)dt,$$

$$\int_0^{d(By,Sy)} \varphi(t)dt, \int_0^{d(Ax,Sy)} \varphi(t)dt, \int_0^{d(By,Sx)} \varphi(t)dt\Big) \geq 0,$$

where $G \in \mathscr{G}$. If the pairs (A,S) and (B,S) are wsc and compatible of type (E), then A and S have a unique common fixed point in X.

Theorem 3.6. *Let A,B,S and T be self maps on a metric space (X,d) into itself satisfying (3.1). If the two pairs $(A,S),(B,T)$ are A-subsequentially continuous and A compatible of type (E)(or S-subsequentially continuous and S compatible of type (E)) and (B,T) is B-subsequentially continuous and B-compatible of type (E)(or T-subsequentially continuous and T-compatible of type (E)), then A,B,S and T have a unique common fixed point in X.*

Example 3.7. Let $X = [0,2]$ and d is the euclidian metric. We define A,B,S and T by

$$Ax = Bx = \begin{cases} 1, & 0 \leq x \leq 2, \\ \frac{1}{2}, & 2 < x \leq 4, \end{cases} \qquad Sx = \begin{cases} x, & 0 \leq x \leq 1, \\ \frac{1}{4}, & 1 < x \leq 2, \end{cases}$$

$$Tx = \begin{cases} 2-x, & 0 \leq x \leq 1, \\ 2, & 1 < x \leq 2. \end{cases}$$

Consider $\{x_n\}$ defined by $x_n = 1 - \frac{1}{n}$, $\forall n \geq 1$. It is clear that $\lim_{n\to\infty} Ax_n = 2$ and $\lim_{n\to\infty} Sx_n = 1$. We also have

$$\lim_{n\to\infty} ASx_n = 1 = A(1)$$
$$= S(1),$$
$$\lim_{n\to\infty} A^2x_n = S(1) = 1.$$

Then (A,S) is A-subsequentially continuous and A-compatible of type (E).

On the other hand, we consider a sequence $\{y_n\}$ defined by: $y_n = 1$, for all $n \geq 0$. It is clear that

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 2,$$

and

$$\lim_{n\to\infty} BTy_n = B(1)$$
$$= T(1) = 1,$$
$$\lim_{n\to\infty} B^2y_n = T(1) = 1.$$

This yields that (B,T) is B-subsequentially continuous and B-compatible of type (E). We utilize Corollary 3.2 with $a = \frac{4}{3}$ that $b = c = \frac{1}{3}, p = 1$ and $\varphi(t) = \frac{3}{2}\sqrt{t}$. Using (3.1), we obtain that

(1) For $x, y \in [0,1]$, we have

$$d(Sx,Ty) = 2-(x+y),$$

$$d(Ax,By) = 0, \;\; d(Ax,Sx) = 1-x, \;\; d(By,Ty) = 1-y,$$

$$\int_0^{d(Sx,Ty)} \varphi(t)dt = (2-(x+y))^{\frac{3}{2}} \geq \frac{1}{3}[(1-x)^3+(1-x)^3]$$

$$= \frac{1}{3}[d^3(Ax,by+d^3(Ax,Sx)+d^3(By,Ty).$$

(2) For $x \in [0,1]$ and $1 < y \leq 2$, we have

$$\int_0^{d(Sx,Ty)} \varphi(t)dt = (2-x)^3 \geq \frac{1}{3}[4(\frac{1}{2})^3+(1-x)^3+(\frac{3}{2})^3]$$

$$= \frac{1}{3}[4d(Ax,By)+d(Ax,Sx)+d(By,Ty)].$$

(3) For $x \in (1,2]$ and $y \in [0,1]$, we have

$$\int_0^{d(Sx,Ty)} \varphi(t)dt = (2-y)^3 \geq \frac{1}{3}[4(\frac{1}{2})^3+(1-y)^3+(\frac{3}{2})^3]$$

$$= \frac{1}{3}[4d(Ax,By)+d(Ax,Sx)+d(By,Ty)].$$

(4) For $x, y \in (2,4]$, we have

$$\int_0^{d(Sx,Ty)} \varphi(t)dt = (\frac{7}{4})^3 \geq (\frac{1}{4})^3+(\frac{3}{2})^3$$

$$= \frac{1}{3}[4d(Ax,By)+d(Ax,Sx)+d(By,Ty)].$$

Consequently all hypotheses of Corollary 3.3 are satisfied. Therefore, 1 is the unique common fixed for A, S and T.

REFERENCES

[1] A. Branciari, A fixed point theorem for mapping satisfying a general contractive condition of integral type, Int. J. Math. Math. Sci. 29 (2002), 531-536.

[2] T. Suzuki, Meir-Keeler contractions of integral type are still Keir-Keeler contractions, Int. J. Math. Math. Sci. 2007 (2007), Article ID 39281.

[3] S.Z. Wang, B.Y. Li, Z.M. Gao, K. Iseki, Some fixed point theorems on expansion mappings, Math Japon. 29 (1984), 631-636.

[4] I. Beg, V. Gupta, A. Kanwar, Fixed points on intuitionistic fuzzy metric spaces using the E.A. property, J. Nonlinear Funct. Anal. 2015 (2015), Article ID 20.

[5] M. Imdad, T.I. Khan, Fixed point theorems for some expansive mappings via implicit relations, Nonlinear. Anal. Forum 9 (2004), 209-218.

[6] A. Sharma, A.H. Ansari, M. Imdad, Even tupled coincidence and common fixed point results for weakly contractive mappings in complete metric spaces via new functions, Commun. Optim. Theory 2017 (2017), Article ID 6.

[7] G. Jungck, Commuting mappings and fixed points, Amer. Math. Monthly, 83 (1976), 261-263.

[8] G. Jungck, Compatible mappings and common fixed points, Internat. J. Math. Math. Sci. 9 (1986), 771-779.

[9] M. A. Al-Thagafi, N. Shahzad, Generalized I-nonexpansive selfmaps and invariant approximations, Acta Math. Sinica 24 (2008), 867-876.

[10] D. Dorić, Z. Kadelburg, S. Radenović, A note on occasionally weakly compatible mappings and common fixed point, Fixed Point Theory 13 (2012), 475-480.

[11] H. Bouhadjera, C.G. Thobie, Common fixed point theorems for pairs of subcompatible maps, arXiv:0906.3159v1 [math.FA].(2009).

[12] M. Imdad, J. Ali, M. Tanveer, Remarks on some recent metrical common fixed point theorems, Appl. Math. Lett. 24 (2011), 1165-1169.

[13] H. K. Pathak, M. S. Khan, Compatible mappings of type (B) and common fixed point theorems of Gregus type, Czechoslovak Math. J. 45 (1995), 685-698.

[14] H.K. Pathak, Y.J. Cho, S.M. Kang, B.S. Lee, Fixed point theorems for compatible mappings of type (P) and application to dynamic programming, Le Matematiche (Fasc. I), 50 (1995), 15-33.

[15] H.K. Pathak, Y.J. Cho, S.M. Khan, B. Madharia, Compatible mappings of type (C) and common fixed point theorems of Gregus type, Demonstratio Math. 31 (1998), 499-518.

[16] G. Jungck, P.P. Murthy, Y.J. Cho, Compatible mappings of type (A) and common fixed points, Math. Japon. 38 (1993), 381-390.

[17] M.R. Singh, Y.M. Singh, Compatible mappings of type (E) and common fixed point theorems of Meir-Keeler type, International J. Math. Sci. Engg. Appl. 1 (2)(2007), 299-315.

[18] M.R. Singh, Y.M. Singh, On various types of compatible maps and common fixed point theorems for non-continuous maps, Hacettepe J. Math. Stat. 40 (2011), 503-513.

[19] S. Beloul, Common fixed point theorems for weakly subsequentially continuous generalized contractions with aplications, Appl. Maths. E-Notes, 15 (2015), 173-186.

[20] A. Djoudi, A unique common fixed point for compatible mappings of type (B) satisfying an implicit relation, Demonstratio Math. 36 (2003), 763-770.