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n-TUPLET FIXED POINTS OF MULTIVALUED MAPPINGS VIA MEASURE OF NONCOMPACTNESS

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Abstract. In this paper, some results on the existence of n-tuplet fixed points for multivalued contraction mappings are proved via measure of noncompactness. As an application, the existence of solutions for a system of integral inclusions is studied.

Keywords. Measure of noncompactness; n-tuplet fixed point; Multivalued set contraction mapping, Differential inclusion.

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1. Introduction

In a wide range of mathematical problems, the existence of a solution is equivalent to the existence of a fixed point of a nonlinear operator. The existence of a fixed point is therefore of paramount importance in several areas of mathematics and other sciences. Fixed point theory as an important research field has been extensively investigated by many authors since it finds a

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lot of applications in real world problems, such as, engineering, economy, optimization, game theory, and medicine. Recently, Kakutani [1] extended the Brouwer's fixed point theorem from single valued mappings to multivalued mappings and Nadler [2] extended the Banach contraction principle from single valued mappings to multivalued mappings using the Hausdorff metric. And the classes of Nadler's fixed point theorem was further extended and generalized for various multivalued mappings in [3, 4]. Measure of noncompactness plays a fundamental role in the study of single valued and multivalued mappings, especially, in the metric and topological fixed point theory. It is a very useful tool to guarantee the existence of fixed points. The measure of noncompactness was defined and studied by Kuratowski [5]. Darbo [6] used this measure to generalize both the Schauder's fixed point theorem and the Banach's contraction principle for condensing operators. Recently, measure of noncompactness has been used in differential equations, integral equations, nonlinear equations; see [7, 8, 9] and the references therein.

Partially ordered metric spaces are an important generalization of metric spaces in the fixed point theory. By using two basic concepts, Guo and Lakshmikantham [10] first gave some existence theorems of the coupled fixed points for both continuous and discontinuous operators. They also provided some applications to the initial value problems of ordinary differential equations with discontinuous right-hand sides. Bhaskar and Lakshmikantham [11] introduced coupled fixed points and established a coupled fixed point theorem in a partially ordered metric space. Berinde and Borcut [12] established a tripled fixed point theorem for nonlinear mappings in partially ordered complete metric spaces. Ertürk and Karakaya [13] introduced the concept of n-tuplet fixed points and studied the existence and uniqueness of fixed points of contractive type mappings in partially ordered metric spaces. Moreover, by using the condensing operators, Aghajani *et al.* [14] obtained some results on the existence of coupled fixed points and Karakaya *et al.* [15] obtained some results concerning the existence of tripled fixed points via measure of noncompactness. Recently, the existence of fixed points for various contractive mappings has been studied by many authors under different conditions. The concept of coupled fixed points for multivalued mappings was introduced by Samet and Vetro [17] and they obtained coupled fixed point theorems for multivalued nonlinear contraction mappings in a partially ordered metric space. Rao, Kishore and Kenan [18] obtained a tripled coincidence fixed point theorem for multivalued mappings in a partially ordered metric space.

In this paper, motivated by the research going in this direction, using condensing operators, we investigate n-tuplet fixed points of multivalued mappings on a Banach space. We also give an application of our result to a system of integral inclusions.

2. Preliminaries

Throughout this paper, E is always assumed to be a Banach space and $P(E)$ (or 2^E) is always assumed to be the set of all subsets of E. We denote the set

$$P_k(E) = \{X \subset E, X \text{ is nonempty and has a property } k\}.$$

$P_{rcp}(E), P_{cl,bd}(E), P_{cl,cv}(E)$ denotes the classes of all relatively compact, closed-bounded and closed-convex subsets of E, respectively.

A mapping $T : E \to P_k(E)$ is called a multivalued mapping or set valued mapping on E into E. A point $x \in E$ is called a fixed point of T if $x \in Tx$.

Definition 2.1. [19] A mapping $\mu : P_{cl,bd}(X) \to \mathbb{R}^+$ is called a measure of noncompactness if it satisfies the following conditions:

(M_1) $\varnothing \neq \mu^{-1}(0) \subset P_{rcp}(X)$,

(M_2) $\mu(\bar{A}) = \mu(A)$, where $\bar{A}$ denotes the closure of A,

(M_3) $\mu(conv\, A) = \mu(A)$, where $conv\, A$ denotes the convex hull of A,

(M_4) μ is nondecreasing,

(M_5) If $\{A_n\}$ is a decreasing sequence of sets in $P_{cl,bd}(X)$ satisfying $\lim_{n\to\infty} \mu(A_n) = 0$, then the intersection $A_\infty = \bigcap_{n=1}^{\infty} A_n$ is nonempty.

If (M_4) holds, then $A_\infty \in P_{rcp}(X)$. For this, let $\lim_{n\to\infty} \mu(A_n) = 0$. As $A_\infty \subseteq A_n$ for each $n = 0, 1, 2, \ldots$; by the monotonicity of μ, we obtain $\mu(A_\infty) \leq \lim_{n\to\infty} \mu(A_n) = 0$. So, by ($M_1$), we get that A_∞ is nonempty and $A_\infty \in P_{rcp}(X)$.

Theorem 2.2. [20] *Let X be a closed and convex subset of a Banach space E. Then every compact, continuous map $T : X \to X$ has at least one fixed point.*

Theorem 2.3. [7] *Let X be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : X \to X$ be a continuous mapping. Suppose that there exists a constant $k \in [0, 1)$ such that$\mu(T(X)) \leq k\mu(X)$ for any subset X of E, then T has a fixed point.*

Definition 2.4. [19] A multivalued mapping $T : E \to P_{cl,bd}(E)$ is said to be *D-set-Lipschitz* if there exists a continuous nondecreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that$\mu(T(X)) \leq \varphi(\mu(X))$ for all $X \in P_{cl,bd}(E)$ with $T(X) \in P_{cl,bd}(E)$, where $\varphi(0) = 0$. Generally, we call function φ to a *D-function* of T on E.

If $\varphi(r) = kr$, $k > 0$, then T is called a *k-set-Lipschitz* mapping. If $k < 1$, then T is called a *k-set-contraction* on E. If $\varphi(r) < r$ for $r > 0$, then T is called a *nonlinear D-set-contraction* on E.

Lemma 2.5. [21] *If φ is a D-function with $\varphi(r) < r$ for $r > 0$, then $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t \in [0,\infty)$.*

Theorem 2.6. [19] *Let X be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : X \to P_{cl,cv}(X)$ be a closed and nonlinear D-set-contraction. Then T has a fixed point.*

Theorem 2.7. [19] *Let X be a bounded, closed and convex subset of a Banach space E and let $T : X \to P_{cl,cv}(X)$ be a closed and k-set-contraction. Then T has a fixed point.*

Definition 2.8. [22] Let X be a topological space, 2^X the family of all subsets of X and T be a mapping of X into 2^X such that Tx is nonempty, for all $x \in X$. Then the mapping T is called upper semicontinuous if for each closed subset C of X, $T^{-1}(C) = \{x \in X : Tx \cap C \neq \varnothing\}$ is closed.

Definition 2.9. [19] A mapping $\mu : P_k(E) \to \mathbb{R}^+$ is said to be nondecreasing if $A, B \in P_k(E)$ are any two sets with $A \subseteq B$, then $\mu(A) \leq \mu(B)$, where $\subseteq$ is order relation of inclusion in $P_k(E)$.

Lemma 2.10. [23] *Let X be a Banach space and F be a Caratheodory multivalued mapping. Let $\Phi : L^1(H;X) \to C(H;X)$ be linear continuous mapping. Then,*

$$\begin{aligned} \Phi \circ S_F : C(H;X) &\to \mathscr{P}_{cl;cv}(C(H;X)) \\ u &\longmapsto (\Phi \circ S_F)\, u := \Phi(S_F(u)), \end{aligned}$$

is a closed graph operator in $C(H;X) \times C(H;X)$.

Lemma 2.11. [24] *(1) Let $A \subseteq C(H;X)$ be bounded. Then $\mu(A(t)) \leqslant \mu(A)$ for all $t \in H$, where $A(t) = \{y(t), y \in A\} \subset X$. Furthermore, if A is equicontinuous on H, then $\mu(A(t))$ is continuous on H and $\mu(A) = \sup\{\mu(A(t)), t \in H\}$. (2) If $A \subset C(H;X)$ is bounded and* *equicontinuous, then*

$$\mu\left(\int_0^t A(s)\,ds\right) \leqslant \int_0^t \mu(A(s))\,ds,$$

for all $t \in H$, *where* $\int_0^t A(s)\,ds = \left\{\int_0^t x(s)\,ds : x \in A\right\}$.

3. n-tuplet fixed point theorems and some related results

In this section, we investigate the n-tuplet fixed point property of a multivalued mapping and give some applications for special cases $n = 2$, that is, coupled fixed points.

Definition 3.1. Let X be a nonempty set and $G : X^n \to P(X)$ be a given mapping. An element $(x_1, x_2, x_3, \ldots, x_n) \in X^n$ is called an n-tuplet fixed point of G if

$$\begin{aligned} x_1 &\in G(x_1, x_2, x_3, \ldots, x_n), \\ x_2 &\in G(x_2, x_3, \ldots, x_n, x_1), \\ &\vdots \\ x_n &\in G(x_n, x_1, x_2, \ldots, x_{n-1}). \end{aligned}$$

Remark 3.2. If we take as special cases $n = 2$ and $n = 3$ in Definition 3.1, respectively, we get coupled fixed points [17] and tripled fixed points [18].

Theorem 3.3. [25] *Let* $\mu_1, \mu_2 \ldots, \mu_n$ *be measures of noncompactness in Banach spaces* $E_1, E_2 \ldots, E_n$ *respectively. Suppose that the function* $F : [0,\infty)^n \to [0,\infty)$ *is convex and* $F(x_1, x_2, \ldots, x_n) = 0$ *if and only if* $x_i = 0$ *for* $i = 1, 2, \ldots, n$. *Then* $\widetilde{\mu}(X) = F(\mu_1(X_1), \mu_2(X_2) \ldots, \mu_n(X_n))$ *defines a measure of noncompactness in* $E_1 \times E_2 \times \ldots \times E_n$ *where* X_i *denotes the natural projection of* X *onto* E_i, *for* $i = 1, 2, \ldots, n$.

Remark 3.4. By taking

$$F(x_1, x_2, x_3, \ldots, x_n) = \max\{x_1, x_2, x_3, \ldots, x_n\},$$

or

$$F(x_1, x_2, x_3, \ldots, x_n) = x_1 + x_2 + x_3 + \ldots + x_n,$$

for any $(x_1, x_2, x_3, \ldots, x_n) \in [0,\infty)^n$, the conditions of Theorem 3.3 are satisfied. Therefore,

$$\widetilde{\mu}(X) := \max(\mu(X_1), \mu(X_2), \ldots, \mu(X_n)),$$

or

$$\widetilde{\mu}(X) := \mu(X_1) + \mu(X_2) + \cdots + \mu(X_n)$$

defines measures of noncompactness in E^n, where X_i, $i = 1, 2, ..., n$ are the natural projections of X on E_i.

We now give the following theorem for the existence of fixed points of multivalued mappings under measure of noncompactness conditions.

Theorem 3.5. *Let X be a nonempty, bounded, closed and convex subset of a Banach space E and let μ be an arbitrary measure of noncompactness in E. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing and upper semicontinuous function such that $\varphi(r) < r$ for all $r > 0$. Suppose that $G : X_1 \times X_2 \times \cdots \times X_n \to P_{cl,cv}(X)$ is continuous multivalued operator satisfying*

$$\mu(G(X_1 \times X_2 \times \cdots \times X_n)) \leqslant \varphi\left(\frac{\mu(X_1) + \mu(X_2) + \cdots + \mu(X_n)}{n}\right)$$

for all $X_1, X_2, ..., X_n \subset X$. Then G has at least one n-tuplet fixed point.

Proof. As in Remark 3.4, we define the measure of noncompactness $\tilde{\mu}$ by

$$\widetilde{\mu}(X) := \mu(X_1) + \mu(X_2) + \cdots + \mu(X_n).$$

Define the mapping $\tilde{G}(X) := G(X_1 \times X_2 \times \cdots \times X_n)$. We prove that $\widetilde{G}$ satisfies all the conditions of Theorem 2.6. Then

$$\begin{aligned}
& \widetilde{\mu}\left(\widetilde{G}(X)\right) \\
= \; & \widetilde{\mu}(G(X_1 \times X_2 \times \cdots \times X_n)) \\
= \; & \widetilde{\mu}(G(x_1, x_2, x_3, ..., x_n), G(x_2, x_3, ..., x_n, x_1), ..., G(x_n, x_1, x_2, ..., x_{n-1})) \\
= \; & \mu(G(x_1, x_2, x_3, ..., x_n)) + \mu(G(x_2, x_3, ..., x_n, x_1)) + \cdots + \mu(G(x_n, x_1, x_2, ..., x_{n-1})) \\
\leqslant \; & \varphi\left(\frac{\mu(X_1) + \mu(X_2) + \cdots + \mu(X_n)}{n}\right) + \varphi\left(\frac{\mu(X_2) + \mu(X_3) + \cdots + \mu(X_1)}{n}\right) \\
& + \cdots + \varphi\left(\frac{\mu(X_n) + \mu(X_1) + \cdots + \mu(X_{n-1})}{n}\right) \\
= \; & n\varphi\left(\frac{\mu(X_1) + \mu(X_2) + \cdots + \mu(X_n)}{n}\right).
\end{aligned}$$

Note that

$$\frac{1}{n}\widetilde{\mu}\left(\widetilde{G}(X)\right) \leq \varphi\left(\frac{\mu(X_1) + \mu(X_2) + \cdots + \mu(X_n)}{n}\right).$$

Taking $\widetilde{\mu}' = \frac{1}{n}\widetilde{\mu}$, we get

$$\widetilde{\mu}'\left(\widetilde{G}(X)\right) \leqslant \varphi\left(\widetilde{\mu}'(X)\right).$$

Also, $\widetilde{\mu}'$ is a measure of noncompactness. Thus, by Theorem 2.6, we obtain that G has at least one n-tuplet fixed point.

Remark 3.6. If we take μ measure of noncompactness in Theorem 3.5 as

$$\widetilde{\mu}(X) := \max\left(\mu(X_1), \mu(X_2), ..., \mu(X_n)\right),$$

we can obtain the same result.

4. Applications to inclusions systems

The multivalued fixed point theorem of this paper has some applications to differential and integral systems of inclusions. As an example, we study the solvability of a system of differential inclusions.

Consider the following differential system

$$\begin{cases} x'(t) \in A(t)x(t) + G(t, x(t), y(t)), \ \ t \in [0,b], \\ y'(t) \in A(t)y(t) + F(t, y(t), x(t)), \ \ t \in [0,b] \end{cases} \tag{4.1}$$

with

$$x(0) = \varphi(x,y), \ y(0) = \varphi(y,x), \tag{4.2}$$

where G is an upper Caratheodory multimap, $\varphi : C([0,b], X) \to X$ is a given multivalued function, $\{A(t) : t \in [0,b]\}$ is a family of linear closed unbounded operators on X with domain $D(A(t))$ independent of t that generate Δ an evolution system of operators $\{U(t,s) : t, s \in \Delta\}$ with $\Delta = \{(t,s) \in [0,b] \times [0,b] : 0 \leqslant s \leqslant t \leqslant b\}$.

Define the set

$$S_G(x,y) = \left\{g \in L^1([0,b], X) : g(t) \in G(t, x(t), y(t))\right\}.$$

Definition 4.1. A family $\{U(t,s)\}_{(t,s)\in\Delta}$ of bounded linear operators $U(t,s) : X \to X$, where $(t,s) \in \Delta := \{(t,s) \in J \times J : 0 \leqslant s \leqslant t < +\infty\}$ for $J = [0,b]$ is called an evolution system if the following properties are satisfied

(1) $U(t,t) = I$, where I is the identity operator in X and $U(t,s)\,U(s,\tau) = U(t,\tau)$ for $0 \leqslant \tau \leqslant s \leqslant t < +\infty$,

(2) The mapping $(t,s) \to U(t,s)\,y$ is strongly continuous, that is, there exists a constant $M > 0$ such that $\|U(t,s)\| \leqslant M$, $\forall (t,s) \in \Delta$.

An evolution system $U(t,s)$ is said to be compact if $U(t,s)$ is compact for any t-$s > 0$. $U(t,s)$ is said to be equicontinuous if $\{U(t,s)x : x \in M\}$ is equicontinuous at $0 \leqslant s < t \leqslant b$ for any bounded subset $B \subset X$. Clearly, if $U(t,s)$ is a compact evolution system, it must be equicontinuous. The converse is not necessarily true.

More details on evolution systems and their properties could be found in the books of Ahmed [26], Engel and Nagel [27] and Pazy [28].

Definition 4.2. We say that the couple $(x(t), y(t)) \in C([0,b], X) \times C([0,b], X)$ is a mild solution of the evolution system (4.1)-(4.2) if it satisfies the following integral system

$$\begin{cases} x(t) = U(t,0)\ \varphi(x,y) + \int_0^t U(t,s)\ g(s)\ ds \quad \text{for } g \in S_G(x,y), \\ y(t) = U(t,0)\ \varphi(y,x) + \int_0^t U(t,s)\ g(s)\ ds \quad \text{for } g \in S_G(y,x) \end{cases} \tag{4.3}$$

for all $t \in [0,b]$.

Theorem 4.3. Assume the following hypotheses

$(H1)$ $\{A(t) : t \in J\}$ is a family of linear operators. $A(t) : D(A) \subset X \to X$ generates an equicontinuous evolution system $\{U(t,s) : (t,s) \in \Delta\}$ and$|U(t,s)| \leqslant M$.

$(H2)$ The multifunction $G : J \times C([0,b] \times X \times X) \longrightarrow \mathscr{P}_{cl,cv}(X)$ is an upper Carathéodory with respect to x and y and $\varphi : C(J;X) \to X$ is compact and

$$\mu(G(t, W \times W)) < k\mu\left(\frac{W \times W}{2}\right), \forall t \in J.$$

$(H3)$ There exists a constant $r > 0$ such that

$$M[\|\varphi(x,y)\| + \{\|g(t)\|_1 : g \in S_G(x,y), x \in A_0\}] \leqslant r$$

and

$$M[\|\varphi(y,x)\| + \{\|g(t)\|_1 : g \in S_G(y,x), y \in A_0\}] \leqslant r,$$

where $A_0 = \{z \in C(J;X) : \|z(t)\| \leqslant r \text{ for all } t \in J\}$ hold. Then the non local system (4.1)-(4.2) has at least one mild solution in the space $C(J,X)$.

Proof. To solve problem given in (4.1)-(4.2), we transform it into the following fixed point problem.

Consider the multivalued operator $N : C([0,b];X;X) \to \mathscr{P}(C([0,b];X))$ defined by,

$$N(x,y) = \left\{ h \in C(J;X) : h(t) = U(t,0)\varphi(x,y) + \int_0^t U(t,s)\ g(s)\ ds, \text{ with } g \in S_G(x,y) \right\}.$$

Clearly, coupled fixed points of the operator N are mild solutions. For each $y \in C([0,b];X)$, set $S_G(x,y)$ is nonempty since G has a measurable selection. Let us show that N has a coupled fixed point. To this end, we need to verify all the conditions of Theorem 3.5 Let $A_0 = \{z \in C([0,b];X) : \|z(t)\| \leqslant r \text{ for all } t \in [0,b]\}$. We notice that A_0 is closed, bounded and convex. To show that $N(A_0 \times A_0) \subseteq A_0$, we need first to prove that the family

$$\left\{ \int_0^t U(t,s)\ f(s)\ ds\ : f \in S_F(y) \text{ and } y \in A_0 \right\}$$

is equicontinuous for $t \in J$, that is, all the functions are continuous and they have equal variation over a given neighbourhood. In view of $(H1)$, we have that functions in $\{U(t,s) : (t,s) \in \Delta\}$ are equicontinuous, i.e, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|t-\tau| < \delta$ implies

$$\|U(t,s) - U(\tau,s)\| < \varepsilon$$

for all $U(t,s) \in \{U(t,s) : (t,s) \in \Delta\}$. Then, given some $\varepsilon > 0$, letting $\delta = \frac{\varepsilon'}{\varepsilon\|g\|_\infty}$ such that $|t-\tau| < \delta$, we have

$$\left| \int_0^t U(t,s)\ g(s)\ ds - \int_0^\tau U(\tau,s) g(s)\ ds \right| \leqslant \int_\tau^t |U(t,s) - U(\tau,s)|\,|g(s)|\,ds.$$

As $\{U(t,s) : (t,s) \in \Delta\}$ is equicontinuous, we have

$$\begin{aligned} \left| \int_0^t U(t,s)\ g(s)\ ds - \int_0^\tau U(\tau,s)\ g(s)\ ds \right| &\leqslant \varepsilon \|g\|_\infty |t-\tau| \\ &< \varepsilon \|g\|_\infty \frac{\varepsilon'}{\varepsilon \|g\|_\infty} = \varepsilon'. \end{aligned}$$

Hence we conclude that $\left\{ \int_0^t U(t,s)\ g(s)\ ds\ : g \in S_G(x,y) \text{ and } (x,y) \in A_0 \times A_0 \right\}$ is equicontinuous for $t \in J$.

Now, we show that $N(A_0 \times A_0) \subseteq A_0$. For $t \in J$, we have

$$\begin{aligned} |h(t)| &= \left| U(t,0)\varphi(x,y) + \int_0^t U(t,s)\, g(s)\, ds \right| \\ &\leqslant |U(t,0)\varphi(x,y)| + \int_0^t |U(t,s)\, g(s)|\, ds \\ &\leqslant M\|\varphi(x,y)\| + M\|g\|_1 \\ &= M[\|\varphi(x,y)\| + \|g\|_1] \leqslant r. \end{aligned}$$

Thus $N(A_0 \times A_0) \subseteq A_0$. Further, it is easy to see that N is convex value.

Now, let us show that N has a closed graph. Letting $x_n \to x$, $y_n \to y$ and $h_n \to h$ such that $h_n(t) \in N(x_n, y_n)$, we show that $h(t) \in N(x,y)$.

Now, there exists a sequence $g_n \in S_G(x_n, y_n)$ such that

$$h_n(t) = U(t,0)\varphi(x_n, y_n) + \int_0^t U(t,s) g_n(s) ds.$$

Consider the linear operator $\Phi : L^1([0,b];X) \to C([0,b];X)$ defined by

$$\Phi f(t) = \int_0^t U(t,s) g_n(s) ds.$$

Clearly, Φ is linear and continuous. So we get that $\Phi \circ S_G(x,y)$ is a closed graph operator. Further, we have

$$h_n(t) - U(t,0)\varphi(x_n, y_n) \in \Phi \circ S_G(x,y).$$

Since $x_n \to x$, $y_n \to y$ and $h_n \to h$, we have

$$h(t) - U(t,0)\varphi(x,y) \in \Phi \circ S_G(x,y).$$

That is, there exists a function $g \in S_G(x,y)$ such that

$$h(t) = U(t,0)\varphi(y) + \int_0^t U(t,s) g(s) ds.$$

Therefore N has a closed graph, hence N has closed values on $C([0,b] \times X \times X, X)$.

We know that the family $\left\{\int_0^t U(t,s)f(s)ds, f\in S_F\left(W\left(t\right)\right)\right\}$ is equicontinuous, hence by Lemma 2.5, we have

$$\begin{aligned}
& \mu\left(\int_0^t U(t,s)g(s)ds,\ g\in S_G\left(W\left(t\right)\times W\left(t\right)\right)\right) \\
\leqslant\ & \int_0^t \mu\left(U(t,s)g(s),\ g\in S_G\left(W\left(t\right)\times W\left(t\right)\right)\right)ds \\
\leqslant\ & M\int_0^t \mu\left(g(s),\ g\in S_G\left(W\left(t\right)\times W\left(t\right)\right)\right)ds \\
\leqslant\ & Mt\mu\left(G\left(t,W\left(t\right)\right)\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
& \mu\left(N\left(W\times W\right)\right) \\
=\ & \mu\Big(N\Big(U\left(t,0\right)\varphi(W\left(t\right)\times W\left(t\right))+\int_0^t U(t,s)g(s)ds, g\in S_G\left(W\left(t\right)\times W\left(t\right)\right)\Big)\Big) \\
\leqslant\ & \mu\left(U\left(t,0\right)\varphi(W\left(t\right)\times W\left(t\right))\right)+\mu\left(\int_0^t U(t,s)g(s)ds, g\in S_G\left(W\left(t\right)\times W\left(t\right)\right)\right) \\
\leqslant\ & M\mu\left(\varphi(W\left(t\right)\times W\left(t\right))\right)+Mt\mu\left(G\left(t,W\left(t\right)\times W\left(t\right)\right)\right).
\end{aligned}$$

In view of $(H2)$, we get

$$\mu\left(N\left(W\times W\right)\right)\leqslant Mbk\mu\left(\frac{W\times W}{2}\right).$$

Therefore, for $Mbk<1$, we obtain that N has at least one coupled fixed point. Hence, system (4.1)-(4.2) has at least one solution.

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