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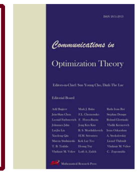

SOME PROPERTIES OF SECOND-ORDER CONTINGENT EPIDERIVATIVES AND ITS APPLICATIONS TO VECTOR EQUILIBRIUM PROBLEMS

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Abstract. In this paper, we study some properties of the contingent epiderivative of a single-valued map from a variational perspective. As an application, we obtain second-order efficiency conditions for weakly efficient solutions of vector equilibrium problems with constraints. Several examples are also provided to illustrate our main results.

Keywords. Second-order contingent epiderivative; Second-order contingent derivative; Second-order contingent set; Weakly efficient solution; Second-order efficiency condition.

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1. Introduction

The (interior) second-order contingent set and the second-order adjacent set have been extensively studied recently in [1, 2, 3, 4]. They have been used to establish second-order necessary and sufficient efficiency conditions for efficient solution types of vector equilibrium problems with constraints, see, for instance, Li, Zhu and Teo [5]. Basing on the concepts of second-order contingent sets, the second-order contingent epiderivatives of a single-valued map are

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established; see [1, 2, 3, 5, 6, 7] and the references therein. It is known that the existence results of the generalized second-order contingent epiderivative has been studied; see [5] and the references therein.

Motivated by the above arguments, we provide some characterizations of the contingent epiderivatives of a single-valued map in Banach spaces. As an application, we derive necessary and sufficient second-order efficiency conditions for weakly efficient solutions of vector equilibrium problems with constraints (which is denoted by CVEP) in terms of contingent epiderivatives. It is known that the vector equilibrium problem provides a unified mathematical model including vector complementarity problems, vector saddle point problems, vector optimization problems and vector variational inequality problems as special cases. A large number of results for vector equilibrium problem have been investigated consists of existences of solutions, see, for instance, Marín and Sama [8] and optimality conditions, see, for instance, [1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16] and the references therein.

Let X,Y,Z and W be real Banach spaces in which Y be partially ordered by a closed pointed convex cone Q with its interior nonempty, and let S be a closed convex cone in Z with its interior nonempty and C be a nonempty subset in X. Given a vector bifunction $F : X \times X \longrightarrow Y$ and the objective functions $g : X \longrightarrow Z$, $h : X \longrightarrow W$. We set

$$K = \{x \in C : g(x) \in -S,\ h(x) = 0\}.$$

Our vector equilibrium problem with constraints (shortly, CVEP) is given as follows: finding a vector $\bar{x} \in K$ such that

$$F(\bar{x},x) \notin -intQ\ \forall x \in K. \tag{CVEP}$$

Vector $\bar{x}$ is called a weakly efficient solution to the CVEP and K is called the feasible set to the CVEP. From now on, for each $\bar{x} \in C$, we denote by $F_{\bar{x}} = F(\bar{x}, .) : X \longrightarrow Y$, $F(\bar{x},K) = \bigcup_{x \in K} F(\bar{x},x)$, and the functions $F_{\bar{x}}$ and g are always assumed to be steady at $\bar{x}$.

We call the profile map of F is $F_+ : X \longrightarrow 2^Y$ with $F_+ = F + Q$ such that

$$F_+(x) = F(x) + Q\ \forall x \in X.$$

In addition, if $F : X \longrightarrow Y$ then $F_+ : X \longrightarrow 2^Y$.

The remainder of this paper is organized as follows. Sections provides some preliminaries and notations. In Section 3, we provide some important properties of second-order contingent epiderivatives of a single-valued map. As applications, Section 4 is devoted to second-order efficiency conditions for weakly efficient solutions of vector equilibrium problems with constraints. Some examples are also provided to illustrate our main results.

2. Preliminaries

Let X, Y, Z and W be given as in Section 1. For each $A \subset X$, as usual we denote by $intA(clA)$ the interior (closure) of the set A, by cone$A$ the cone generated by A, where

$$\text{cone}A = \{ta : a \in A, t \geq 0\},$$

and $t_n \to 0^+$ instead of a sequence of positive real numbers with limit 0. Let $F : X \longrightarrow 2^Y$ be a set-valued mapping from X into Y. Let us recall that the effective domain, graph and epigraph of F are:

$$\text{dom}F = \{x \in X \,|\, F(x) \neq \emptyset\},$$

$$\text{graph}(F) = \{(x,y) \in X \times Y \,|\, x \in \text{dom}F,\, y \in F(x)\},$$

$$\text{epi}(F) = \{(x,y) \in X \times Y \,|\, x \in \text{dom}F,\; y \in F(x) + Q\}.$$

Let Y^* be the topological dual space of Y and the dual cone of Q be

$$Q^+ = \{\xi \in Y^* \,|\, \langle \xi, q \rangle \geq 0 \;\forall\, q \in Q\},$$

where $\langle .,. \rangle$ denotes the coupling between Y^* and Y. Let M be a nonempty subset of Y, a direction $u \in Y$, and a vector $\bar{z} \in clM$. The following second-order contingent sets will be used in this article (see [1, 2, 3, 5, 7, 15]).

- The second-order contingent set $T^2(M, \bar{z}, u)$ of M at $(\bar{z}, u)$ is defined as

$$T^2(M, \bar{z}, u) = \{y \in Y : \exists t_n \longrightarrow 0^+, \exists y_n \longrightarrow y \text{ such that } \bar{z} + t_n u + \frac{1}{2} t_n^2 y_n \in M \;\forall\, n \geq 1\}.$$

- The interior second-order contingent set $IT^2(M, \bar{z}, u)$ of M at $(\bar{z}, u)$ is defined as

$$IT^2(M, \bar{z}, u) = \{y \in Y : \forall t_n \longrightarrow 0^+, \forall y_n \longrightarrow y \text{ such that } \bar{z} + t_n u + \frac{1}{2} t_n^2 y_n \in M \text{ for all n large enough}\}.$$

- The second-order adjacent set $A^2(M, \bar{z}, u)$ of M at $(\bar{z}, u)$ is defined as

$$A^2(M, \bar{z}, u) = \{y \in Y : \forall t_n \longrightarrow 0^+, \exists y_n \longrightarrow y \text{ such that } \bar{z} + t_n u + \frac{1}{2} t_n^2 y_n \in M \;\forall\, n \geq 1\}.$$

Let $f : X \longrightarrow Y$ be a single-valued map in which Y be partially ordered by cone Q and let $\bar{x} \in X$ and $(u, v) \in X \times Y$. We recall that (see [1, 2, 3, 5, 7, 15]):

• The second-order contingent derivative of f (resp., f_+) at $(\bar{x},u,v)$ is the set-valued map $D_c^2 f(\bar{x},u,v)$ (resp., $D_c^2 f_+(\bar{x},u,v)$) from X to 2^Y defined as

$$\text{graph}\Big(D_c^2 f(\bar{x},u,v)\Big) = T^2\Big(\text{graph}(f),(\bar{x},f(\bar{x})),(u,v)\Big)$$

$$\Big(\text{resp., graph}\Big(D_c^2 f_+(\bar{x},u,v)\Big) = T^2\Big(\text{graph}(f_+),(\bar{x},f(\bar{x})),(u,v)\Big)\Big).$$

• The second-order contingent epiderivative of f at $(\bar{x},u,v)$ is the single-valued map $\underline{D}^2 f(\bar{x},u,v)$ from X to Y defined as

$$\text{epi}\Big(\underline{D}^2 f(\bar{x},u,v)\Big) = T^2\Big(\text{epi}\,(f),(\bar{x},f(\bar{x})),(u,v)\Big).$$

Definition 2.1. ([5, 7, 8, 10, 11, 12]) We say that $y \in M$ is an ideal minimal point (resp. maximal point) of M with respect to Q if $a \geq y$ (resp. $y \geq a$) for any $a \in M$. The set of all ideal minimal points (resp. maximal points) is denoted by $IMin(M)$ (resp. $IMax(M)$). Where $a \geq y$ (resp. $y \geq a$) is equivalent to $a - y \in Q$ (resp. $y - a \in Q$).

It is not hard to check that

$$IMin(M) = \{m \in M : M \subset m + Q\},$$

$$IMax(M) = \{m \in M : M \subset m - Q\}.$$

The following definitions are necessary to the paper.

Definition 2.2. ([7, 9]) A function $f : X \longrightarrow Y$ is called steady at $\bar{x} \in X$ in the direction $v \in X$ (shortly, f is steady $(\bar{x},v)$), if

$$\lim_{(t,u)\longrightarrow(0^+,v)} \frac{f(\bar{x}+tu)-f(\bar{x}+tv)}{t} = 0.$$

If f is steady at $\bar{x}$ in all the directions, then f is called steady at $\bar{x}$.

Definition 2.3. ([7, 9]) A function $f : X \longrightarrow Y$ is called stable at $\bar{x} \in X$ if there exists a neighborhood U of $\bar{x}$ and $L > 0$ such that

$$\|f(x) - f(\bar{x})\| \leq L\|x - \bar{x}\| \ \forall x \in U.$$

It is well known that f is steady at $\bar{x}$ in all the direction 0 if and only if f is stable at $\bar{x}$. Therefore, if f is steady at $\bar{x} \in X$ then f is stable at $\bar{x}$.

3. Variational characterization of the contingent epiderivative of order-2

Let X, Y and f be as in Section 2 and let Q be a closed convex pointed cone in Y. Let K be a nonempty subset in X and $\bar{x} \in K$. For any $(u,v) \in X \times Y$, we denote by $L(u,v)$ $(resp., L_Q(u,v))$ instead of the following set

$$L(u,v) = \{(x,z) \in X \times Y \mid D_c^2 f(\bar{x},u,v)(x) \subset z + Q\}$$

$$\Big(\text{resp.,} \quad L_Q(u,v) = \{(x,z) \in X \times Y \mid D_c^2 f_+(\bar{x},u,v)(x) \subset z + Q\}\Big).$$

It is easily seen that

$$L_Q(u,v) \subset L(u,v) \text{ for all } (u,v) \in X \times Y.$$

In fact, from the fact that

$$\text{graph}(f) \subset \text{epi}(f) = \text{graph}(f_+),$$

it follows that $D_c^2 f(\bar{x},u,v)(x) \subset D_c^2 f_+(\bar{x},u,v)(x)$ for any $x \in X$.

Proposition 3.1. *Let $(u,v) \in X \times Y$ and $x \in dom D_c^2 f_+(\bar{x},u,v)$. Then, if $\underline{D}^2 f(\bar{x},u,v)(x)$ exists then $L_Q(u,v) \neq \emptyset$. Moreover, $(x, \underline{D}^2 f(\bar{x},u,v)(x)) \in L_Q(u,v)$.*

Proof. By the definition of second-order contingent epiderivative of f at $\bar{x}$ in the direction (u,v), we have that

$$\text{epi}(\underline{D}^2 f(\bar{x},u,v)) = T^2(\text{epi}(f), (\bar{x}, f(\bar{x})), (u,v)) = T^2(\text{graph}(f_+), (\bar{x}, f(\bar{x})), (u,v)).$$

It follows that

$$D_c^2 f_+(\bar{x},u,v)(x) \subset \underline{D}^2 f(\bar{x},u,v)(x) + Q. \tag{3.1}$$

Therefore $(x, \underline{D}^2 f(\bar{x},u,v)(x)) \in L_Q(u,v)$, which completes the proof.

Proposition 3.2. *Let $(u,v) \in X \times Y$ and $x \in dom D_c^2 f_+(\bar{x},u,v)$ and assume, in addition, that $\underline{D}^2 f(\bar{x},u,v)(x)$ exists. Then we have the following statements hold*

(i) $(x,z) \in L_Q(u,v)$ if and only if $\underline{D}^2 f(\bar{x},u,v)(x) \in z + Q$.

If, in addition, Q has a compact base then

(ii) $(x,z) \in L(u,v)$ if and only if $\underline{D}^2 f(\bar{x},u,v)(x) \in z + Q$.

(iii) $(x,z) \in L(u,v)$ if and only if $(x,z) \in L_Q(u,v)$.

Proof. Case (i). By the definitions of second-order contingent derivative and epiderivative of f and f_+ at point $\bar{x}$ in the direction (u,v) we have

$$\underline{D}^2 f(\bar{x},u,v)(x) \in D_c^2 f_+(\bar{x},u,v)(x).$$

If $(x,z) \in L_Q(u,v)$, then

$$\underline{D}^2 f(\bar{x},u,v)(x) \in D_c^2 f_+(\bar{x},u,v)(x) \subset z+Q,$$

which find the result. On the other hand, if $\underline{D}^2 f(\bar{x},u,v)(x) \in z+Q$, by making use of condition (3.1), one obtains

$$D_c^2 f_+(\bar{x},u,v)(x) \subset \underline{D}^2 f(\bar{x},u,v)(x)+Q \subset z+Q+Q = z+Q,$$

which yields that (i) holds.

Case (ii). We assume that Q has a compact base. By applying the obtained results in Proposition 3.3 (see [7], p. 6) to deduce that

$$\underline{D}^2 f(\bar{x},u,v)(x) = IMin(D_c^2 f(\bar{x},u,v)(x)). \tag{3.2}$$

By using Definition 2.1, we conclude the result.

Case (iii). It follows from the cases (i) and (ii), which completes the proof.

Theorem 3.3. *Let $(u,v) \in X \times Y$ and $x \in dom D_c^2 f_+(\bar{x},u,v)$ and assume that $\underline{D}^2 f(\bar{x},u,v)(x)$ exists. Then the following assertions hold*

(i) If Q has a compact base then $(x,z) \in L_Q(u,v) \cap T^2(graph(f),(\bar{x},f(\bar{x})),(u,v))$ if and only if $z = \underline{D}^2 f(\bar{x},u,v)(x)$.

(ii) $(x,z) \in L_Q(u,v) \cap T^2(epi(f),(\bar{x},f(\bar{x})),(u,v))$ if and only if $z = \underline{D}^2 f(\bar{x},u,v)(x)$.

(iii) If Q has a compact base then $(x,z) \in L(u,v) \cap T^2(graph(f),(\bar{x},f(\bar{x})),(u,v))$ if and only if $z = \underline{D}^2 f(\bar{x},u,v)(x)$.

(iv) If $z = \underline{D}^2 f(\bar{x},u,v)(x)$ then $(x,z) \in L(u,v) \cap T^2(epi(f),(\bar{x},f(\bar{x})),(u,v))$. The converse case is still true if, in addition, Q has a compact base.

Proof. Case (i). By virtue of Proposition 3.2, we have

$$\underline{D}^2 f(\bar{x},u,v)(x) \in z+Q.$$

Making use of the definition of second-order contingent derivative of f at $(\bar{x},u,v)$, we get from (3.2) that

$$z \in D_c^2 f(\bar{x},u,v)(x) \subset \underline{D}^2 f(\bar{x},u,v)(x)+Q.$$

Moreover, as Q is pointed, $z = \underline{D}^2 f(\bar{x},u,v)(x)$. If, in addition, Q has a compact base, making use of (3.2), we find that $(x,z) \in \text{graph}(D_c^2 f(\bar{x},u,v))$, where $z = \underline{D}^2 f(\bar{x},u,v)(x)$. This implies that $(x,z) \in T^2(\text{graph}(f),(\bar{x},f(\bar{x})),(u,v))$. Using Proposition 3.1, we find the result.
Case (ii). It is obvious.
Case (iii). Its proof follows from Proposition 3.2 (ii).
Case (iv). If $z = \underline{D}^2 f(\bar{x},u,v)(x)$ then, as $L_Q(u,v) \subset L(u,v)$, thus

$$(x,z) \in L(u,v) \cap T^2(\text{epi}(f),(\bar{x},f(\bar{x})),(u,v)).$$

Conversely, we can apply to Proposition 3.2 (iii), and arrive at the conclusion.

Corollary 3.4. *Let $(u,v) \in X \times Y$ and $x \in dom D_c^2 f_+(\bar{x},u,v)$ and assume that $\underline{D}^2 f(\bar{x},u,v)(x)$ exists. Consider the following variational system (P_x) :*

$$(P_x) \begin{cases} \textit{Find } z \in Y \textit{ such that} \\ \langle \lambda, z \rangle = inf\,\{\langle \lambda, w \rangle \,:\, w \in D_c^2 f_+(\bar{x},u,v)(x)\} \textit{ for any } \lambda \in Q^+. \end{cases}$$

Then we have the following assertions hold

(i) $(x,z) \in L_Q(u,v) \cap T^2(epi(f),(\bar{x},f(\bar{x})),(u,v))$ if and only if z is a solution of the variational system (P_x).

If in addition Q has a compact base then

(ii) $(x,z) \in L_Q(u,v) \cap T^2(graph(f),(\bar{x},f(\bar{x})),(u,v))$ if and only if z is a solution of the variational system (P_x).

(iii) z is a solution of the variational system (P_x) if and only if, either $(x,z) \in L(u,v) \cap T^2(graph(f),(\bar{x},f(\bar{x})),(u,v))$ or $(x,z) \in L(u,v) \cap T^2(epi(f),(\bar{x},f(\bar{x})),(u,v))$.

Proof. According to L. Marín and M. Sama [8], variational system (P_x) has at most one solution, say z, and moreover it is given by $z = IMin(D_c^2 f_+(\bar{x},u,v)(x))$. By directly applying Proposition 3.1 ([7], p. 5), we see that $z = \underline{D}^2 f(\bar{x},u,v)(x)$. Therefore, the proof of Corollary 3.4 is inferred from Theorem 3.3, which completes the proof.

Corollary 3.5. *Under the assumptions of Theorem 3.1 and in addition the cone Q has a compact base. Consider the following variational system* (Q_x) :

$$(Q_x)\begin{cases} \text{Find } z \in Y \text{ such that} \\ \langle \lambda, z\rangle = inf\{\langle \lambda, w\rangle : w \in D^2_c f(\bar{x},u,v)(x)\} \text{ for any } \lambda \in Q^+. \end{cases}$$

Then we have the following statements valid

(i) z is a solution of (Q_x) *if and only if* (x,z) *belongs to, either* $L_Q(u,v)\cap T^2(epi(f),(\bar{x},f(\bar{x})),(u,v))$, *or* $L_Q(u,v)\cap T^2(graph(f),(\bar{x},f(\bar{x})),(u,v))$.

(ii) z is a solution of (Q_x) *if and only if* (x,z) *belongs to, either* $L(u,v)\cap T^2(graph(f),(\bar{x},f(\bar{x})),(u,v))$ *or* $L(u,v)\cap T^2(epi(f),(\bar{x},f(\bar{x})),(u,v))$.

Proof. If z is a solution of variational system (Q_x), then

$$z = IMin(D^2_c f(\bar{x},u,v)(x)) = \underline{D}^2 f(\bar{x},u,v)(x)$$

(can see in Proposition 3.3 [10, p.p 6]). Making use of preceding Corollary 3.4, we find the desired conclusion.

Theorem 3.6. *Let* $(u,v)\in X\times Y$, $x\in dom D^2_c(f+Q)(\bar{x},u,v)$ *and assume, moreover, that* $\underline{D}^2 f(\bar{x},u,v)(x)$ *exists. If z is solution of* (P_x), *then z is also solution of* (Q_x). *The reverse case is still holds if, in addition, Q has a compact base.*

Proof. Firstly, we prove that the relation of

$$D^2_c f(\bar{x},u,v)(x)+Q \subset D^2_c f_+(\bar{x},u,v)(x) \tag{3.3}$$

holds. In fact, by taking $q\in Q$ such that for any $y\in D^2_c f(\bar{x},u,v)(x)$, we have

$$(x,y-q)\in T^2(\text{graph}(f),(\bar{x},f(\bar{x})),(u,v)).$$

It follows from the definition of second-order contingent set of f at $(\bar{x},u,v)$ that there exists sequence $(t_n,x_n,y_n)\longrightarrow(0^+,x,y-q)$ such that

$$(\bar{x},f(\bar{x}))+t_n(u,v)+\frac{1}{2}t_n^2(x_n,y_n)\in \text{graph}(f) \quad \forall n\geq 1,$$

which implies that

$$f(\bar{x})+t_nv+\frac{1}{2}t_n^2(y_n+q)\in f(\bar{x}+t_nu+\frac{1}{2}t_n^2x_n)+Q = f_+(\bar{x}+t_nu+\frac{1}{2}t_n^2x_n) \quad \forall n\geq 1.$$

Therefore $y \in D_c^2 f_+(\bar{x},u,v)(x)$ because $y_n + q \to y$. So, condition (3.3) holds. A consequence is

$$D_c^2 f(\bar{x},u,v)(x) \subset D_c^2 f_+(\bar{x},u,v)(x).$$

From there we conclude. If in addition Q has a compact base then the variational systems (P_x) and (Q_x) has a common solution $z = \underline{D}^2 f(\bar{x},u,v)(x)$, as was to be shown.

Corollary 3.7. *Suppose that dim$(Y) < +\infty$, $Q = \mathbb{R}_+^p$ and all the assumptions of Theorem 3.2 are fulfilled. Then the variational systems (P_x) and (Q_x) has an unique common solution.*

Proof. It is clear that Q has a bounded closed base. Repeating the proof of preceding Theorem 3.6, we find the desired conclusion immediately.

Let g and S be defined as in Section 1 and moreover $g_+ = g + S$. For any $(u,v) \in X \times Z$, we also denote by $M(u,v)$ (resp.,$M_S(u,v)$) instead of the following set

$$M(u,v) = \{(x,z) \in X \times Z \,:\, D_c^2 g(\bar{x},u,v)(x) \subset z + S\}$$

$$\Big(\text{resp.} \quad M_s(u,v) = \{(x,z) \in X \times Z \,:\, D_c^2 g_+(\bar{x},u,v)(x) \subset z + S\}\Big).$$

We define the function $(f,g) : X \longrightarrow Y \times Z$ by

$$(f,g)(x) = (f(x), g(x)),\ \forall x \in X.$$

For simplicity of statements, we denote $(f+Q, g+S) = (f_+, g_+) = (f,g)_+$. In addition, in space $Y \times Z$, we take as norm $\|(y,z)\| = \|y\| + \|z\|$ for any $(y,z) \in Y \times Z$.

Definition 3.8. If (f,Q) is replaced by (g,S) (resp., $\big((f,g),(Q,S)\big)$) then $(P_x),(Q_x)$ is called (P_x^1) and (Q_x^1) (resp., (P_x^2) and (Q_x^2)), respectively.

Proposition 3.9. *Let $(u,(v_1,v_2)) \in X \times (Y \times Z)$, $x \in dom D_c^2(f_+,g_+)(\bar{x},u,(v_1,v_2))$ and let z_1, z_2 be solutions of (P_x) and (P_x^1), respectively. Then, if $z = (z_1,z_2) \in D_c^2(f_+,g_+)(\bar{x},u,v)(x)$, then z is a solution of (P_x^2).*

Proof. By the definition of second-order contingent derivative of (f_+,g_+) at $(\bar{x},u,(v_1,v_2))$, we obtain the following inclusion

$$D_c^2(f_+,g_+)(\bar{x},u,v)(x) \subset D_c^2 f_+(\bar{x},u,v_1)(x) \times D_c^2 g_+(\bar{x},u,v_2)(x). \tag{3.4}$$

By hypothesis, for all $(\xi,\eta) \in Q^+ \times S^+$, it follows that

$$\langle \xi, z_1 \rangle = inf\,\{\langle \xi, w_1 \rangle \mid w_1 \in D_c^2 f_+(\bar{x},u,v_1)(x)\},$$

$$\langle \eta, z_2 \rangle = inf\,\{\langle \eta, w_2 \rangle \mid w_2 \in D_c^2 g_+(\bar{x}, u, v_2)(x)\}.$$

Therefore,

$$\begin{aligned}
&inf\,\{\langle (\xi, \eta), (w_1, w_2) \rangle \mid (w_1, w_2) \in D_c^2(f_+, g_+)(\bar{x}, u, (v_1, v_2))(x)\} \\
&\geq inf\,\{\langle \xi, w_1 \rangle + \langle \eta, w_2 \rangle \mid (w_1, w_2) \in D_c^2 f_+(\bar{x}, u, v_1)(x) \times D_c^2 g_+(\bar{x}, u, v_2)(x)\} \\
&\geq inf\,\{\langle \xi, w_1 \rangle \mid w_1 \in D_c^2 f_+(\bar{x}, u, v_1)(x)\} + inf\,\{\langle \eta, w_2 \rangle \mid w_2 \in D_c^2 g_+(\bar{x}, u, v_2)(x)\} \\
&= \langle \xi, z_1 \rangle + \langle \eta, z_2 \rangle.
\end{aligned} \tag{3.5}$$

On the other hand, it follows from assumptions that $(z_1, z_2) \in D_c^2(f_+, g_+)(\bar{x}, u, v)(x)\}$ and together this with (3.5), yields that $z = (z_1, z_2)$ is solution of (P_x^2). This completes the proof.

Proposition 3.10. *Let $(u, v_1, v_2) \in X \times Y \times Z$, $x \in dom D_c^2(f_+, g_+)(\bar{x}, u, v_1, v_2)$ and let $z = (z_1, z_2) \in D_c^2(f_+, g_+)(\bar{x}, u, v_1, v_2)(x)$. Then, if $(x, z_1) \in L_Q(u, v_1)$ and $(x, z_2) \in M_S(u, v_2)$ then z is a solution of (P_x^2).*

Proof. It is easy to see that $z_1 \in D_c^2 f_+(\bar{x}, u, v_1)(x)$, which together with the fact that $(x, z_1) \in L_Q(u, v_1)$ implies that $z_1 = IMin(D_c^2 f_+(\bar{x}, u, v_1)(x))$. Therefore, z_1 is a solution of (P_x). In the same way as above, z_2 is also solution of (P_x^1). By virtute of preceding Proposition 3.7, we conclude that z is a solution of (P_x^2). This completes the proof.

4. Applications

By using the results obtained in Section 3, we derive second-order efficiency conditions for weakly efficient solution of CVEP in terms of contingent derivatives and epiderivatives with the class of steady functions. In this section, for each $\bar{x} \in C$, $F(\bar{x}, \bar{x}) = 0$ and $h = 0$, we set

$$N(-S, g(\bar{x})) = \{\eta \in Y^* \mid \langle \eta, z \rangle \leq 0 \;\forall z \in T(-S, g(\bar{x}))\}.$$

A necessary second-order optimality condition for weakly efficient solution of CVEP via contingent epiderivatives can be stated as follows.

Theorem 4.1. *Assume that all the following conditions are fulfilled*

(i) $\bar{x} \in K$ is a weakly efficient solution to the CVEP.

(ii) The mappings $F_{\bar{x}}$ and g are steady at $\bar{x}$, and $(u, v_1, v_2) \in X \times (-Q) \times (-S)$.

(iii) For every $x \in A^2(C,\bar{x},u)$ *there exists* $z=(z_1,z_2) \in D_c^2(F_{\bar{x}+},g_+)(\bar{x},u,v_1,v_2)(x)$ *such that* $(x,z_1) \in L_Q(u,v_1)$ *and* $(x,z_2) \in M_S(u,v_2)$.

Then, for every $x \in A^2(C,\bar{x},u)$*, there exist* $(\xi,\eta) \in Y^* \times Z^*$ *with* $(\xi,\eta) \neq (0,0)$ *such that*

$$\xi \in Q^+,\ \eta \in N(-S,g(\bar{x})), \tag{4.1}$$

$$\langle \xi, w_1\rangle + \langle \eta, w_2\rangle \geq 0 \quad \forall\, (w_1,w_2) \in D_c^2(F_{\bar{x}+},g_+)(\bar{x},u,v_1,v_2)(x). \tag{4.2}$$

Proof. Fix $x \in A^2(C,\bar{x},u)$. It follows from hypothesis (iii) that $x \in \operatorname{dom} D_c^2(f_+,g_+)(\bar{x},u,v_1,v_2)$, and, moreover, there exists $z=(z_1,z_2) \in D_c^2(f_+,g_+)(\bar{x},u,v_1,v_2)(x)$ such that $(x,z_1) \in L_Q(u,v_1)$ and $(x,z_2) \in M_S(u,v_2)$, where $f := F_{\bar{x}}$. In view of Proposition 3.9, we have the result that z is solution of the variational system (P_x^2). This follows that for every $(\xi,\eta) \in (Q^+,S^+)$,

$$\langle \xi, z_1\rangle + \langle \eta, z_2\rangle = inf\{\langle \xi, w_1\rangle + \langle \eta, w_2\rangle \,|(w_1,w_2) \in D_c^2(f_+,g_+)(\bar{x},u,v_1,v_2)(x)\}. \tag{4.3}$$

If the following inclusion holds

$$D_c^2(f_+,g_+)(\bar{x},u,v_1,v_2)(x) \cap (-intQ) \times IT(-S,g(\bar{x})) = \emptyset \quad \forall\, x \in A^2(C,\bar{x},u), \tag{4.4}$$

then $z=(z_1,z_2) \notin (-intQ) \times IT(-S,g(\bar{x}))$. By the definition of IT (see [9]), we have

$$(-intQ) \times IT(-S,g(\bar{x})) = (-intQ) \times (-int\, cone(S+g(\bar{x})))$$

and this implies that $(-intQ) \times IT(-S,g(\bar{x}))$ is an open convex set in the product space $Y \times Z$. By applying a trong separation theorem for the disjoint convex sets $(-intQ) \times IT(-S,g(\bar{x}))$ and $\{z\}$ yields the existence of $(\xi,\eta) \in Y^* \times Z^*, (\xi,\eta) \neq (0,0)$, such that (4.1) and (4.5) hold, where

$$\langle \xi, z_1\rangle + \langle \eta, z_2\rangle \geq 0. \tag{4.5}$$

Combining (4.3) and (4.5), we find that (4.2) holds. To finish the proof of Theorem 4.1, it suffices to show that (4.4) holds. In fact, we may assume that there exists $x_0 \in A^2(C,\bar{x},u)$ such that

$$D_c^2(f_+,g_+)(\bar{x},u,v_1,v_2)(x_0) \cap (-intQ) \times IT(-S,g(\bar{x})) \neq \emptyset.$$

Then there exists $c=(a,b) \in D_c^2(f_+,g_+)(\bar{x},u,v_1,v_2)(x_0)$ such that $a \in -intQ$ and $b \in IT(-S,g(\bar{x}))$. Adapting the definition of second-order contingent derivative of (f_+,g_+) at $(\bar{x},u,v_1,v_2)$ yields the existence of sequence $(t_n, x_n, a_n, b_n) \longrightarrow (0^+, x_0, a, b)$ such that

$$(\bar{x}, f(\bar{x}), g(\bar{x})) + t_n(u, v_1, v_2) + \frac{1}{2}t_n^2(x_n, a_n, b_n) \in \text{graph}(f_+, g_+) \quad \forall\, n \geq 1. \tag{4.6}$$

By the initial assumptions $v_1 \in -Q$, $v_2 \in -S$ and $(f_+, g_+)(x) = (f, g)(x) + (Q, S)$ $(\forall\, x \in X)$, and letting $n \to +\infty$, we find from (4.6) that

$$\lim_{n \longrightarrow +\infty} \frac{(f,g)(\bar{x} + t_n u + \frac{1}{2}t_n^2 x_n) - (f,g)(\bar{x})}{\frac{1}{2}t_n^2} \in (a, b) - Q \times S. \tag{4.7}$$

As $x_0 \in A^2(C, \bar{x}, u)$, there exists $x'_n \longrightarrow x_0$ satisfying $\bar{x} + t_n u + \frac{1}{2}t_n^2 x'_n \in C$, $\forall n \geq 1$. Since f and g are steady at $\bar{x}$ in the direction u, we see that (f, g) is steady at $(\bar{x}, u)$ (see [9]). By an argument similar as in the proof of Lemma 4.6 [[9], p. 461], one also obtains the following result

$$\lim_{n \longrightarrow +\infty} \frac{(f,g)(\bar{x} + t_n u + \frac{1}{2}t_n^2 x'_n) - (f,g)(\bar{x})}{\frac{1}{2}t_n^2} \in (a, b) - Q \times S. \tag{4.8}$$

It follows from (4.8) that

$$\lim_{n \longrightarrow +\infty} \frac{f(\bar{x} + t_n u + \frac{1}{2}t_n^2 x'_n) - f(\bar{x})}{\frac{1}{2}t_n^2} \in a - Q \subset -intQ - Q = -intQ, \tag{4.9}$$

$$\lim_{n \longrightarrow +\infty} \frac{g(\bar{x} + t_n u + \frac{1}{2}t_n^2 x'_n) - g(\bar{x})}{\frac{1}{2}t_n^2} \in b - S \subset IT(-S, g(\bar{x})) - S. \tag{4.10}$$

This implies from (4.9) that

$$f(\bar{x} + t_n u + \frac{1}{2}t_n^2 x'_n) \in -intQ \quad \text{for n sufficiently large} \tag{4.11}$$

From (4.10), we have

$$g(\bar{x} + t_n u + \frac{1}{2}t_n^2 x'_n) \in -S \quad \text{for n sufficiently large.} \tag{4.12}$$

By the definition of $IT(-S, g(\bar{x}))$, it implies that $g(\bar{x}) + \frac{1}{2}t_n^2\big(\frac{g(\bar{x}+t_n u + \frac{1}{2}t_n^2 x'_n) - g(\bar{x})}{\frac{1}{2}t_n^2} + s\big) \in -S$ for n sufficiently large and $s \in S$. Thus for n large enough $g(\bar{x} + t_n u + \frac{1}{2}t_n^2 x'_n) \in -\frac{1}{2}t_n^2 s - S \subset -S$. So, we obtain (4.12). Therefore, for n sufficiently large, $\bar{x} + t_n u + \frac{1}{2}t_n^2 x'_n \in K$. So $\bar{x} \in K$ is not weakly efficient solution for VEPC, which conflicts with assumption (i). This completes the proof.

The following example is used to illustrate for Theorem 4.1

Example 4.2. Let $X = Y = Z = \mathbb{R}^2, W = \mathbb{R}, C = \mathbb{R}^2_+, Q = S = \mathbb{R}^2_+$ and $\bar{x} = (0, 0) \in C$, where $\mathbb{R}^2_+ = -\mathbb{R}^2_- = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0,\ y \geq 0\}$. The mappings $f = F(\bar{x}, .),\ g\ : X \longrightarrow \mathbb{R}^2$ are defined respectively by

$$f(x_1, x_2) = (x_1^2 + |x_1| + |x_2|,\ x_1^2 - x_2), \quad \forall (x_1, x_2) \in X,$$

$$g(x_1,x_2)=(-x_1^2-x_1,\ -x_1^2-x_2),\quad forall\,(x_1,x_2)\in X.$$

Then the feasible set K of CVEP is given by

$$K=\{x=(x_1,x_2)\in C\mid g(x)\in -S,\ h(x)=0\}=C.$$

We pick $u=(u_1,u_2)=(0,1), v=(v_1,v_2)=(0,-1)$. It is easy to verify that conditions (i) and (ii) in Theorem 4.1 are fulfilled. By a direct computation, one obtains the results as follows $A^2(C,\bar{x},u)=[0,+\infty)\times\mathbb{R}$,

$$D_c^2(f_+,g_+)(\bar{x},u,v)(x)=\big([|x_1|+|x_2|,+\infty)\times[-x_2,+\infty)\big)\times\big([-x_1,+\infty)\times[-x_2,+\infty)\big),$$

$$\underline{D}^2 f(\bar{x},u,v_1)(x)=(|x_1|+|x_2|,-x_2),$$

$$\underline{D}^2 g(\bar{x},u,v_2)(x)=(-x_1,-x_2),\quad \forall x=(x_1,x_2)\in A^2(C,\bar{x},u).$$

Thus, the second-order contingent epiderivatives $\underline{D}^2 f(\bar{x},u,v_1)(x)$ and $\underline{D}^2 g(\bar{x},u,v_2)(x)$ exist for every $x\in A^2(C,\bar{x},u)$. By taking

$$z=(z_1,z_2)=\big((|x_1|+|x_2|,-x_2),(-x_1,-x_2)\big)$$

for each $x=(x_1,x_2)\in A^2(C,\bar{x},u)$. According to Theorem 3.1, it follows that

$$(x,z_1)\in L_Q(u,v_1),\ \ (x,z_2)\in M_S(u,v_2).$$

From here we conclude that all the assumptions of Theorem 4.1 are fulfilled. We pick $(\xi_1,\xi_2)=(1,1)\in Q^+$, $(\eta_1,\eta_2)=(0,0)\in N(-S,g(\bar{x}))$, by directly calculating, we see that the inequality of (4.2) is satisfied, as was to be checked. Note that if the second-order contingent epiderivative of (f,g) at $(\bar{x},u,v)$ exists, then we set

$$(\overline{z_1}(x),\overline{z_2}(x))=\underline{D}^2(f,g)(\bar{x},u,v_1,v_2)(x),\quad \forall\, x\in X.$$

Corollary 4.3. *Assume that (i) and (ii) in Theorem 4.1 are fulfilled and the second-order contingent epiderivative* $\underline{D}^2(F_{\bar{x}},g)(\bar{x},u,v_1,v_2)(x)$ *exists for all* $x\in A^2(C,\bar{x},u)$*. Then for any* $x\in A^2(C,\bar{x},u)$*, there exists* $(\xi,\eta)\in Y^*\times Z^*$ *with* $(\xi,\eta)\neq(0,0)$ *satisfying* $\xi\in Q^+$, $\eta\in N(-S,g(\bar{x}))$ *and the following inequality*

$$\langle\xi,\overline{z_1}(x)\rangle+\langle\eta,\overline{z_2}(x)\rangle\geq 0. \tag{4.13}$$

Proof. By using Theorem 4.1 and the definitions of second-order adjacent sets, we find the desired conclusion immediately.

Theorem 4.4. *Assume that the assumptions (i), (ii) in Theorem 4.1 are fulfilled and for every* $x \in A^2(C,\bar{x},u)$. *There exists* $z = (z_1, z_2) \in D_c^2(F_{\bar{x}}, g)(\bar{x}, u, v_1, v_2)(x)$ *such that* $(x, z_1) \in L(u, v_1)$ *and* $(x, z_2) \in M(u, v_2)$. *Then for each* $x \in A^2(C,\bar{x},u)$ *there exist* $\xi \in Q^+,\ \eta \in N(-S, g(\bar{x}))$ *with* $(\xi, \eta) \neq (0,0)$ *such that*

$$\langle \xi, w_1 \rangle + \langle \eta, w_2 \rangle \geq 0, \quad \forall\, (w_1, w_2) \in D_c^2(F_{\bar{x}}, g)(\bar{x}, u, v_1, v_2)(x).$$

Proof. Since he proof is similarly with the proof in Theorem 4.4, we here omit the proof.

Proposition 4.5. *We have the following assertions*

(i) If all the assumptions of Theorem 4.1 are fulfilled, then there exist $\xi \in Q^+, \eta \in N(-S, g(\bar{x}))$ *with* $(\xi, \eta) \neq (0,0)$ *such that*

$$inf\{\langle \xi, w_1 \rangle + \langle \eta, w_2 \rangle \,|\, (w_1, w_2) \in D_c^2(F_{\bar{x}+}, g_+)(\bar{x}, u, v_1, v_2)(x)\} \geq 0 \ \forall x \in A^2(C,\bar{x},u).$$

Furthermore, if $\underline{D}^2(F_{\bar{x}}, g)(\bar{x}, u, v_1, v_2)(x)$ *exists for all* $x \in A^2(C,\bar{x},u)$ *then*

$$\langle \xi, \overline{z_1}(x) \rangle + \langle \eta, \overline{z_2}(x) \rangle \geq 0 \ \ \forall x \in A^2(C,\bar{x},u).$$

(ii) If all the assumptions of Theorem 4.2 are fulfilled, then there exist $\xi \in Q^+, \eta \in N(-S, g(\bar{x}))$ *with* $(\xi, \eta) \neq (0,0)$ *such that*

$$inf\{\langle \xi, w_1 \rangle + \langle \eta, w_2 \rangle \,|\, (w_1, w_2) \in D_c^2(F_{\bar{x}}, g)(\bar{x}, u, v_1, v_2)(x)\} \geq 0 \ \forall x \in A^2(C,\bar{x},u).$$

Proof. It is a consequence from Theorem 4.1 and Theorem 4.4. We omit the proof.

Proposition 4.6. *Let* $\bar{x} \in K$ *and* $(u, v) \in X \times Y$. *Assume, in addition, that* $\underline{D}^2 F_{\bar{x}}(\bar{x}, u, v)(x)$ *exist for all* $x \in dom D_c^2 F_{\bar{x}+}(\bar{x}, u, v)$. *Then*

(i) $IMin\Big(D_c^2 F_{\bar{x}+}(\bar{x}, u, v)(x)\Big) \neq \emptyset$ *and* $\underline{D}^2 F_{\bar{x}}(\bar{x}, u, v)(x) = IMin\Big(D_c^2 F_{\bar{x}+}(\bar{x}, u, v)(x)\Big)$ *for all* $x \in dom D_c^2 F_{\bar{x}+}(\bar{x}, u, v)$.

(ii) If, in addition, Q *has a compact base* B, *then* $IMin\Big(D_c^2 F_{\bar{x}}(\bar{x}, u, v)(x)\Big) \neq \emptyset$ *and*

$$\underline{D}^2 F_{\bar{x}}(\bar{x}, u, v)(x) = IMin\Big(D_c^2 F_{\bar{x}}(\bar{x}, u, v)(x)\Big), \forall x \in dom D_c^2 F_{\bar{x}+}(\bar{x}, u, v).$$

Proof. The proof is obvious. We here omit the proof.

Next, we give another example.

Example 4.7. Let $X=\mathbb{R}$, $Y=\mathbb{R}^3$ and $Q=\{(x,y,z)\in\mathbb{R}^3 : x\geq 0, y\geq 0, z\geq 0\}$ be a pointed convex closed cone in $\mathbb{R}^3$. Taking $u\in\mathbb{R}$ such that $v=(u,2u,3u)$ and $\bar{x}=1$. The function $F_1:\mathbb{R}\longrightarrow\mathbb{R}^3$ is given as $F_1(x)=(x-1,x^2-1,x^3-1)$ for all $x\in\mathbb{R}$. Notice that Q has a compact base B. By directly calculating, one obtains the results as follows

$$\operatorname{dom} D_c^2 F_{1+}(\bar{x},u,v)=\mathbb{R},$$

$\underline{D}^2 F_1(\bar{x},u,v)(x)$ exists for all $x\in\mathbb{R}$, and moreover $\underline{D}^2 F_1(\bar{x},u,v)(x)=(x,2x,3x+3u^2)\ \forall x\in\mathbb{R}$. For any $x\in\mathbb{R}$, we always have

$$D_c^2 F_1(\bar{x},u,v)(x)=\{(x,2x,3x+3u^2)\},$$

$$D_c^2 F_{1+}(\bar{x},u,v)(x)=[x,+\infty)\times[2x,+\infty)\times[3x+3u^2,+\infty).$$

Therefore,

$$IMin\Big(D_c^2 F_1(\bar{x},u,v)(x)\Big)=IMin\Big(D_c^2 F_{1+}(\bar{x},u,v)(x)\Big)=\underline{D}^2 F_1(\bar{x},u,v)(x).$$

To provide sufficient second-order efficiency conditions for weakly efficient solution to the CVEP, we recall the following define

▶ Recall that an extended-real-valued function l, defined on a set $C\subset X$, is said to be quasiconvex at $\bar{x}\in C$ with respect to C if and only if, for each $x\in C$,

$$l(x)\leq l(\bar{x}) \quad\Longrightarrow\quad \forall\, t\in(0,1),\quad l(tx+(1-t)\bar{x})\leq l(\bar{x}).$$

l is said to be quasiconvex on C if and only if it is quasiconvex at each $x\in Q$. l is called quasilinear at $\bar{x}$ with respect to C is and only if l and $-l$ are quasiconvex at $\bar{x}$ with respect to C.

Theorem 4.8. *Let $(u,v_1,v_2)\in X\times(-Q)\times(-S)$ be arbitrary. Suppose, in addition, that Y,Z be finite dimensional spaces, $F_{\bar{x}}$ and g be steady at $\bar{x}$, and for every $x\in A^2(C,\bar{x},u)\setminus\{0\}$ there exist $(\xi,\eta)\in(Y\times Z)^*$ satisfying (4.1) and the following inequality*

$$\langle\xi,w_1\rangle+\langle\eta,w_2\rangle>0\quad\forall\,(w_1,w_2)\in D_c^2(F_{\bar{x}},g)(\bar{x},u,v_1,v_2)(x). \tag{4.14}$$

Assume that C is convex subset, $\xi_0 f$ and $\eta_0 g$ are quasiconvex functions at $\bar{x} \in K$ with respect to C. Then $\bar{x}$ is a weakly efficient solution to the CVEP.

Proof. We prove that $\bar{x}$ is weakly efficient solution to the CVEP. In fact, if not, then there would exists $x \in K \setminus \{\bar{x}\}$ such that $f(x) := F_{\bar{x}}(x) \in -intQ \subset -Q$ and $g(x) \in -S$. Take sequence $t_n \in (0,1)\ \forall n \geq 1$ such that $t_n \to 0^+$ and sequence $x_n \longrightarrow x - \bar{x}$. By the hypothesis, we can choose $(u, v_1, v_2) = (0,0,0)$. It is clear that (f,g) is stable at $\bar{x}$ (can seen in [9]). Without loss of generality, we may assume that

$$\lim_{n \longrightarrow +\infty} \frac{(f,g)(\bar{x} + t_n u + \frac{1}{2} t_n^2 x_n) - (f,g)(\bar{x})}{\frac{1}{2} t_n^2} = (a,b) \in D_c^2(f,g)(\bar{x}, u, v_1, v_2)(x - \bar{x}). \tag{4.15}$$

From the fact that (f,g) is also steady at $\bar{x}$, we find from (4.15) that

$$\lim_{n \longrightarrow +\infty} \frac{(f,g)(\bar{x} + t_n u + \frac{1}{2} t_n^2 (x - \bar{x})) - (f,g)(\bar{x})}{\frac{1}{2} t_n^2} = (a,b) \in D_c^2(f,g)(\bar{x}, u, v_1, v_2)(x - \bar{x}). \tag{4.16}$$

In view of Proposition 2.3 (i) [4, p. 302], we find that $A^2(C, \bar{x}, u) = clcone(C - \bar{x})$. Consequently $x - \bar{x} \in A^2(C, \bar{x}, u)$ because

$$x - \bar{x} \in K - \bar{x} \subset C - \bar{x} \subset cl\, cone(C - \bar{x}) = A^2(C, \bar{x}, u).$$

By the hypothesis, we see that there exist $(\xi, \eta) \in Q^+ \times N(-S, g(\bar{x})) \setminus \{(0,0)\}$ such that (4.14) is fulfilled for $x = x - \bar{x}$. A consequence is

$$\langle \xi, a \rangle + \langle \eta, b \rangle > 0. \tag{4.17}$$

From (4.16), we see that

$$a = \lim_{n \longrightarrow +\infty} \frac{f(\bar{x} + t_n u + \frac{1}{2} t_n^2 (x - \bar{x})) - f(\bar{x})}{\frac{1}{2} t_n^2}.$$

By the initial hypothesis it yields that $\xi \in Q^+$ with $\xi_0 f$ is quasiconvex at $\bar{x}$ with respect to C and moreover $\langle \xi, f(x) \rangle \leq \langle \xi, f(\bar{x}) \rangle$ because $f(\bar{x}) = 0$, $f(x) \in -Q$. Adapting the definition of quasiconvexity of f at $\bar{x}$ with respect to C, it leads to

$$\left\langle \xi, f(\bar{x} + t_n u + \frac{1}{2} t_n^2 (x - \bar{x})) \right\rangle \leq \langle \xi, f(\bar{x}) \rangle \quad \forall\, n \geq 1.$$

It follows that

$$\langle \xi, a \rangle = \lim_{n \longrightarrow +\infty} \frac{\left\langle \xi, f(\bar{x} + t_n u + \frac{1}{2} t_n^2 (x - \bar{x})) \right\rangle - \langle \xi, f(\bar{x}) \rangle}{\frac{1}{2} t_n^2} \leq 0. \tag{4.18}$$

In the same way, we also obtain

$$\langle \eta, b\rangle = \lim_{n\longrightarrow+\infty} \frac{\left\langle \eta, g(\bar{x}+t_n u+\frac{1}{2}t_n^2(x-\bar{x}))\right\rangle - \langle \eta, g(\bar{x})\rangle}{\frac{1}{2}t_n^2} \leq 0. \tag{4.19}$$

Combining (4.18)-(4.19) yields that $\langle \xi, a\rangle + \langle \eta, b\rangle \leq 0$ holds, which conflicts with (4.17). Hence, $\bar{x}$ is a weakly efficient solution for VEPC, which completes the proof.

Example 4.9. Let $X, Y, Z, W, C, Q, S, K, \bar{x}, F_{\bar{x}}, g, h$ be given as in Example 4.2. It is obvious that $A^2(C, \bar{x}, u) = [0, +\infty) \times \mathbb{R}$ and C is a convex subset in X. For all $w = (w_1, w_2) \in D_c^2(F_{\bar{x}}, g)(\bar{x}, u, v)(x)$ where $x = (x_1, x_2) \in [0, +\infty) \times \mathbb{R} \setminus \{(0,0)\}$. It is easy to see that

$$w = (w_1, w_2) \in D_c^2(F_{\bar{x}+}, g_+)(\bar{x}, u, v)(x),\ \forall\, x = (x_1, x_2) \in [0, +\infty) \times \mathbb{R} \setminus \{(0,0)\}.$$

Consequently

$$w_1 = (w_{11}, w_{12}), \quad w_{11} \geq |x_1| + |x_2|, w_{12} \geq -x_2,$$

$$w_2 = (w_{21}, w_{22}), \quad w_{21} \geq -x_1, w_{22} \geq -x_2.$$

By choosing $(\xi, \eta) = ((1,0), (0,0))$, we obtain the following strict inequality

$$\langle \xi, w_1\rangle + \langle v\rangle = w_{11} \geq |x_1| + |x_2| > 0,$$

because $(x_1, x_2) \neq (0,0)$. In view of the result of Theorem 4.4, it leads to $\bar{x} = (0,0)$ being a weakly efficient solution to the CVEP.

Theorem 4.10. *Let $(u, v_1, v_2) \in X \times (-Q) \times (-S)$ be arbitrary. Suppose, in addition, that Y, Z be finite dimensional spaces, $F_{\bar{x}}$ and g be steady at $\bar{x}$, and for every $x \in A^2(C, \bar{x}, u) \setminus \{0_X\}$ and there exist $(\xi, \eta) \in (Y \times Z)^*$ satisfying (4.1) and the following inequality*

$$\langle \xi, w_1\rangle + \langle \eta, w_2\rangle > 0\ \forall\, (w_1, w_2) \in D_c^2(F_{\bar{x}+}, g_+)(\bar{x}, u, v_1, v_2)(x). \tag{4.20}$$

Assume that C is convex subset and $\xi_0 f$ and $\eta_0 g$ are quasiconvex at $\bar{x} \in K$ with respect to C. Then $\bar{x}$ is a weakly efficient solution to the CVEP.

Proof. It follows from Theorem 4.4, which completes the proof.

Corollary 4.11. *Let $(u, v_1, v_2) \in X \times (-Q) \times (-S)$ be arbitrary. Suppose, in addition, that Y, Z be finite dimensional spaces, $F_{\bar{x}}$ and g be steady at $\bar{x}$, and for every $x \in A^2(C, \bar{x}, u) \setminus \{0_X\}$ there exist $(\xi, \eta) \in (Y \times Z)^*$ satisfying (4.1) and the following inequality*

$$inf\left\{\, \langle \xi, w_1\rangle + \langle \eta, w_2\rangle \,|(w_1, w_2) \in D_c^2(F_{\bar{x}+}, g_+)(\bar{x}, u, v_1, v_2)(x)\right\} > 0\ \forall x \in A^2(C, \bar{x}, u) \setminus \{0\}.$$

Assume that C is convex subset and $\xi_0 f$ and $\eta_0 g$ are quasiconvex at $\bar{x} \in K$ with respect to C. Then $\bar{x}$ is a weakly efficient solution to the CVEP.

Proof. It follows from Theorem 4.4, which completes the proof.

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