Commun. Optim. Theory 2017 (2017), Article ID 22 https://doi.org/10.23952/cot.2017.22

Communications in Optimization Theory
Available online at http://cot.mathres.org

FIXED POINTS OF GENERALIZED (ψ, s, α)-CONTRACTIVE MAPPINGS IN DISLOCATED AND b-DISLOCATED METRIC SPACES

K. ZOTO[1,*], S. RADENOVIĆ[2,3], J. DINE[1], I. VARDHAMI[4]

[1]Department of Mathematics and Computer Sciences,

Faculty of Natural Sciences, University of Gjirokastra, Albania

[2]Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Beograd, Serbia

[3]Department of Mathematics, State University of Novi Pazar, Novi Pazar, Serbia

[4]Department of Mathematics, Faculty of Natural Science, University of Tirana, Albania

Abstract. In this paper, we prove unique fixed point results for a self mapping satisfying (ψ, s, α)-contractive conditions in the setting of complete dislocated and b-dislocated metric spaces. Our results extend and generalize some several known results to the framework of b-spaces.

Keywords. b-dislocated metric; Dislocated metric, (ψ, s, α)-contraction; Fixed point.

2010 Mathematics Subject Classification. 47H09, 47H10.

1. Introduction

In 1922, Banach established the celebrated Banach's Contraction Principle. Since then, fixed point theory has received much attention due to its applications in pure mathematics and applied sciences. Recently, a number of generalizations of metric spaces were introduced and

[*]Corresponding author.

E-mail addresses: zotokastriot@yahoo.com (K. Zoto), J. Dine (jani_dine@yahoo.com), radens@beotel.net (S. Radenović), ivardhami@yahoo.com (I. Vardhami).

Received December 17, 2016; Accepted April 21, 2017.

extensively studied. In 1989, Bakhtin [1] (and also Czerwik [2]) introduced the concept of b-metric spaces and presented contraction mappings in b-metric spaces that is a generalization of Banach contraction principle in metric spaces. For fixed point theory in b-metric spaces, see [3]-[11] and the references therein. In 2000, Hitzlerand and Seda [12] introduced the notion of dislocated metric spaces, in which the self distance of a point need not be equal to zero. Such spaces play an important role in topology and logical programming. Recently, Hussain *et al.* [3] introduced the notion of b-dislocated metric spaces. Some topological properties of dislocated and b-dislocated metrics were investigated in [3]. Very recently, many results on fixed points, coincidence points and common fixed points of mappings which satisfy certain contractive conditions in dislocated metric spaces where obtained, see [13]-[19] and references therein.

In this paper, we establish some fixed point theorems for contractive mappings in the setting of dislocated and b-dislocated metric spaces, using altering distance functions and (ψ, s, α)-contractions. Our main results extend and generalize some known results in the literature to general metric spaces.

2. Preliminaries

Definition 2.1. [12] Let X be a nonempty set. A mapping $d_l : X \times X \to [0, \infty)$ is said to be a dislocated metric (or simply d_l-metric) if for all $x, y, z \in X$, the following conditions are satisfied:

(1) If $d_l(x,y) = 0$, then $x = y$;
(2) $d_l(x,y) = d_l(y,x)$;
(3) $d_l(x,y) \leq d_l(x,z) + d_l(z,y)$.

The pair (X, d_l) is called a dislocated metric space (or d_l-metric space for short).

Note that if $x = y$, then $d_l(x,y)$ may not be 0.

Example 2.2. If $X = \mathbb{R}$, then $d(x,y) = |x| + |y|$ defines a dislocated metric on X.

Definition 2.3. [12] A sequence $\{x_n\}$ in d_l-metric space (X, d_l) is said to be

(1) a Cauchy sequence if, for given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m, n > n_0$, we have $d_l(x_m, x_n) < \varepsilon$ or $\lim_{n,m\to\infty} d_l(x_m, x_n) = 0$,

(2) convergent with respect to d_l if there exists $x \in X$ such that $d_l(x_n,x) \to 0$ as $n \to \infty$. In this case, x is called the limit of $\{x_n\}$ and we write $x_n \to x$.

A d_l-metric space is called complete if every Cauchy sequence in X converges to a point in X.

Definition 2.4. [3] Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $b_d : X \times X \to [0,\infty)$ is said to be a b-dislocated metric (or simply b_d-metric for short) if for all $x,y,z \in X$, the following conditions are satisfied:

(1) If $b_d(x,y) = 0$, then $x = y$;
(2) $b_d(x,y) = b_d(y,x)$;
(3) $b_d(x,y) \leq s\,[b_d(x,z) + b_d(z,y)]$.

(X,b_d) is called a b-dislocated metric space (with parameter $s \geq 1$). And the class of b-dislocated metric space is larger than that of dislocated metric spaces, since a b-dislocated metric is a dislocated metric when $s = 1$.

In [3], it was showed that each b_d-metric on X generates a topology τ_{b_d} whose base is the family of open b_d-balls $B_{b_d}(x,\varepsilon) = \{y \in X : b_d(x,y) < \varepsilon\}$.

Definition 2.5. [7] Let (X,b_d) be a b_d-metric space, and $\{x_n\}$ be a sequence of points in X . A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim\limits_{n\to\infty} b_d(x_n,x) = 0$ and we say that the sequence $\{x_n\}$ is b_d-convergent to x and denote it by $x_n \to x$ as $n \to \infty$.

The limit of a b_d-convergent sequence in a b_d-metric space is unique; seee ([8], Proposition 1.27).

Definition 2.6. [3] A sequence $\{x_n\}$ in a b_d-metric space (X,b_d) is called a b_d-Cauchy sequence iff, given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n,m > n_0$, we have $b_d(x_n,x_m) < \varepsilon$ or $\lim\limits_{n,m\to\infty} b_d(x_n,x_m) = 0$. Every b_d-convergent sequence in a b_d-metric space is a b_d-Cauchy sequence.

Remark 2.7. The sequence $\{x_n\}$ in a b_d-metric space (X,b_d) is called a b_d-Cauchy sequence iff $\lim\limits_{n,m\to\infty} b_d(x_n,x_{n+p}) = 0$ for all $p \in \mathbb{N}^*$.

Definition 2.8. [3] A b_d-metric space (X,b_d) is called complete if every b_d-Cauchy sequence in X is b_d-convergent.

Definition 2.9. [21] A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:

(1) ψ is monotone increasing and continuous,

(2) $\psi(t) = 0$ if and only if $t = 0$.

We denote the set of altering distance functions by Ψ .

Lemma 2.10. [3] *Let (X, b_d) be a b-dislocated metric space with parameter $s \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are b_d-convergent to $x, y \in X$, respectively. Then we have*

$$\frac{1}{s^2} b_d(x,y) \leq \liminf_{n\to\infty} b_d(x_n, y_n) \leq \limsup_{n\to\infty} b_d(x_n, y_n) \leq s^2 b_d(x,y).$$

In particular, if $b_d(x,y) = 0$, then we have $\lim_{n\to\infty} b_d(x_n, y_n) = 0 = b_d(x,y)$. Moreover, for each $z \in X$, we have

$$\frac{1}{s} b_d(x,z) \leq \liminf_{n\to\infty} b_d(x_n, z) \leq \limsup_{n\to\infty} b_d(x_n, z) \leq s b_d(x,z).$$

In particular, if $b_d(x,z) = 0$, then we have $\lim_{n\to\infty} b_d(x_n, z) = 0 = b_d(x,z)$.

Example 2.11. Let $X = \mathbb{R}^+ \cup \{0\}$ and a constant α with $\alpha > 0$. Define the function $d_l : X \times X \to [0,\infty)$ by $d_l(x,y) = \alpha(x+y)$. Then (X, d_l) is a dislocated metric space.

Example 2.12. If $X = \mathbb{R}^+ \cup \{0\}$, then $b_d(x,y) = (x+y)^2$ defines a b-dislocated metric on X with parameter $s = 2$.

We prove the following lemma which is used to prove our results.

Lemma 2.13. *Let (X, b_d) be complete b-dislocated metric space with parameter $s \geq 1$ and let $\{x_n\}$ be a sequence such that*

$$\lim_{n\to\infty} b_d(x_n, x_{n+1}) = 0. \tag{2.1}$$

If $\{x_n\}$ is not Cauchy, then there exists $\varepsilon > 0$ and two subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ with $n_k > m_k > k$ (positive integers) such that $b_d(x_{m_k}, x_{n_k}) \geq \varepsilon$, $b_d(x_{m_k}, x_{n_k-1}) < \varepsilon$, and

$$\begin{aligned}
\frac{\varepsilon}{s^2} \leq & \limsup_{k\to\infty} b_d(x_{m_k-1}, x_{n_k-1}) & \leq \varepsilon s, \\
\frac{\varepsilon}{s} \leq & \limsup_{k\to\infty} b_d(x_{n_k-1}, x_{m_k}) & \leq \varepsilon s^2, \\
\frac{\varepsilon}{s} \leq & \limsup_{k\to\infty} b_d(x_{m_k-1}, x_{n_k}) & \leq \varepsilon s^2.
\end{aligned}$$

Proof. If $\{x_n\}$ is not a b_d-Cauchy sequence, then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ such that n_k is smallest index for which

$$n_k > m_k > k, \qquad b_d(x_{m_k}, x_{n_k}) \geq \varepsilon. \tag{2.2}$$

This means that

$$b_d(x_{m_k}, x_{n_k-1}) < \varepsilon. \tag{2.3}$$

From (2.2) and property (c) of Definition 2.4, we have

$$\begin{aligned} \varepsilon \leq b_d(x_{m_k}, x_{n_k}) &\leq s b_d(x_{m_k}, x_{m_k-1}) + s b_d(x_{m_k-1}, x_{n_k}) \\ &\leq s b_d(x_{m_k}, x_{m_k-1}) + s^2 b_d(x_{m_k-1}, x_{n_k-1}) + s^2 b_d(x_{n_k-1}, x_{n_k}). \end{aligned} \tag{2.4}$$

Taking the upper limit as $k \to \infty$ in (2.4) and using (2.1)-(2.3), we get

$$\frac{\varepsilon}{s^2} \leq \limsup_{k\to\infty} b_d(x_{m_k-1}, x_{n_k-1}). \tag{2.5}$$

By triangular inequality, we have

$$b_d(x_{m_k-1}, x_{n_k-1}) \leq s b_d(x_{m_k-1}, x_{m_k}) + s b_d(x_{m_k}, x_{n_k-1}).$$

Taking the upper limit as $k \to \infty$, we get

$$\limsup_{k\to\infty} b_d(x_{m_k-1}, x_{n_k-1}) \leq \varepsilon s. \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$\frac{\varepsilon}{s^2} \leq \limsup_{k\to\infty} b_d(x_{m_k-1}, x_{n_k-1}) \leq \varepsilon s, \tag{2.7}$$

and

$$\varepsilon \leq b_d(x_{m_k}, x_{n_k}) \leq s b_d(x_{m_k}, x_{m_k-1}) + s b_d(x_{m_k-1}, x_{n_k}).$$

Taking the upper limit as $k \to \infty$, we get

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} b_d(x_{m_k-1}, x_{n_k}), \tag{2.8}$$

and

$$\varepsilon \leq b_d(x_{m_k}, x_{n_k}) \leq s b_d(x_{m_k}, x_{n_k-1}) + s b_d(x_{n_k-1}, x_{n_k}).$$

They imply

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} b_d(x_{n_k-1}, x_{m_k}). \tag{2.9}$$

Since $b_d(x_{n_k-1},x_{m_k}) \leq sb_d(x_{n_k-1},x_{m_k-1})+sb_d(x_{m_k-1},x_{m_k})$, we find (2.1) and (2.7) that

$$\limsup_{k\to\infty} b_d(x_{n_k-1},x_{m_k}) \leq s\limsup_{k\to\infty} b_d(x_{n_k-1},x_{m_k-1}) \leq \varepsilon s^2. \tag{2.10}$$

Consequently, we have

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} b_d(x_{n_k-1},x_{m_k}) \leq \varepsilon s^2 \tag{2.11}$$

and

$$b_d(x_{m_k-1},x_{n_k}) \leq sb_d(x_{m_k-1},x_{n_k-1})+sb_d(x_{n_k-1},x_{n_k}).$$

Using (2.1), (2.7) and (2.8), we have

$$\limsup_{k\to\infty} b_d(x_{m_k-1},x_{n_k}) \leq s\limsup_{k\to\infty} b_d(x_{m_k-1},x_{n_k-1}) \leq \varepsilon s^2.$$

Consequently, we arrive at

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} b_d(x_{m_k-1},x_{n_k}) \leq \varepsilon s^2.$$

This completes the proof.

3. Main results

Based on the definitions of Cirić contraction in [23], we introduce the definitions of (ψ,s,α)-contractive mappings in the setting of b-dislocated metric spaces. Some fixed point results for these contractions are obtained in such spaces.

Definition 3.1. *Let (X,b_d) be a b-dislocated metric space with parameter $s\geq 1$. A self-mapping $T:X\to X$ is called a $s-\alpha$ contraction if it satisfies the following condition:*

$$sb_d(Tx,Ty) \leq \alpha b_d(x,y)$$

for all $x,y\in X$, where $\alpha\in[0,1)$.

Definition 3.2. *Let (X,b_d) be a b-dislocated metric space with parameter $s\geq 1$. A self-mapping $T:X\to X$ is called a (ψ,s,α)-contraction if there exist $\psi\in\Psi$ such that*

$$\psi(sb_d(Tx,Ty)) \leq \alpha\psi(b_d(x,y))$$

for all $x,y\in X$, where $\alpha\in[0,1)$.

Definition 3.3. *Let (X, b_d) be a b-dislocated metric space with parameter $s \geq 1$. If $T : X \to X$ is a self mapping that satisfies:*

$$sb_d(Tx,Ty) \leq \max\left\{b_d(x,y), b_d(x,Tx), b_d(y,Ty), \frac{b_d(x,Ty)+b_d(y,Tx)}{4s}\right\}$$

for all $x, y \in X$, where $\alpha \in [0,1)$. Then T is called a $s-\alpha$-generalized contraction.

Definition 3.4. *Let (X, b_d) be a b-dislocated metric space with parameter $s \geq 1$. We say that a self-mapping $T : X \to X$ is a (ψ, s, α)-generalized contractive mapping if there exist $\psi \in \Psi$ such that*

$$\psi\left(sb_d\left(Tx,Ty\right)\right) \leq \alpha\psi\left(\max\left\{b_d\left(x,y\right), b_d\left(x,Tx\right), b_d\left(y,Ty\right)\frac{b_d\left(x,Ty\right)+b_d\left(y,Tx\right)}{4s}\right\}\right) \quad (3.1)$$

for all $x, y \in X$, where $0 \leq \alpha < 1$.

Theorem 3.5. *Let (X, b_d) be a complete b-dislocated metric space with parameter $s \geq 1$ and let $T : X \to X$ be a (ψ, s, α)-generalized contractive mapping. Then T has a unique fixed point in X.*

Proof. Let x_0 be an arbitrary point in X. Define an iterative sequence $\{x_n\}$ as follows:

$$x_1 = T(x_0), x_2 = T(x_1), \ldots, x_{n+1} = T(x_n), \ldots, \qquad n \geq 0.$$

If $b_d(x_n, x_{n+1}) = 0$ for some $n \in \mathbb{N}$, then $x_{n+1} = x_n$, that is $x_n = x_{n+1} = T(x_n)$. Hence, x_n is a fixed point of T and the proof is completed. Consequently, we assume that $b_d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$, that is, $x_{n+1} \neq x_n$. By condition (3.1), we have

$$\begin{aligned}
\psi\left(b_d\left(x_n,x_{n+1}\right)\right) &\leq \psi\left(sb_d\left(x_n,x_{n+1}\right)\right)\\
&= \psi\left(sb_d\left(Tx_{n-1},Tx_n\right)\right)\\
&\leq \alpha\psi\left(\max\left\{\begin{array}{l} b_d\left(x_{n-1},x_n\right), b_d\left(x_{n-1},Tx_{n-1}\right), b_d\left(x_n,Tx_n\right),\\ \frac{b_d(x_{n-1},Tx_n)+b_d(x_n,Tx_{n-1})}{4s}\end{array}\right\}\right)\\
&= \alpha\psi\left(\max\left\{\begin{array}{l} b_d\left(x_{n-1},x_n\right), b_d\left(x_{n-1},x_n\right), b_d\left(x_n,x_{n+1}\right),\\ \frac{b_d(x_{n-1},x_{n+1})+b_d(x_n,x_n)}{4s}\end{array}\right\}\right) \qquad (3.2)\\
&\leq \alpha\psi\left(\max\left\{\begin{array}{l} b_d\left(x_{n-1},x_n\right), b_d\left(x_{n-1},x_n\right), b_d\left(x_n,x_{n+1}\right),\\ \frac{s[b_d(x_{n-1},x_n)+b_d(x_n,x_{n+1})]+2sb_d(x_{n-1},x_n)}{4s}\end{array}\right\}\right).
\end{aligned}$$

If $b_d(x_{n-1},x_n) \leq b_d(x_n,x_{n+1})$ for some $n \in \mathbb{N}$, then we find from inequality (3.2) that

$$\psi(b_d(x_n,x_{n+1})) \leq \alpha\psi(b_d(x_n,x_{n+1})). \tag{3.3}$$

It follows that $\psi(b_d(x_n,x_{n+1})) = 0$. Using the property of function ψ, we have $b_d(x_n,x_{n+1}) = 0$, which is a contradiction. Hence,

$$b_d(x_n,x_{n+1}) < b_d(x_{n-1},x_n)$$

for each $n \in \mathbb{N}$, that is, sequence $\{b_d(x_n,x_{n+1})\}$ is nonincreasing and bounded below. Thus there exists $r \geq 0$ such that $\lim_{n\to\infty} b_d(x_n,x_{n+1}) = r$. We claim that $r = 0$. If not, that is, $\lim_{n\to\infty} b_d(x_n,x_{n+1}) = r > 0$. Applying contractive condition (3.2) yields that

$$\psi\left(b_d\left(x_n,x_{n+1}\right)\right) \leq \alpha\psi\left(\max\left\{b_d\left(x_n,x_{n+1}\right), b_d\left(x_{n-1},x_n\right)\right\}\right). \tag{3.4}$$

Taking limit as $n \to \infty$ in (3.4), we get $\psi(r) \leq \alpha\psi(r)$ which is a contradiction. Hence

$$\lim_{n\to\infty} b_d(x_n,x_{n+1}) = 0. \tag{3.5}$$

Next, we show that $\{x_n\}$ is a b_d-Cauchy sequence in X. Suppose the contrary, that is, $\{x_n\}$ is not a b_d-Cauchy sequence. Then by Lemma 2.14, there exists $\varepsilon > 0$ and two subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ with $n_k > m_k > k$ such that $b_d(x_{m_k},x_{n_k}) \geq \varepsilon$, $b_d(x_{m_k},x_{n_k-1}) < \varepsilon$ and

$$\begin{aligned} \frac{\varepsilon}{s^2} &\leq \limsup_{k\to\infty} b_d(x_{m_k-1},x_{n_k-1}) \leq \varepsilon s, \\ \frac{\varepsilon}{s} &\leq \limsup_{k\to\infty} b_d(x_{n_k-1},x_{m_k}) \leq \varepsilon s^2, \\ \frac{\varepsilon}{s} &\leq \limsup_{k\to\infty} b_d(x_{m_k-1},x_{n_k}) \leq \varepsilon s^2. \end{aligned} \tag{3.6}$$

From contractive condition (3.1), we have

$$\begin{aligned} \psi(sb_d(x_{m_k},x_{n_k})) &= \psi(sb_d(Tx_{m_k-1},Tx_{n_k-1})) \\ &\leq \alpha\psi\left(\begin{array}{c} \max\{,b_d(x_{m_k-1},x_{n_k-1}), b_d(x_{m_k-1},Tx_{m_k-1}), b_d(x_{n_k-1},Tx_{n_k-1}), \\ \frac{b_d(x_{m_k-1},Tx_{n_k-1})+b_d(x_{n_k-1},Tx_{m_k-1})}{4s}\} \end{array}\right) \\ &= \alpha\psi\left(\begin{array}{c} \max\{b_d(x_{m_k-1},x_{n_k-1}), b_d(x_{m_k-1},x_{m_k}), b_d(x_{n_k-1},x_{n_k}), \\ \frac{b_d(x_{m_k-1},x_{n_k})+b_d(x_{n_k-1},x_{m_k})}{4s}\} \end{array}\right). \end{aligned} \tag{3.7}$$

Taking the upper limit as $k\to\infty$ in (3.7) and using (3.6), we get

$$\psi(s\varepsilon)\leq\alpha\psi(\max\{\varepsilon s,0,0,\frac{\varepsilon s}{2}\})\leq\alpha\psi(\varepsilon s),$$

which finds a contradiction due to the property of ψ. Thus the sequence is a b_d-Cauchy sequence in (X,b_d). So there exist some $u\in X$, such that $\{x_n\}$ is convergent to u. If T is a continuous mapping, we get

$$T(u)=T(\lim_{n\to\infty}x_n)=\lim_{n\to\infty}T(x_n)=\lim_{n\to\infty}(x_{n+1})=u.$$

Thus u is a fixed point of T. If T is not continuous then, we consider

$$\begin{aligned}\psi\left(sb_d\left(x_{n+1},Tu\right)\right)&=\psi\left(sb_d\left(Tx_n,Tu\right)\right)\\&\leq\alpha\psi\left(\max\left\{\begin{array}{l}b_d\left(x_n,u\right),b_d\left(x_n,Tx_n\right),b_d\left(u,Tu\right),\\\frac{b_d(x_n,Tu)+b_d(u,Tx_n)}{4s}\end{array}\right\}\right)\\&=\alpha\psi\left(\max\left\{\begin{array}{l}b_d\left(x_n,u\right),b_d\left(x_n,x_{n+1}\right),b_d\left(u,Tu\right),\\\frac{b_d(x_n,Tu)+b_d(u,x_{n+1})}{4s}\end{array}\right\}\right).\end{aligned}\tag{3.8}$$

Taking the upper limit in (3.8) and using Lemma 2.10 and (3.5), we find that

$$\begin{aligned}\psi\left(b_d\left(u,Tu\right)\right)&=\psi\left(s\frac{1}{s}b_d\left(u,Tu\right)\right)\\&\leq\alpha\psi\left(\max\left\{0,0,b_d\left(u,Tu\right),\frac{sb_d\left(u,Tu\right)+0}{4s}\right\}\right)\\&\leq\alpha\psi\left(b_d\left(u,Tu\right)\right).\end{aligned}\tag{3.9}$$

From inequality (3.9) and property of function ψ, we have $b_d(u,Tu)=0$ and $Tu=u$. Hence u is a fixed point of T.

Now, we are in a position to show the uniqueness. Let u and v be two fixed points of T, where $Tu=u$ and $Tv=v$. Let us prove that if u is a fixed point of T. Then we have $b_d(u,u)=0$. From contractive condition (3.1), we see that

$$\begin{aligned}
\psi(b_d(u,u)) &\leq \psi(sb_d(u,u)) \\
&= \psi(sb_d(Tu,Tu)) \\
&\leq \alpha\psi\left(\max\left\{\begin{array}{l} b_d(u,u), b_d(u,Tu), b_d(u,Tu), \\ \frac{b_d(u,Tu)+b_d(u,Tu)}{4s} \end{array}\right\}\right) \\
&= \alpha\psi\left(\max\left\{\begin{array}{l} b_d(u,u), b_d(u,u), b_d(u,u), \\ \frac{b_d(u,u)+b_d(u,u)}{4s} \end{array}\right\}\right) \\
&\leq \alpha\psi(b_d(u,u)).
\end{aligned}$$

From the inequality above and the property of function ψ, we have $b_d(u,u) = 0$. Again, using condition (3.1), we have

$$\begin{aligned}
\psi(b_d(u,v)) &\leq \psi(sb_d(u,v)) = \psi(sb_d(Tu,Tv)) \\
&\leq \alpha\psi\left(\max\left\{\begin{array}{l} b_d(u,v), b_d(u,Tu), b_d(v,Tv), \\ \frac{b_d(u,Tv)+b_d(v,Tu)}{4s} \end{array}\right\}\right) \\
&= \alpha\psi\left(\max\left\{\begin{array}{l} b_d(u,v), b_d(u,u), b_d(v,v), \\ \frac{b_d(u,v)+b_d(v,u)}{4s} \end{array}\right\}\right) \\
&\leq \alpha\psi\left(\max\left\{b_d(u,v), 0, 0, \frac{b_d(u,v)}{2s}\right\}\right) \\
&\leq \alpha\psi(b_d(u,v)).
\end{aligned}$$

So, by the above inequality and property of ψ, we get $b_d(u,v) = 0$. Therefore $u = v$, and the fixed point is unique. This completes the proof.

The following example illustrates Theorem 3.5.

Example 3.6. Let $X = [0,1]$ and $b_d(x,y) = (x+y)^2$ for all $x,y \in X$. It is clear that, b_d is a b-dislocated metric on X with parameter $s = 2$. Also, b_d is not a dislocated metric or a b-metric or a metric on X. Define the self-mapping $T : X \to X$ by $Tx = \frac{x}{5}$. For all $x,y \in [0,1]$, and the function $\psi(t)=2t$, we have

$$\begin{aligned} & \psi\left(sb_d\left(Tx,Ty\right)\right) \\ = \; & \psi\left(2\left(\frac{x}{5}+\frac{y}{5}\right)^2\right)=\psi\left(2\frac{(x+y)^2}{25}\right)=\frac{4}{25}(x+y)^2 \\ = \; & \frac{2}{25}2(x+y)^2=\frac{2}{25}2b_d\left(x,y\right)=\frac{2}{25}\psi\left(b_d\left(x,y\right)\right)\leq\alpha\psi\left(b_d\left(x,y\right)\right) \\ \leq \; & \alpha\psi\left(\max\left\{b_d\left(x,y\right),b_d\left(x,Tx\right),b_d\left(y,Ty\right),\frac{b_d\left(x,Ty\right)+b_d\left(y,Tx\right)}{4s}\right\}\right). \end{aligned}$$

All of the conditions of theorem are satisfied, and $x=0$ is a unique fixed point of T.

If $s=1$ in Theorem 3.5, we deduce the following theorem in the setting of dislocated metric spaces.

Corollary 3.7. *Let (X,d_l) be a complete b-dislocated metric space, and $T:X\to X$ is a self mapping that satisfies:*

$$\psi(d_l\left(Tx,Ty\right))\leq\alpha\psi\left(\max\left\{d_l\left(x,y\right),d_l\left(x,Tx\right),d_l\left(y,Ty\right),\frac{d_l\left(x,Ty\right)+d_l\left(y,Tx\right)}{4}\right\}\right)$$

for all $x,y\in X$, where $0\leq\alpha<1$, $\psi\in\Psi$. Then T has a unique fixed point in X.

The following example shows that Theorem 3.5 is a proper generalization.

Example 3.8. Let $X=[0,1]$ and $d_l:X^2\to\mathbb{R}^+$ by $d_l(x,y)=(x+y)$ for all $x,y\in X$. It is clear that d_l is a dislocated metric on X and (X,d_l) is complete. Also d_l is not a metric on X. Define a self-mapping $T:X\to X$ by

$$Tx=\begin{cases} \frac{x}{8}, & 0\leq x<1, \\ \frac{1}{16}, & x=1. \end{cases}$$

We have the following cases:

Case 1. If $x=y=0$, then

$$\psi(d_l(Tx,Ty))=\psi(d_l(0,0))=\psi(0)\leq\alpha\psi(0)=\alpha\psi(d_l(0,0)).$$

Case 2. If $1>x=y>0$, then

$$\psi(d_l(Tx,Ty))=\psi\left(d_l\left(\frac{x}{8},\frac{x}{8}\right)\right)=\psi\left(\frac{2x}{8}\right)=\psi\left(\frac{1}{8}2x\right)=\psi\left(\frac{1}{8}d_l\left(x,y\right)\right)\leq\alpha\psi(d_l(x,y)).$$

Case 3. If $0 < x < y < 1$, then

$$\begin{aligned}\psi(d_l(Tx,T1)) &= \psi\left(d_l\left(\frac{x}{8},\frac{1}{16}\right)\right) = \psi\left(\left(\frac{x}{8}+\frac{1}{16}\right)\right) < \psi\left(\left(\frac{x}{8}+\frac{1}{8}\right)\right) \\ &= \psi\left(\frac{1}{8}d_l(x,1)\right) \leq \alpha\psi(d_l(x,1)).\end{aligned}$$

Case 4. If $1 > x > y > 0$, then

$$\begin{aligned}\psi(d_l(Tx,Ty)) &= \psi\left(d_l\left(\frac{x}{8},\frac{y}{8}\right)\right) = \psi\left(\left(\frac{x}{8}+\frac{y}{8}\right)\right) = \psi\left(\frac{1}{8}(x+y)\right) < \psi\left(\frac{1}{8}d_l(x,y)\right) \\ &\leq \alpha\psi(d_l(x,y)).\end{aligned}$$

We see that α exists since in all cases have $\psi\left(\frac{1}{8}d_l(x,y)\right) \leq \psi(d_l(x,y))$. That means

$$\frac{\psi\left(\frac{1}{8}d_l(x,y)\right)}{\psi(d_l(x,y))} < 1.$$

Thus all of the conditions of Theorem 3.5 are satisfied and T has a unique fixed point in X. Therefore, if we see the special case (corollary) of above theorem as

$$\psi(d_l(Tx,Ty)) \leq \alpha\psi(d_l(x,y)).$$

It is noted that, for $x = 1$ and $y = \frac{99}{100}$ in the usual metric space (X,d) where $d(x,y) = |x-y|$ we have

$$\psi\left(d\left(T(1),T\left(\frac{99}{100}\right)\right)\right) = \psi\left(d\left(\frac{1}{16},\frac{99}{800}\right)\right) = \psi\left(\frac{49}{800}\right) = \psi\left(\frac{49}{400}\right)$$

$$\psi(d(x,y)) = \psi\left(d\left(1,\frac{99}{100}\right)\right) = \psi\left(\frac{1}{100}\right).$$

We can see that inequality $\psi\left(\frac{49}{800}\right) \leq \alpha\psi\left(\frac{1}{100}\right)$ holds for $\alpha \geq 1$ since function ψ is increasing and $\psi\left(\frac{1}{100}\right) < \psi\left(\frac{49}{800}\right)$. So the contractive condition is not true in the usual metric on X. Also, we can say the same in the setting of b-metric space (X,d), where $d(x,y) = |x-y|^2$.

Theorem 3.9. *Let (X,b_d) be a complete b-dislocated metric space with parameter $s \geq 1$ and $T : X \to X$ a self-mapping satisfying the following condition*

$$\psi(sb_d(Tx,Ty)) \leq \alpha\psi\left(\max\left\{b_d(x,y),\frac{b_d(x,Tx)+b_d(y,Ty)}{4s}\right\}\right)$$

for all $xy \in X$, where $0 \leq \alpha < 1$, $\psi \in \Psi$. Then T has a unique fixed point in X.

Proof. Note that

$$\max\left\{b_d(x,y), \frac{b_d(x,Tx)+b_d(y,Ty)}{4s}\right\}$$
$$\leq \max\left\{b_d(x,y), b_d(x,Tx), b_d(y,Ty)\frac{b_d(x,Ty)+b_d(y,Tx)}{4s}\right\}$$

holds for all $x,y \in X$. Using the monotonic property of function ψ, we find from Theorem 3.5 the desired conclusion immediately.

In the following we are giving some periodic point results. Obviously, if T is a map which has a fixed point u, then u is also a fixed point of T^n for every $n \in \mathbb{N}$, that is, $Fix(T) \subset Fix(T^n)$. However the converse need not be true. If a self-mapping $T : X \to X$ satisfies: $Fix(T) = Fix(T^n)$ for each $n \in \mathbb{N}$, then T is said to have property P.

Theorem 3.10. *Let (X, b_d) be a complete b-dislocated metric space with parameter $s \geq 1$ and a self-mapping $T : X \to X$ is a (ψ, s, α)-generalized contractive mapping. Then T satisfies the property P.*

Proof. From Theorem 3.5, we see that T has a unique fixed point in X. Letting $u \in Fix(T^n)$, we find from condition (3.1) that

$$\begin{aligned}
&\psi(b_d(u,Tu)) \\
&< \psi(sb_d(u,Tu)) \\
&= \psi\left(sb_d\left(T^m u, T^{m+1}u\right)\right) \\
&= \psi\left(sb_d\left(TT^{m-1}u, TT^m u\right)\right) \\
&\leq \alpha\psi\left(\max\left\{\begin{array}{l} b_d\left(T^{m-1}u,T^m u\right), b_d\left(T^{m-1}u,T^m u\right), b_d\left(T^m u,T^{m+1}u\right), \\ \frac{b_d\left(T^{m-1}u,T^{m+1}u\right)+b_d\left(T^m u,T^m u\right)}{4s} \end{array}\right\}\right) \qquad (3.10)\\
&= \alpha\psi\left(\max\left\{\begin{array}{l} b_d\left(T^{m-1}u,u\right), b_d\left(T^{m-1}u,u\right), b_d(u,Tu), \\ \frac{b_d\left(T^{m-1}u,Tu\right)+b_d(u,u)}{4s} \end{array}\right\}\right) \\
&\leq \alpha\psi\left(\max\left\{\begin{array}{l} b_d\left(T^{m-1}u,u\right), b_d\left(T^{m-1}u,u\right), b_d(u,Tu), \\ \frac{sb_d\left(T^{m-1}u,u\right)+sb_d(u,Tu)+2sb_d\left(T^{m-1}u,u\right)}{4s} \end{array}\right\}\right).
\end{aligned}$$

If $b_d(T^{m-1}u,u) \leq b_d(u,Tu)$, then we find from inequality (3.10) that

$$\psi(b_d(u,Tu)) \leq \alpha\psi(b_d(u,Tu)),$$

which implies $\psi(b_d(u,Tu))=0$. So $b_d(u,Tu)=0$. Hence u is a fixed point of T. If $b_d(u,Tu) < b_d(T^{m-1}u,u)$, then

$$\psi(b_d(u,Tu)) \leq \alpha\psi(b_d(T^{m-1}u,u)). \tag{3.11}$$

Applying condition (3.1), we have

$$\begin{aligned}
&\psi\left(b_d\left(u,T^{m-1}u\right)\right)\\
&\leq \psi\left(sb_d\left(T^m u,T^{m-1}u\right)\right)=\psi\left(sb_d\left(TT^{m-1}u,TT^{m-2}u\right)\right)\\
&\leq \alpha\psi\left(\max\left\{\begin{array}{l} b_d\left(T^{m-1}u,T^{m-2}u\right),b_d\left(T^{m-1}u,T^m u\right),\\ b_d\left(T^{m-2}u,T^{m-1}u\right),\frac{b_d\left(T^{m-1}u,T^{m-1}u\right)+b_d\left(T^{m-2}u,T^m u\right)}{4s}\end{array}\right\}\right)\\
&= \alpha\psi\left(\max\left\{\begin{array}{l} b_d\left(T^{m-1}u,T^{m-2}u\right),b_d\left(T^{m-1}u,u\right),b_d\left(T^{m-2}u,T^{m-1}u\right),\\ \frac{b_d\left(T^{m-1}u,T^{m-1}u\right)+b_d\left(T^{m-2}u,u\right)}{4s}\end{array}\right\}\right)\\
&\leq \alpha\psi\left(\max\left\{\begin{array}{l} b_d\left(T^{m-1}u,T^{m-2}u\right),b_d\left(T^{m-1}u,u\right),b_d\left(T^{m-2}u,T^{m-1}u\right),\\ \frac{2sb_d\left(T^{m-1}u,u\right)+sb_d\left(T^{m-2}u,T^{m-1}u\right)+sb_d\left(T^{m-1}u,u\right)}{4s}\end{array}\right\}\right).
\end{aligned} \tag{3.12}$$

If $b_d(T^{m-2}u,T^{m-1}u) < b_d(T^{m-1}u,u)$, then we find from (3.12) that

$$\psi(b_d(T^{m-1}u,u)) < \alpha\psi(b_d(T^{m-1}u,u)).$$

Using (3.10), we get $b_d(u,Tu)=0$. Hence u is a fixed point of T. If

$$b_d(T^{m-1}u,u) < b_d(T^{m-2}u,T^{m-1}u),$$

we find from (3.11) that

$$\psi(b_d(T^{m-1}u,u)) < \alpha\psi(b_d(T^{m-2}u,T^{m-1}u)). \tag{3.13}$$

By virtue of (3.11) and (3.13), we get

$$\psi(b_d(u,Tu)) < \alpha\psi\left(b_d\left(T^{m-1}u,u\right)\right) < \alpha^2\psi\left(b_d\left(T^{m-2}u,T^{m-1}u\right)\right) < \cdots < \alpha^n\psi\left(b_d(u,Tu)\right).$$

As a result, we have $b_d(u,Tu)=0$. Hence $Tu=u$ and u is a fixed point of T.

If $s=1$ in Theorem 3.10, we deduce the following corollary in the setting of dislocated metric spaces.

Corollary 3.11. *Let (X, d_l) be a complete b-dislocated metric space and a self-mapping $T : X \to X$ that satisfies the following condition:*

$$\psi\left(d_l\left(Tx, Ty\right)\right) \leq \alpha\psi\left(\max\left\{\begin{array}{l} d_l\left(x, y\right), d_l\left(x, Tx\right), d_l\left(y, Ty\right), \\ \frac{d_l\left(x, Ty\right) + d_l\left(y, Tx\right)}{4} \end{array}\right\}\right)$$

for all $x, y \in X$, where $0 \leq \alpha < 1$, $\psi \in \Psi$. Then T satisfies property P.

Remark 3.12. Since every b-metric space is a b-dislocated metric space with the same parameter, our results can be seen as a generalization of several corresponding results in metric and b-metric spaces.

REFERENCES

[1] I.A. Bakhtin, The contraction mapping principle in quasimetric spaces, Func. An. Gos. Ped. Inst. Unianowsk, 30 (1989), 26-37.

[2] S. Czerwik, Contraction mappings in b-metric spaces, Acta Math. Inform. Univ. Ostrav. 1 (1993), 5-11.

[3] N. Hussain, J. R. Roshan, V. Parvaneh, M. Abbas, Common fixed point results for weak contractive mappings in ordered b-dislocated metric spaces with applications, J. Inequal. Appl. 2013 (2013), Article ID 486.

[4] M.A. Alghmandi, N. Hussain, P. Salimi, Fixed point and coupled fixed point theorems on b-metric-like spaces, J. Iequal. Appl. 2013 (2013), Article ID 402.

[5] R. George, K.P. Reshma, A. Padmavati, Fixed point theorems for cyclic contractions in b-metric spaces, J. Nonlinear Funct. Anal. 2015 (2015), Article ID 5.

[6] N. Hussain, V. Parvaneh, J.R. Roshan, Z. Kadelburg, Fixed points of cyclic weakly (ψ, φ, L, A, B)-contractive mappings in ordered b-metric spaces with applications, Fixed Point Theory Appl. 2013 (2013), Article ID 256.

[7] S.K. Malhotra, J.B. Sharma, Satish Shukla, Some fixed point theorems for G-contractions in cone b-metric spaces over a Banach algebra, J. Nonlinear Funct. Anal. 2017 (2017), Article ID 9.

[8] Z. Mustafa, J.R. Roshan, V. Parvaneh, Z. Kadelburg, Some common fixed point results in ordered partial b-metric spaces, J. Inequal. Appl. 2013 (2013), Article ID 562.

[9] J.R. Roshan, N. Shobkolaei, S. Sedghi, M. Abbas, Common fixed point of four maps in b-metric spaces, Hacet. J. Math. Stat. 43 (2014), 613-624.

[10] G. S. Saluja, Some fixed point theorems for generalized contractions involving rational expressions in b-metric spaces, Commun. Optim. Theory 2016 (2016), Article ID 17.

[11] W. Sintunavarat, S. Plubtieng, P. Katchang, Fixed point result and applications on b-metric space endowed with an arbitrary binary relation, Fixed Point Theory Appl. 2013 (2013), Article ID 296.

[12] P. Hitzler, A.K. Seda, Dislocated topologies, J. Electr. Eng. 51 (2000), 3-7.

[13] M.A. Ahamed, F.M. Zeyada, G.F. Hasan, Fixed point theorems in generalized types of dislocated metric spaces and its applications, Thai J. Math. 11 (2013), 67-73.

[14] M. Arshad, A. Shoaib, P. Vetro, Common fixed points of a pair of Hardy Rogers type mappings on a closed ball in ordered dislocated metric spaces, J. Funct. Space Appl. 2013 (2013), Article ID 638181.

[15] M. Arshad, A. Shoaib, I. Beg, Fixed point of a pair of contractive dominated mappings on a closed ball in an ordered dislocated metric space, Fixed Point Theory Appl. 2013 (2013), Article ID 115.

[16] E. Karapinar, P. Salimi, Dislocated metric space to metric spaces with some fixed point theorems, Fixed Point Theory Appl. 2013 (2013), Articl ID 22.

[17] Y. Ren, J. Li, Y. Yu, Common fixed point theorems for nonlinear contractive mappings in dislocated metric spaces, Abst. Appl. Anal. 2013 (2013), Article ID 483059.

[18] K. Zoto, E. Hoxha, Fixed point theorems in dislocated and dislocated quasi-metric space, J. Adv. Stud. Topology 3 (2012), 119-124.

[19] M. A. Kutbi, M. Arshad, J. Ahmad, A. Azam, Generalized common fixed point results with applications, Abst. Appl. Anal. 2014 (2014), Article ID 363925.

[20] E. Karapinar, P. Salimi, Dislocated metric space to metric spaces with some fixed point theorems, Fixed Point Theory Appl. 2013 (2013), Article ID 222.

[21] M.S. Khan, M. Swaleh, S. Sessa, Fixed points theorems by altering distances between the points, Bull. Austral. Math. Soc. 30 (1984), 1-9.