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FIXED POINTS FOR (α, ψ)-KHAN-RATIONAL GERAGHTY CONTRACTIVE MAPPINGS

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Abstract. In this paper, we study some results on the existence of fixed points for a class of (α, ψ)-Khan-rational Geraghty contractive mappings. Our main results extend and unify the corresponding results in Fisher [3] and Shahi *et al.* [5].

Keywords. Contractive mapping; Fixed point; Metric space; Khan theorem.

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1. Introduction and preliminaries

In the mid-sixties ten, fixed points results dealing with general contractive conditions with rational expressions were appeared. On of the well-known works in this direction were established by Khan [4]. After that, Fisher [3] gave a revised version of Khan result as follows.

Theorem 1.1. *Let (X, d) be a complete metric space and let $T : X \to X$ satisfy*

$$d(Tx,Ty) \leq \begin{cases} k\frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{d(x,Ty)+d(Tx,y)}, & if \;\; d(x,Ty)+d(Tx,y) \neq 0, \\ 0, & if \;\; d(x,Ty)+d(Tx,y) = 0, \end{cases}$$

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where $k \in [0,1)$ and $x,y \in X$. Then T has a unique fixed point $x^* \in X$. Moreover, for all $x \in X$, the sequence $\{T^n x\}$ converges to x^*.

Definition 1.2. Let X be a nonempty set, $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ be two mappings. We say that T is α-admissible if for all $x,y \in X$, $\alpha(x,y) \geq 1$ implies that $\alpha(Tx,Ty) \geq 1$.

Definition 1.3. Let X be a nonempty set and $\alpha : X \times X \to [0,\infty)$ be a mapping. We say that α is transitive if for all $x,y,z \in X$, $\alpha(x,y) \geq 1$ and $\alpha(y,z) \geq 1$ implies that $\alpha(x,z) \geq 1$.

Let Ψ be a family of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(Ψ_1) ψ is nondecreasing.

(Ψ_2) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the n-th iterate of ψ.

It can be easily verified that if $\psi \in \Psi$, then $\psi(t) < t$ for any $t > 0$.

Define $\Phi = \{\varphi \mid \varphi : [0,\infty) \to [0,\infty)\}$ such that φ is Lebesgue integrable and satisfies

$$\int_0^{\varepsilon} \varphi(t)dt > 0, \qquad \forall \varepsilon > 0.$$

Very recently, Shahi *et al.* [5] gave the integral version of (α,ψ)-contractive type mappings and proved some related fixed point theorems.

Definition 1.4. Let (X,d) be a metric space and $T : X \to X$ be a given mapping. We say that T is an (α,ψ)-contractive mapping of integral type if there exist two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that for each $x,y \in X$,

$$\alpha(x,y) \int_0^{d(Tx,Ty)} \varphi(t)d(t) \leq \psi\left(\int_0^{d(x,y)} \varphi(t)d(t)\right),$$

where $\varphi \in \Phi$.

Theorem 1.5. *Let (X,d) be a complete metric space and $\alpha : X \times X \to [0,\infty)$ be a transitive mapping. Suppose that $T : X \to X$ is an (α,ψ)-contractive mapping of integral type and satisfies the following conditions:*

(i) T is α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$;

(ii) T is continuous.

Then T has a fixed point, that is, there exists $z \in X$ such that $Tz = z$.

2. Main results

Definition 2.1. Let (X,d) be a metric space with constant $s \geq 1$ and $T : X \to X$ be a given mapping. We say that T is an (α,ψ)- rational Geraghty contractive mapping of integral type, if there are a function $\alpha : X \times X \to [0,\infty)$ and a continuous function $\psi \in \Psi$ such that for all distinct $x,y \in X$. If $\max\{d(x,Ty),d(Tx,y)\} \neq 0$, then

$$\alpha(x,y)\int_0^{d(Tx,Ty)} \varphi(t)d(t) \leq \psi\left(\int_0^{M_T(x,y)} \varphi(t)d(t)\right), \tag{2.1}$$

where, $\varphi \in \Phi$ and

$$M_T(x,y) = \max\left\{\ d(x,y), \frac{d(x,Tx)d(y,Ty)}{d(x,y)}, \frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{\max\{d(x,Ty),d(Tx,y)\}},\ \right\},$$

and if $\max\{d(x,Ty),d(Tx,y)\} = 0$, then $Tx = Ty$.

Theorem 2.2. *Let (X,d) be a complete b-metric space with constant $s \geq 1$ and $\alpha : X \times X \to [0,\infty)$ be a transitive mapping. Suppose that $T : X \to X$ be an (α,ψ)-rational Geraghty contractive mapping of integral type I and satisfies the following conditions:*

- (i) *T is α-admissible;*
- (ii) *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$;*
- (ii) *T is continuous.*

Then T has a fixed point $x^* \in X$.

Proof. Put $x_{n+1} = Tx_n = T^{n+1}x_0$ for all $n \in \mathbb{N}_0$. If there exists $n \in \mathbb{N}$ such that $x_n = x_{n-1}$, then x_{n-1} is a fixed point of T. This completes the proof. Therefore, we suppose that

$$d(x_n,x_{n-1}) > 0, \quad \forall n \in \mathbb{N}. \tag{2.2}$$

Due to the fact that T is α-admissible, we find that for all $n \in \mathbb{N}_0$

$$\alpha(x_n,x_{n+1}) \geq 1. \tag{2.3}$$

We shall divide the proof into two cases.

Case 1. Assume that

$$\max\{d(x_m,Tx_n),d(Tx_m,x_n)\} \neq 0, \quad \forall m \in \mathbb{N}, \forall n \in \mathbb{N}_0. \tag{2.4}$$

By applying inequality (2.1) with $x = x_{n-1}$ and $y = x_n$ and using (2.3), we deduce that

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)d(t) \leq \psi\left(\int_0^{M_T(x_{n-1},x_n)} \varphi(t)d(t)\right). \tag{2.5}$$

Since

$$M_T(x_{n-1},x_n) = \max\left\{ \begin{array}{c} d(x_{n-1},x_n), \dfrac{d(x_{n-1},Tx_{n-1})d(x_n,Tx_n)}{d(x_{n-1},x_n)}, \\ \dfrac{d(x_{n-1},Tx_{n-1})d(x_{n-1},Tx_n)+d(x_n,Tx_n)d(x_n,Tx_{n-1})}{\max\{d(x_{n-1},Tx_n),d(Tx_{n-1},x_n)\}}, \end{array} \right\} \tag{2.6}$$

$$\leq \max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\}.$$

If $d(x_n,x_{n+1}) > d(x_{n-1},x_n)$, then $M_T(x_{n-1},x_n) \leq d(x_n,x_{n+1})$. From (2.5), (2.6) and (Ψ_1), we get

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)d(t) \leq \psi\left(\int_0^{d(x_n,x_{n+1})} \varphi(t)d(t)\right) < \int_0^{d(x_n,x_{n+1})} \varphi(t)d(t),$$

which is a contradiction (from the property of ψ, we have $\psi(t) < t$ for any $t > 0$). Thus, we conclude that $d(x_n,x_{n+1}) \leq d(x_{n-1},x_n)$. By utilizing (2.6) and (Ψ_1), we derive from (2.5) that

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)d(t) \leq \psi^n\left(\int_0^{d(x_0,x_1)} \varphi(t)d(t)\right). \tag{2.7}$$

From (2.2) and (Ψ_2), we find that

$$\lim_{n\to\infty} \psi^n\left(\int_0^{d(x_0,x_1)} \varphi(t)d(t)\right) = 0. \tag{2.8}$$

which implies that

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{2.9}$$

Now, we claim that

$$\lim_{n,m\to\infty} d(x_n,x_m) = 0.$$

Arguing by contradiction, we assume that there exist $\varepsilon > 0$, the sequences $\{p(n)\}_{n=1}^{\infty}$ and $\{q(n)\}_{n=1}^{\infty}$ of natural numbers such that

$$p(n) > q(n) > n,\ \ d(x_{p(n)},x_{q(n)}) \geq \varepsilon,\ \ d(x_{p(n)-1},x_{q(n)}) < \varepsilon,\quad \forall n \in \mathbb{N}. \tag{2.10}$$

Let $u,v \in \mathbb{N}$ and $u < v$. In view of triangular inequality and using (2.9), we deduce that

$$\lim_{n\to\infty} d(x_{p(n)+u},x_{p(n)+v}) = 0. \tag{2.11}$$

From (2.10), we get

$$\varepsilon \leq d(x_{p(n)},x_{p(n)-1}) + d(x_{p(n)-1},Tx_{q(n)-1}),\quad \forall n \in \mathbb{N}. \tag{2.12}$$

It follows from (2.8) and (2.12) that

$$\liminf_{n\to\infty} d(x_{p(n)-1},Tx_{q(n)-1}) \geq \varepsilon.$$

So, there exists $N\in\mathbb{N}$ such that for all $n\geq N$,

$$\max\{d(x_{p(n)-1},Tx_{q(n)-1}),d(Tx_{p(n-1)},x_{q(n)-1})\}\geq d(x_{p(n)-1},Tx_{q(n)-1})\geq\frac{\varepsilon}{2}. \tag{2.13}$$

Since ψ is a continuous and nondecreasing, so by applying inequality (2.1) with $x=x_{p(n)}$ and $y=x_{q(n)}$ and using (2.3) and (2.10), we deduce that

$$\int_0^{\varepsilon}\varphi(t)d(t)\leq\psi\left(\limsup_{n\to\infty}\int_0^{M_T(x_{p(n)-1},x_{q(n)-1})}\varphi(t)d(t)\right). \tag{2.14}$$

On the other hand, for all $n\geq N_1$, we have

$$\begin{aligned}
&M_T(x_{p(n)-1},x_{q(n)-1})\\
&=\max\left\{\begin{array}{c} d(x_{p(n)-1},x_{q(n)-1}),\dfrac{d(x_{p(n)-1},Tx_{p(n)-1})d(x_{p(n)-1},Tx_{q(n)-1})}{d(x_{p(n)-1},Tx_{q(n)-1})}\\ +\dfrac{d(x_{q(n)-1},Tx_{q(n)-1})d(x_{q(n)-1},Tx_{p(n)-1})}{\max\{d(x_{p(n)-1},Tx_{q(n)-1}),d(Tx_{p(n)-1},x_{q(n)-1})\}}\end{array}\right\}\\
&\leq\max\left\{\begin{array}{c}\dfrac{\varepsilon+d(x_{q(n)},x_{q(n)-1})}{\varepsilon},\dfrac{2d(x_{p(n)-1},x_{p(n)})d(x_{q(n)-1},x_{q(n)})}{\varepsilon},\\ \dfrac{d(x_{p(n)-1},x_{p(n)})d(x_{p(n)-1},x_{q(n)})+d(x_{q(n)-1},x_{q(n)})d(x_{q(n)-1},x_{p(n)})}{\varepsilon}\end{array}\right\},
\end{aligned}$$

which implies from (2.11) that

$$\limsup_{n\to\infty}M_T(x_{p(n)-1},x_{q(n)-1})\leq\varepsilon.$$

Since, the function $s\to\int_0^s\varphi(t)d(t)$ and ψ are increasing and continuous, so we get

$$\psi\left(\limsup_{n\to\infty}\int_0^{M_T(x_{p(n)-1},x_{q(n)-1})}\varphi(t)d(t)\right)\leq\psi\left(\int_0^{\varepsilon}\varphi(t)d(t)\right) \tag{2.15}$$

Since $\psi(t)<t$, for all $t>0$, so from (2.14) and (2.15), we get a contradiction. This implies that $\lim_{n,m\to\infty}d(x_n,x_m)=0$. Hence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X,d). Due to the completeness of (X,d), there exists $x^*\in X$ such that $x_n\to x^*$ as $n\to\infty$. The continuity of T yields that $Tx_n\to Tx^*$ as $n\to\infty$, that is, $x_{n+1}\to Tx^*$ as $n\to\infty$. By the uniqueness of the limit, we obtain $x^*=Tx^*$. Therefore, x^* is a fixed point of T.

Case 2. Assume that there exist $m\in\mathbb{N}$ and $n\in\mathbb{N}_0$ such that

$$\max\{d(x_m,Tx_n),d(Tx_m,x_n)\}=0. \tag{2.16}$$

Since T is (α,ψ)-rational Geraghty contractive mapping of integral type, so we have $Tx_m=Tx_n$. It follows from (2.16) that, $x_n=Tx_m=Tx_n=x_m$. This completes the proof.

Theorem 2.3. *Let (X,d) be a complete metric space and let $T: X \to X$ be a self-mapping such that for all $x, y \in X$*

$$d(Tx,Ty) \leq \begin{cases} \lambda M_T(x,y), & if \ \max\{d(x,Ty), d(Tx,y)\} \neq 0, \\ 0, & if \ \max\{d(x,Ty), d(Tx,y)\} = 0, \end{cases}$$

where, $\lambda \in (0,1)$ and $N_T(x,y)$ are as in Definition 2.1. Then T has a unique fixed point $x^* \in X$.

Proof. It is suffice to take $\varphi(t) = 1$, for all $t \geq 0$ and $\alpha(x,y) = 1$, for all $x, y \in X$ in Theorem 2.2.

Remark 2.4. Obviously, Theorem 2.2 is a generalization of Theorem 2.1 of [5] and Theorem 2.3 is a generalization of the main result of [3].

REFERENCES

[1] A. Branciari, A fixed point theorem for mappings satisfying a general contractive condition of integral type, Int. J. Math. Math. Sci. 29 (2002), 531-536.

[2] M. Berzig, M. Rus, Fixed point theorems for α-contractive mappings of Meir-Keeler type and applications. arXiv:1303.5798.

[3] B. Fisher, On a theorem of Khan, Riv. Math. Univ. Parma. 4 (1978), 135-137.

[4] M. S. Khan, A fixed point theorem for metric spaces, Rend. Inst. Math. Univ. Trieste 8 (1976), 69-72.

[5] P. Shahi, J. Kaur, S.S. Bhatia, Fixed point theorems for α-ψ-contractive type mappings of integral type with applications, J. Nonlinear Convex. Anal. 16 (2015), 745-760.