Commun. Optim. Theory 2017 (2017), Article ID 20 https://doi.org/10.23952/cot.2017.20

MATHRES

Communications in Optimization Theory

Available online at http://cot.mathres.org

GENERAL NORM INEQUALITIES FOR BOUNDED LINEAR OPERATORS IN HILBER SPACES

MOHAMMED ALI*, MOHAMMED AL-DOLAT, DOAA OBEIDAT

Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid, Jordan

Abstract. Let $B(H)$ be the space of C^*-algebra of all bounded linear operators on a complex Hilbert space H. The norm of the sum of bounded linear operators on H has been attracted the attentions of many mathematicians for along time. In this work, we study the upper bound of the sum of operators belong in $B(H)$ under the usual operator norm given by $\|A\| = \sup_{\|x\|=1} < Ax, Ax >; x \in H$. Moreover, we establish and generalize inequalities for the operator norm of sums of bounded linear operators in Hilbert spaces.

Keywords. Norm inequality; Bounded linear operator; Hilbert space; C^*-algebra.

2010 Mathematics Subject Classification. 47A05, 47A12.

1. Introduction

Let H be a complex Hilbert space with inner product $\langle .,. \rangle$ and let $B(H)$ be the space of all bounded linear operators on H. For $A \in B(H)$, let A^* denote the adjoint operator of A, and $\|A\|$ denote the usual operator norm given by

$$\|A\| = \sup_{\|x\|=1} \|Ax\|,$$

*Corresponding author.

E-mail addresses: myali@just.edu.jo (M. Ali), mmaldolat@just.edu.jo (M. Al-Dolat), dhobeida13@sci.just.edu.jo (D. Obeidat).

Received September 19, 2016; Accepted March 22, 2017.

where $\|Ax\| = \langle Ax, Ax\rangle^{\frac{1}{2}}$.

Several inequalities that provide alternative upper bounds for sum of bounded linear operators have been established by many authors (we refer the readers to [1], [2], [4], [5], [6], as well as [8], and the references therein).

Let us present some known basic lemmas that we need to prove our results. The proofs of our sequels mainly depend on [1] as well as the following lemmas.

Lemma 1.1. [9] *Let $A \in B(H)$ be a positive operator, and let $x \in H$ be a unit vector. Then*

$$\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle \quad \text{for } r \geq 1,$$

and

$$\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r \quad \text{for } r \geq 1.$$

The next following lemma is a generalized from of the mixed Schwarz inequality, which has been proved by Kittaneh in [9].

Lemma 1.2. [9] *Let $A \in B(H)$, and let f, g be non-negative continuous functions on $[0,\infty)$ satisfying $f(t)g(t) = t$ for all $t \in [0,\infty)$. Then, for all $x, y \in H$,*

$$|\langle Ax, y\rangle| \leq \|f(|A|)x\|\|g(|A^*|)y\|.$$

The next lemma is a simple consequence of the classical Jensen's inequality concerning the convexity or the concavity of certain power functions. It is a special case of Schlilch's inequality for the weighted means of non-negative real numbers.

Lemma 1.3. [7]. *For $a, b \geq 0$, $0 < \alpha < 1$, and $r \neq 0$. Let $M_r(a,b,\alpha) = (\alpha a^r + (1-\alpha)b^r)^{\frac{1}{r}}$ and $M_0(a,b,\alpha) = a^{\alpha}b^{1-\alpha}$. Then*

$$M_r(a,b,\alpha) \leq M_s(a,b,\alpha) \quad \text{for } r \leq s.$$

The purpose of this work is to establish some new norm inequalities of bounded linear operators in Hilbert spaces. Also, we generalize and refine some known results.

2. Main results

In this section, we prove some new inequalities for the operator norm of sums of bounded linear operators in Hilbert spaces. Our results are described in the following theorems.

Theorem 2.1. *Let $A_i, B_i, X_i \in B(H) (i = 1, 2, ...n)$ and $m \in \mathbb{N}$. Assume that $\alpha \in (t, t+1)$ with $t \geq 0$. Let f, g be given functions as in Lemma 1.2. Then for any $r \geq 1$,*

$$\left\|\sum_{i=1}^{n} X_i A_i^m B_i\right\|^r \leq \frac{n^{r-1}}{m}\sum_{j=1}^{m}\left(\left\|\sum_{i=1}^{n}\frac{\alpha}{2(t+1)}M_{ij}^{\frac{r(t+1)}{\alpha}}\right\|+\left\|\sum_{i=1}^{n}(1-\frac{\alpha}{2(t+1)})N_{ij}^{\frac{r}{2-\frac{\alpha}{t+1}}}\right\|\right), \quad (2.1)$$

where

$$M_{ij} = B_i^* f^2\left(\left|A_i^{m-j}\right|\right)B_i \text{ and } N_{ij} = A_i^j X_i g^2\left(\left|A_i^{*^{m-j}}\right|\right)A_i^{*^j} X_i^*.$$

Proof. Let $x, y \in H$ be any unit vectors. Then it is trivial to obtain that

$$\begin{aligned}\left|\left\langle\sum_{i=1}^{n} X_i A_i^m B_i x, y\right\rangle\right|^r &\leq \frac{1}{m}\sum_{j=1}^{m}\left(\sum_{i=1}^{n}\left|\langle X_i A_i^m B_i x, y\rangle\right|\right)^r \\ &= \frac{1}{m}\sum_{j=1}^{m}\left(\sum_{i=1}^{n}\left|\left\langle A_i^{m-j}B_i x, A_i^{*^j}X_i^* y\right\rangle\right|\right)^r.\end{aligned}$$

Thus, by Lemma 1.1, Lemma 1.2, Lemma 1.3, and [Lemma 2.4, [1]], we deduce that

$$\begin{aligned}&\left|\left\langle\sum_{i=1}^{n} X_i A_i^m B_i x, y\right\rangle\right|^r \\ \leq\ & \frac{n^{r-1}}{m}\sum_{j=1}^{m}\left(\sum_{i=1}^{n}\left\|f(\left|A_i^{m-j}\right|)B_i x\right\|^r\left\|g(\left|A_i^{*^{m-j}}\right|)A_i^{*^j}X_i^* y\right\|^r\right) \\ =\ & \frac{n^{r-1}}{m}\sum_{j=1}^{m}\left(\sum_{i=1}^{n}\left\langle f\left(\left|A_i^{m-j}\right|\right)B_i x, f\left(\left|A_i^{m-j}\right|\right)B_i x\right\rangle^{\frac{r}{2}}\right. \\ \times\ & \left.\left\langle g\left(\left|A_i^{*^{m-j}}\right|\right)A_i^{*^j}X_i^* y, g\left(\left|A_i^{*^{m-j}}\right|\right)A_i^{*^j}X_i^* y\right\rangle^{\frac{r}{2}}\right) \\ \leq\ & \frac{n^{r-1}}{m}\sum_{j=1}^{m}\left(\sum_{i=1}^{n}\left\langle M_{ij}x, x\right\rangle^{\frac{r}{2}}\left\langle N_{ij}y, y\right\rangle^{\frac{r}{2}}\right) \\ \leq\ & \frac{n^{r-1}}{m}\sum_{j=1}^{m}\left(\sum_{i=1}^{n}\left\langle (M_{ij})^r x, x\right\rangle^{\frac{1}{2}}\left\langle (N_{ij})^r y, y\right\rangle^{\frac{1}{2}}\right) \\ \leq\ & \frac{n^{r-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left\langle (M_{ij})^{r\frac{t+1}{\alpha}}x, x\right\rangle^{\frac{\alpha}{2(t+1)}}\times\left\langle (N_{ij})^{\frac{r}{2-\frac{\alpha}{t+1}}}y, y\right\rangle^{1-\frac{\alpha}{2(t+1)}}\right) \\ \leq\ & \frac{n^{r-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\frac{\alpha}{2(t+1)}\left\langle (M_{ij})^{r\frac{t+1}{\alpha}}x, x\right\rangle+(1-\frac{\alpha}{2(t+1)})\left\langle (N_{ij})^{\frac{r}{2-\frac{\alpha}{t+1}}}y, y\right\rangle\right).\end{aligned}$$

We finish the proof of this theorem by taking the supremum over all unit vectors $x, y \in H$.

We point out that inequality 2.1 includes several norm inequalities as special cases. Samples of inequalities are demonstrated in what follows; by taking $t = 0$ and letting α goes to 1, then our result approaches to [Theorem 3.1, [1]]. Furthermore, for $f(x) = x^{\alpha}$, $g(x) = x^{1-\alpha}$ with $t = 0$; take $X_i = B_i = I$; and replace $(m - j)$ by j plus A^* by A, we obtain [Corollary 3.3, [1]].

Following the same arguments used in the proof of Theorem 2.1, we achieve the following theorem.

Theorem 2.2. *Let $A_i, B_i, X_i \in B(H) (i = 1, 2, ...n), m \in \mathbb{N}$, and let f and g be given as in Lemma 1.2. Assume that $\sum_{i=1}^{n} w_i = 1$ with $w_i > 0$. Then the inequality*

$$\left\|\sum_{i=1}^{n} w_i A_i X_i^m B_i\right\|^r \leq \frac{1}{2m} \sum_{j=1}^{m} \left(\left\|\sum_{i=1}^{n} w_i M_{ij}^r\right\| + \left\|\sum_{i=1}^{n} w_i N_{ij}^r\right\| \right) \tag{2.2}$$

holds for all $r \geq 1$, where $M_{ij} = B_i^* f^2\left(\left|X_i^j\right|\right) B_i$ and $N_{ij} = A_i X_i^{m-j} g^2\left(\left|X_i^{*j}\right|\right) X_i^{*^{m-j}} A_i^*$.

Proof. For any unit vectors $x, y \in H$, by using Lemma 1.2, we have

$$\begin{aligned}
\left|\sum_{i=1}^{n} \langle w_i A_i X_i^m B_i x, y\rangle\right|^r &\leq \frac{1}{m} \sum_{j=1}^{m} \left(\sum_{i=1}^{n} w_i \left|\langle A_i X_i^m B_i x, y\rangle\right| \right)^r \\
&= \frac{1}{m} \sum_{j=1}^{m} \left(\sum_{i=1}^{n} w_i \left|\left\langle X_i^j B_i x, X_i^{*^{m-j}} A_i^* y\right\rangle\right| \right)^r \\
&\leq \frac{1}{m} \sum_{j=1}^{m} \left(\sum_{i=1}^{n} w_i \left\langle M_{ij} x, x\right\rangle^{\frac{1}{2}} \left\langle N_{ij} y, y\right\rangle^{\frac{1}{2}} \right)^r.
\end{aligned}$$

Hence, by Lemma 1.1 as well as Lemma 1.3, and thanks to [3], we get that

$$\begin{aligned}
\left|\sum_{i=1}^{n} \langle w_i A_i X_i^m B_i x, y\rangle\right|^r &\leq \frac{1}{m} \sum_{j=1}^{m} \sum_{i=1}^{n} w_i \left\langle M_{ij} x, x\right\rangle^{\frac{r}{2}} \left\langle N_{ij} y, y\right\rangle^{\frac{r}{2}} \\
&\leq \frac{1}{m} \sum_{j=1}^{m} \sum_{i=1}^{n} w_i \left\langle M_{ij}^r x, x\right\rangle^{\frac{1}{2}} \left\langle N_{ij}^r y, y\right\rangle^{\frac{1}{2}} \\
&\leq \frac{1}{2m} \sum_{j=1}^{m} \sum_{i=1}^{n} w_i \left(\left\langle M_{ij}^r x, x\right\rangle + \left\langle N_{ij}^r y, y\right\rangle \right).
\end{aligned}$$

By taking the supremum over all unit vectors $x, y \in H$, we complete the proof.

For all $i = 1,2,\cdots,n$, take $A_i = B_i = I$ plus $w_i = \frac{1}{n}$, and also take $f(t) = t^{\alpha}$, $g(t) = t^{1-\alpha}$, then we reach [Corollary 3.3, [1]]. However, when we take $f(t) = t$, $g(t) = 1$ and $r = 1$, we get exactly [Theorem 3.4, [1]]

Using [Theorem 2, [10]] and applying the same manner used in proving the above theorems, we derive the following result.

Theorem 2.3. *Suppose that $A_i, B_i, X_i \in B(H)(i = 1,2,\dots,n)$. Then for any $p > 1$ and $\lambda \geq \max\left\{(n-1)^{p-1},(p-1)^{p-1}\right\}$, then the inequality*

$$\left\|\sum_{i=1}^{n} A_i X_i^m B_i\right\|^p \leq \frac{1}{m}\sum_{j=1}^{m}\left(\lambda \sum_{i=1}^{n}\left(\frac{\left\|M_{ij}^{pr}\right\| + \left\|N_{ij}^{pr}\right\|}{2}\right)^{\frac{1}{r}} + \left(\frac{n^p - n\lambda}{2^p}\right)\left(\prod_{i=1}^{n}\left(\|M_{ij}\| + \|N_{ij}\|\right)\right)^{p/n}\right)$$

holds for any $r \geq 1$, where $M_{ij} = f^2\left(\left|X_i^j B_i\right|\right)$, $N_{ij} = A_i X_i^{m-j} g^2\left(\left|\left(X_i^j B_j\right)^*\right|\right) X_i^{*\,m-j} A_i^*$.

Proof. Following the same above procedure, it is easy to get that for any non zero vectors $x, y \in H$,

$$\begin{aligned}\left|\left\langle\sum_{i=1}^{n} A_i X_i^m B_i x, y\right\rangle\right|^p &\leq \frac{1}{m}\sum_{j=1}^{m}\left(\sum_{i=1}^{n}\left|\left\langle X_i^j B_i x, X_i^{*\,m-j} A_i^* y\right\rangle\right|\right)^p \\ &\leq \frac{1}{m}\sum_{j=1}^{m}\left(\sum_{i=1}^{n}\left\langle M_{ij}x, x\right\rangle^{\frac{1}{2}}\left\langle N_{ij}y, y\right\rangle^{\frac{1}{2}}\right)^p.\end{aligned}$$

Take $\lambda \geq \max\left\{(n-1)^{p-1},(p-1)^{p-1}\right\}$, then by [Theorem 2, [10]], we obtain that

$$\begin{aligned}\left|\left\langle\sum_{i=1}^{n} A_i X_i^m B_i x, y\right\rangle\right|^p &\leq \frac{1}{m}\sum_{j=1}^{m}\left(\lambda\sum_{i=1}^{n}\left(\left\langle M_{ij}x, x\right\rangle^{\frac{p}{2}}\left\langle N_{ij}y, y\right\rangle^{\frac{p}{2}}\right.\right. \\ &\qquad \left.\left. + (n^p - n\lambda)\left(\prod_{i=1}^{n}\left\langle M_{ij}x, x\right\rangle^{\frac{1}{2}}\left\langle N_{ij}y, y\right\rangle^{\frac{1}{2}}\right)^{\frac{p}{n}}\right)\right. \\ &\leq \frac{1}{m}\sum_{j=1}^{m}\left(\lambda\sum_{i=1}^{n}\left(\left\langle M_{ij}^p x, x\right\rangle^{\frac{1}{2}}\left\langle N_{ij}^p y, y\right\rangle^{\frac{1}{2}}\right.\right. \\ &\qquad \left.\left. + (n^p - n\lambda)\left(\prod_{i=1}^{n}\left\langle M_{ij}x, x\right\rangle^{\frac{1}{2}}\left\langle N_{ij}y, y\right\rangle^{\frac{1}{2}}\right)^{\frac{p}{n}}\right).\right.\end{aligned}$$

Hence,

$$\begin{aligned}\left|\left\langle \sum_{i=1}^{n} A_i X_i^m B_i x, y\right\rangle\right|^p &\leq \frac{1}{m}\sum_{j=1}^{m}\left(\lambda \sum_{i=1}^{n}\left(\left(\frac{\left\langle M_{ij}^{pr}x,x\right\rangle + \left\langle N_{ij}^{pr}y,y\right\rangle}{2}\right)^{\frac{1}{r}}\right.\right.\\ &\quad \left.\left. + \frac{(n^p - n\lambda)}{2^p}\left(\prod_{i=1}^{n}\left\langle M_{ij}x,x\right\rangle + \left\langle N_{ij}y,y\right\rangle\right)^{\frac{p}{n}}\right)\right)\\ &\leq \frac{1}{m}\sum_{j=1}^{m}\left(\lambda \sum_{i=1}^{n}\left(\left(\frac{\left\langle M_{ij}^{pr}x,x\right\rangle + \left\langle N_{ij}^{pr}y,y\right\rangle}{2}\right)^{\frac{1}{r}}\right.\right. .\end{aligned}$$

Now the result follows by taking supremum over all unit vectors x, y in H.

REFERENCES

[1] M. Al-Dolat, K. Al-Zoubi, Numarical radius inequalities for Hilbert space operators, J. Math. Inequal. 10 (2016), 1041-1049.

[2] M. Al-Dolat, M. Ali, Norm inequalities for sequences of bounded linear operators in Hilbert spaces, Int. J. Pure App. Math. 22 (2007), 61-75.

[3] P. Bullen, D. Mitrinovic, Vasic, Means and Their Inequalities, D. Reidel, 1988.

[4] S. Dragomir, Norm and numerical radius inequalities for sums of bounded linear operators in Hilbert spaces, Facta Univ. Ser. Math. Inform, 22 (2007), 61-75.

[5] S. Dragomir, Norm inequalities for sequences of operators related to the Schwarz inequality, J. Inequal. Pure Appl. Math. 7 (2006) Article ID 97.

[6] S. Dragomir, Some Schwarz type inequalities for sequences of operators in Hilbert spaces, Bull. Aust. Math. Soc. 73 (2006), 17-26.

[7] G. Hardy, J. Littlewood, G. Pólya, Inequalities, 2nd ed., Cambrridge University Press, Cambridge, 1988.

[8] F. Kittaneh, Norm inequalities for sums and differences of positive operators, Linear Algebra Appl. 383 (2004), 85-91.

[9] F. Kittaneh, Notes on some inequalities for Hilbert space operators, Pub1. Res. Inst. Math. Sci. 24 (1988), 283-293.

[10] S. Wu, Generalization and sharpness of thr power means inequalities and their applications, J. Math. Anal. Appl. 312 (2005), 637-652.