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# IMPLICIT ALGORITHMS FOR A DEMICONTINUOUS SEMIGROUP OF PSEUDOCONTRACTIONS

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**Abstract.** In this paper, we study a demicontinuous semigroup of pseudocontractions based on an implicit algorithm. The results obtained in this paper improve and extend many corresponding results announced recently.

**Keywords.** Banach space; Convergence, Implicit algorithm; Semigroup.

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## 1. Introduction

Fixed point problems, which find a lot of applications in many real world applications, cover variational inequality problem, saddle point problems and complementary problems as special cases; see [1-5] and the references therein. During last four decades, many existence of fixed points of nonlinear operators has been investigated. From the viewpoint of real world applications, we need to find their fixed points via explicit and implicit iterative algorithm. Implicit iterative algorithms, which have been studied extensively, are powerful to treat pseudocontractive operators; see [6-9] and the references therein.

In this paper, we study an implicit hybrid algorithm for fixed points of pseudocontractive operators. Strong convergence theorems are established in the framework of real Hilbert spaces.

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## 2. Preliminaries

Let $\mathscr{H}$ be a real Hibert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$ and let $T: C \to C$ be a nonlinear mapping, where $C$ is a closed convex subset of $\mathscr{H}$. The set of fixed points of $T$ is denoted by $\mathscr{F}(T)$; that is, $\mathscr{F}(T) := \{x \in C : Tx = x\}$. We remark that $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively. Recall that a mapping $T$ is said to be nonexpansive iff

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall\, x,\ y \in C.$$

$T$ is said to be strictly pseudocontractive iff there exists a constant $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall\, x,\ y \in C.$$

$T$ is said to be pseudocontractive iff

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2, \quad \forall\, x,\ y \in C; \tag{2.1}$$

$T$ is pseudocontractive iff $I - T$ is monotone. $T$ is said to be strongly pseudocontractive if there exists a constant $\alpha \in (0,1)$ such that

$$\langle Tx - Ty, x - y \rangle \leq \alpha\|x - y\|^2, \quad \forall\, x,\ y \in C;$$

$T$ is said to be $\alpha-$dissipative with $\alpha \in \mathbb{R}$ iff

$$\langle Tx - Ty, x - y \rangle \leq \alpha\|x - y\|^2, \quad \forall\, x,\ y \in C.$$

We remark here that (2.1) is equivalent to

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall\, x,\ y \in C.$$

The class of strict pseudocontractive mappings falls into the one between classes of nonexpansive mappings and pseudocontractive mappings; and the class of strongly pseudocontractive mappings is independent of the class of strict pseudocontractive ones. The class of strongly pseudocontractive mapping is a subset of the ones of $\alpha-$dissipative mapping with $\alpha \in \mathbb{R}$.

In the following, we give some examples for strict pseudocontractive mapping, strongly pseudocontractive mapping and pseudocontractive mapping, which can be obtained in Zhou [10].

**Example 2.1.** *Take* $C=\mathbb{R}^1$ *and define* $T:C\to C$ *by*

$$T(x)=\begin{cases}1, & \text{if } x\in(-\infty,-1),\\ \sqrt{1-(1+x)^2}, & \text{if } x\in[-1,0),\\ -\sqrt{1-(x-1)^2}, & \text{if } x\in[0,1],\\ -1, & \text{if } x\in(1,+\infty).\end{cases}$$

*Then* $T$ *is a strongly pseudocontractive mapping but not a strict pseudocontractive mapping.*

**Example 2.2.** *Take* $C=(0,\infty)$ *and define* $T:C\to C$ *by*

$$Tx=\frac{x^2}{1+x}$$

*Then* $T$ *is a strict pseudocontractive mapping but not a strongly pseudocontractive one.*

**Example 2.3.** *Take* $\mathscr{H}=\mathbb{R}^2$ *and* $B=\{x\in\mathbb{R}^2:\|x\|\le 1\}$, $B_2=\{x\in B:\|x\|\le\frac{1}{2}\}$, $B_2=\{x\in B:\frac{1}{2}\le\|x\|\le 1\}$. *If* $x\in(a,b)\in\mathscr{H}$, *we define* $x^\perp$ *to be* $(b,-a)\in\mathscr{H}$. *Define* $T:B\to B$ *by*

$$T(x)=\begin{cases}x+x^\perp, & \text{if } x\in B_1,\\ \frac{x}{\|x\|}-x+x^\perp, & \text{if } x\in B_2.\end{cases}$$

*Then* $T$ *is a Lipschitz pseudocontractive mapping but not a strict pseudocontractive one.*

**Definition 2.4.** A pseudocontraction semigroup is a family $\mathfrak{F}:=\{T(t):t\ge 0\}$ of self-mappings of $C$ such that

(1) $T(0)x=x$ for all $x\in C$;

(2) $T(s+t)=T(s)T(t)$ for all $s,\ t\ge 0$;

(3) $\lim_{t\to 0^+}T(t)x=x$ for all $x\in C$;

(4) for each $t>0$, $T(t)$ is pseudocontractive; that is,

$$\langle T(t)x-T(t)y,j(x-y)\rangle\le\|x-y\|^2,\quad \forall\, x,\ y\in C.$$

Obviously, the class of pseudocontractive semigroups includes the class of nonexpansive semigroup as a special case. In this paper, we use $\mathscr{F}$ to denote the set of common fixed points of $\mathfrak{F}$ ; that is,

$$\mathscr{F} := \{x \in C : T(t)x = x, \quad t > 0\} = \bigcap_{t>0} F(T(t)).$$

**Definition 2.5.** $T$ is said to be demicontinuous on $C$, if $\{x_n\} \subset C$ and $x_n \to x \in C$ together imply $Tx_n \rightharpoonup Tx$.

The following lemmas also play an important role in this paper.

**Lemma 2.6.** *Let $C$ be a closed convex subset of a real Hilbert space $\mathscr{H}$. and let $P_C$ be the metric projection from $\mathscr{H}$ onto $C$ (i.e., for $x \in \mathscr{H}$, $P_C$ is the only point in $C$ such that $\|x - P_C x\| = \inf\{\|x - z\| : z \in C\}$). Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if there holds the relations:*

$$\langle x - z, y - z\rangle \leq 0 \ for \ all \ y \in C.$$

**Lemma 2.7.** *Let $\mathscr{H}$ be a real Hilbert space. Then the following equations hold:*

(1) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$ *for all* $x,\ y \in \mathscr{H}$;

(2) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$ *for all* $t \in [0,1]$;

(3) *If $\{x_n\}$ is a sequence in $H$ weakly convergent to $z$, then*

$$\limsup_{n\to\infty} \|x_n - y\| = \limsup_{n\to\infty} \|x_n - z\| + \|y - z\| \ for \ all \ y \in \mathscr{H}.$$

**Definition 2.8.** [11] $T : D \subset K \to \mathscr{H}$ is said to be weakly inward (relative to $K$) if $Tx \in \bar{I}_K(x)$ for $x \in D$, where $\bar{I}_K(x)$ is the closure of the inward set $\bar{I}_K(x) := \{x + c(z-x) : z \in K \ and \ c \geq 1\}$.

**Lemma 2.9.** [12] *Let $C$ be a closed convex subset of $\mathscr{H}$. Assume that $T : C \to \mathscr{H}$ is a demicontinuous weakly inward $\alpha$-dissipative mapping with $\alpha < 1$. Then $T$ has a unique fixed point in $C$.*

**Lemma 2.10.** [13] *Let $C$ be a nonempty closed convex subset of a real Hilbert space $\mathscr{H}$ and $T : C \to C$ be a demicontinuous pseudocontractive mapping. Then $\mathscr{F}(T)$ is closed convex subset of $C$ and $I - T$ is demiclosed at 0.*

## 3. Main results

**Theorem 3.1.** *Let $\mathscr{H}$ be a real Hilbert space and let $C$ be a nonempty closed and convex subset of $\mathscr{H}$. Let $\mathfrak{T} := \{T(t) : t \geq 0\}$ be a demicontinuous semigroup of pseudocontractions on $C$. Let $\{x_n\}$ be a sequence generated in the following iterative process: $x_0 \in H$ is chosen arbitrarily and*

$$\begin{cases} C_1(t) = C, \\ C_1 = \bigcap_{t\geq 0} C_1(t), \\ x_1 = P_{C_1} x_0, \\ y_n(t) = \alpha_n(t) x_n + (1-\alpha_n(t)) T(t) y_n(t), \\ C_{n+1}(t) = \{z \in C_n(t) : \|x_n - z\| \geq \|y_n(t) - z\|\}, \\ C_{n+1} = \bigcap_{t\geq 0} C_{n+1}(t), \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \geq 0. \end{cases}$$

*Assume that $\{\alpha_n(t)\}$ in [0,1] satisfies the condition $\limsup_{n\to\infty} \alpha_n(t) < 1$ for all $t \geq 0$. If $\mathscr{F} := \bigcap_{t\geq 0} \mathscr{F}(T(t)) \neq \emptyset$, then $\{x_n\}$ converges strongly to $P_{\mathscr{F}} x_0$.*

**Proof.** For each $n \geq 0$, defined a mapping $S_n : C \to C$ as follows:

$$S_n x = (1-\alpha_n(t)) T(t) x + \alpha_n(t) x_n.$$

We see that $S_n$ is a demi-continuous and strong psedocontraction. Indeed, for every $x,\ y \in C$, we have

$$\begin{aligned} (1-\alpha_n(t))\|x-y\|^2 &\geq (1-\alpha_n(t))\langle S(t)x - S(t)y, x-y\rangle \\ &= \langle S_n x - S_n y, x-y\rangle. \end{aligned}$$

This shows that $S_n$ has a unique fixed point. Next, we prove that $C_n$ is closed and convex for each $n \geq 1$. Obviously, $C_n$ is closed for each $n \geq 1$. We observe that $C_n$ is convex. Indeed, let $z_1,\ z_2 \in C_{n+1}(t)$ for each $n \geq 0$ and $t \geq 0$. Take $z = \alpha z_1 + (1-\alpha) z_2$ for $\alpha \in (0,1)$. Notice that

$$\|y_n(t) - z_1\| - \|x_n - z_1\| \leq 0, \tag{3.1}$$

and

$$\|y_n(t) - z_2\| - \|x_n - z_2\| \leq 0. \tag{3.2}$$

Using (3.1) and (3.2), we find that

$$\begin{aligned}\|y_n(t)-z\|-\|x_n-z\| &= \|y_n(t)-\alpha z_1-(1-\alpha)z_2\|-\|x_n-\alpha z_1-(1-\alpha)z_2\|\\ &\le \alpha\|y_n(t)-z_1\|+(1-\alpha)\|y_n(t)-z_2\|\\ &\quad -\alpha\|x_n-z_1\|-(1-\alpha)\|x_n-z_2\|\\ &\le 0.\end{aligned}$$

This shows that $C_{n+1}(t)$ is convex for each $n\ge 0$ and $t\ge 0$, therefore, $C_{n+1}=\bigcap_{t\ge 0}C_{n+1}(t)$ is also convex for each $n\ge 0$ and $t\ge 0$. That is, $C_n$ is convex for all $n\ge 1$. $\forall\, p\in\mathscr{F}$, we have

$$\begin{aligned}\|y_n(t)-p\|^2 &= \langle \alpha_n(t)x_n+(1-\alpha_n(t))T(t)y_n(t)-p, y_n(t)-p\rangle\\ &= (1-\alpha_n(t))\langle T(t)y_n(t)-p, y_n(t)-p\rangle+\alpha_n(t)\langle x_n-p, y_n(t)-p\rangle\\ &\le \alpha_n(t)\|y_n(t)-p\|\|x_n-p\|+(1-\alpha_n(t))\|y_n(t)-p\|^2.\end{aligned}$$

It follows that $\|x_n-p\|\ge\|y_n(t)-p\|$. This shows that $\mathscr{F}\subset C_{n+1}(t)$ for all $n\ge 0$ and $t\ge 0$. It follows that $\mathscr{F}\subset C_{n+1}$ for all $n\ge 0$, that is, $\mathscr{F}\subset C_n$ for all $n\ge 1$. In view of $x_n=P_{C_n}x_0$, we see that

$$\langle x_n-x, x_0-x_n\rangle\ge 0,\quad \forall\, x\in C_n. \tag{3.3}$$

and since $\mathscr{F}\subset C_n$, we also have that

$$\langle x_n-w, x_0-x_n\rangle\ge 0,\quad \forall\, w\in\mathscr{F}. \tag{3.4}$$

It follows that

$$\begin{aligned}0 &\le \langle x_n-P_{\mathscr{F}}x_0, x_0-x_n\rangle\\ &= \langle x_n-x_0+x_0-P_{\mathscr{F}}x_0, x_0-x_n\rangle\\ &\le \|x_0-x_n\|\|x_0-x_n\|\|x_0-P_{\mathscr{F}}x_0\|-\|x_0-x_n\|^2.\end{aligned}$$

This shows that

$$\|x_0-P_{\mathscr{F}}x_0\|\ge\|x_0-x_n\|. \tag{3.5}$$

There $\{x_n\}$ is a bounded sequence. In view of the construction of $C_n$, we see that $C_{n+1}\subset C_n$ and $x_{n+1}\in C_n$. It follows from (3.5) that $\|x_0-x_{n+1}\|\ge\|x_0-x_n\|$. Hence, we have $\lim_{n\to\infty}\|x_0-x_n\|$

exists. On the other hand, we have $\langle x_0 - x_n, x_n - x_{n+1}\rangle \geq 0$. It follows that

$$\begin{aligned}\|x_0 - x_{n+1}\|^2 - \|x_n - x_0\|^2 &\geq \|x_0 - x_{n+1}\|^2 - \langle x_0 - x_n, x_n - x_{n+1}\rangle - \|x_n - x_0\|^2 \\ &= \|x_n - x_0\|^2 - 2\langle x_0 - x_n, x_0 - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_{n+1}\|^2.\end{aligned}$$

Letting $n \to \infty$, we see that $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$. Now, we are in a position to show that $\lim_{n\to\infty}\|y_n(t) - T(t)y_n(t)\| = 0$. In view of $x_{n+1} \in C_{n+1}$, one has $x_{n+1} \in C_{n+1}(t)$ for all $t \geq 0$. On the other hand, we have $\|x_n - x_{n+1}\| \geq \|y_n(t) - x_{n+1}\|$. It follows that $\lim_{n\to\infty}\|y_n(t) - x_n\| = 0$. On the other hand, from $(*)$ in (2.1), one has $y_n(t) - x_n = (1 - \alpha_n(t))(T(t)y_n(t) - x_n)$. Since $\limsup_{n\to\infty}\alpha_n(t) < 1$ for all $t \geq 0$ and (2.7), we obtain $\|x_n - T(t)y_n(t)\| \to 0, \quad as\ n \to \infty$. It follows that $\lim_{n\to\infty}\|y_n(t) - T(t)y_n(t)\| = 0$ for all $t \geq 0$. Since both $\{x_n\}$ and $\{y_n(t)\}$ are bounded, we obtain that $\emptyset \neq \omega_\omega(x_n) \subset \mathscr{F}$, where $\omega_\omega(x)$ denotes the weak $\omega-$limit set of the sequence $\{x_n\}$. In view of the weak lower semicontinuity of the norm, we obtain that $\|x_0 - p\| \leq \|x_0 - P_{\mathscr{F}}x_0\|$, $\forall\ p \in \omega_\omega(x_n)$. Since $\omega_\omega(x_n) \subset \mathscr{F}$, we arrive at $p = P_{\mathscr{F}}x_0$, which in turn implies that $\omega_\omega(x_n) = \{P_{\mathscr{F}}x_0\}$. It follows that $\{x_n\}$ converges weakly to $p = P_{\mathscr{F}}x_0$. Finally, we show that $\{x_n\}$ converges strongly to $p = P_{\mathscr{F}}x_0$. From the weak lower semicontinuity of the norm, we obtain that

$$\|x_0 - P_{\mathscr{F}}x_0\| \geq \limsup_{n\to\infty}\|x_0 - x_n\| \geq \liminf_{n\to\infty}\|x_0 - x_n\| \geq \|x_0 - P_{\mathscr{F}}x_0\|.$$

This shows that $\lim_{n\to\infty}\|x_0 - x_n\| = \|x_0 - P_{\mathscr{F}}x_0\|$. It follows that $\{x_n\}$ converges strongly to $P_{\mathscr{F}}x_0$. This completes the Proof.

From Theorem 3.1, we have the following result on a single demicontinuous pseudocontractive operator.

**Corollary 3.2.** *Let $C$ be a nonempty closed and convex subset of a real Hilbert space $\mathscr{H}$. Let $T : C \to C$ be a demicontinuous pseudocontraction. Let $\{x_n\}$ be a sequence generated in the* *following iterative process:*

$$\begin{cases} x_0 \in H \text{ chosen arbitrarily}, \\ C_1 = C, \ x_1 = P_{C_1}x_0, \\ y_n = \alpha_n x_n + (1-\alpha_n)Ty_n, \\ C_{n+1} = \{z \in C_n(t) : \|x_n - z\| \geq \|y_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad \forall n \geq 0. \end{cases}$$

*Assume that the control sequences* $\{\alpha_n\}$ *in [0,1] satisfies the condition* $\limsup_{n\to\infty} \alpha_n < 1$. *If* $\mathscr{F}(T) \neq \emptyset$, *then* $\{x_n\}$ *converges strongly to* $P_{\mathscr{F}}x_0$.

**Remark 3.3.** It is of interest to remove the demicontinuous restrictions imposed on the operators.

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