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A COMMON FIXED POINT THEOREM IN STRICTLY CONVEX FM-SPACES

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Abstract. The purpose of this paper is to define a notion of strictly convex and normal structures in fuzzy metric spaces. Also, a theorem that provides existence of a common fixed point theorem for two self-mappings defined on strictly convex fuzzy metric spaces is proved. In the proof of the main results, topological methods for characterization spaces with nondeterministic distance is used. Moreover, we present the concept of proximal nonexpansive mappings on star-shaped sets in complete fuzzy metric spaces. We also derive some results on the best proximity for these mappings in complete fuzzy metric spaces. Finally, we provide some examples to illustrate our main results.

Keywords. Convex structure; Fuzzy normed space; Normal structure; Proximal nonexpansive mapping.

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1. Introduction-Preliminaries

The theory of fuzzy sets was introduced by Zadeh [23]. Since 1965, the research in many branches of fuzzy mathematics has received much attention. In particular, in the framework of fuzzy topologies, one of the main problems in the theory of fuzzy topological spaces is to obtain an appropriate and a consistent notion of a fuzzy metric space. Many authors have investigated this problem and several different notions of a fuzzy metric space have been defined and studied. In [17], Kramosil and Michalek introduced and studied an interesting notion of fuzzy metric spaces which is closely related to a class of probabilistic metric spaces, the so-called (generalized) Menger spaces. By using the notion of a fuzzy metric space in the sense of Kramosil and Michalek [17], Grabiec [9] proved fuzzy versions of the celebrated Banach fixed

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point theorem and the Edelstein fixed point theorem, respectively. To this end, Grabiec introduced a notion of complete fuzzy metric spaces and compact fuzzy metric spaces, respectively. Later on, George and Veeramani [7] started the study of a stronger form of metric fuzziness. Further, they modified the definition of the Cauchy sequence given by Gabriec in [9], because of the fact that the set of real numbers is not complete with the definition given in [9].

On the other hand, fixed point theory is one of the most famous mathematical theories with applications in several branches of sciences, especially, in chaos theory, game theory, theory of differential equations and so on. The first theorem of fixed point theory for non-expansive mappings was proved independently by Browder [2] and Göhde [8], and by Kirk [14] in a more general form.

The concept of normal structures was introduced in 1948 by Brodskii and Milman [1]. In 1970, Takahashi [22] defined convex and normal structures for sets in metric spaces. In 1987, a convex structure for sets in probabilistic metric spaces was defined in [10].

The first result on the fixed point theory in probabilistic metric spaces was obtained by Sehgal and Bharucha-Reid [19]. Hadžić proved fixed point theorems for mappings in probabilistic metric spaces with a convex structure. For more details on convexity and fixed point results for mappings defined on metric, probabilistic metric spaces and fuzzy metric spaces; see [1, 2, 4, 5, 8, 9, 10, 12, 13, 22] and the references therein.

Recently, Ješić [12] observed a wide class of non-expansive mappings defined on intuitionists fuzzy metric spaces with convex, strictly convex and normal structures and proved the existence of a fixed point for the class of non-expansive mappings in strictly convex intuitionistic fuzzy metric spaces. More recently, Ješić, Nikolic and Babačev [13] defined a notions of a strictly convex and normal structure in Menger PM-spaces. Also, they proved the existence of a common fixed point theorem for two self-mappings defined on strictly Menger PM-spaces.

Let K be any nonempty subset of a metric space X and $T : K \to X$ be a non-self mapping. A fixed point problem is to find a point $z \in K$ such that $d(z,Tz) = 0$. A point $z \in K$, where $\inf\{d(y,Tz) : y \in K\}$ is attained, i.e., $d(z,Tz) = \inf\{d(y,Tz) : y \in K\}$ holds, is called an approximate fixed point of T or an approximate solution of an equation $Tx = x$. In case if it is impossible to solve fixed point problem, it could be interesting to study the conditions that assure existence and uniqueness of approximate fixed point of a mapping T. A well known best approximation theorem, which is due to Fan [6], states if K is a nonempty compact convex subset of a Hausdorff locally convex topological vector space E and $T : K \to E$ is a continuous mapping, then there exists an element $z \in K$ such that $d(z,Tz) = \inf_{y\in K} d(y,Tz) = d(Tz,K)$.

The purpose of this paper is to define a notions of strictly convex and normal structures in fuzzy metric spaces. Also, a theorem that provides the existence of a common fixed point theorem for two self-mappings defined on strictly convex fuzzy metric spaces will be proved. In the proof of the main result topological methods for characterization spaces with nondeterministic distance will be used. As a consequence of the main result, we will give fuzzy variant of Browder's result [2]. Moreover, we present the concept of proximal nonexpansive mappings on star-shaped sets in complete fuzzy metric spaces. We also also derive some results about the best proximity for these mappings in complete fuzzy metric space. Finally, we provide some examples illustrate our main results.

Definition 1.1. [20] A binary operation $\Delta : [0,1]\times[0,1] \to [0,1]$ is said to be a continuous t-norm if $([0,1],\Delta)$ is a topological monoid with unit 1 such that $\Delta(a,b) \le \Delta(c,d)$ whenever $a \le c, b \le d$ for all $a,b,c,d \in [0,1]$.

Some typical examples of t-norm are the following:

$$\Delta(a,b) = ab, \quad \text{(product)}$$

$$\Delta(a,b) = \min\{a,b\}, \quad \text{(minimum)}$$

$$\Delta(a,b) = \max\{a+b-1,0\}, \quad \text{(Lukasiewicz)}$$

$$\Delta(a,b) = \frac{ab}{a+b-ab}. \quad \text{(Hamacher)}$$

Definition 1.2. [17] A triple (X,M,Δ) is called a fuzzy metric space (briefly, a FM-space) if X is an arbitrary (non-empty) set, Δ is a continuous t-norm and M is a fuzzy set on $X\times X\times[0,\infty)$ such that the following axioms hold:

(FM-1) $M(x,y,0) = 0$ for all $x,y \in X$,

(FM-2) $M(x,y,t) = 1$ for every $t > 0$ if and only if $x = y$,

(FM-3) $M(x,y,t) = M(y,x,t)$ for all $x,y \in X$ and $t > 0$,

(FM-4) $M(x,y,.) : [0,\infty) \to [0,1]$ is left continuous for all $x,y \in X$,

(FM-5) $M(x,z,t+s) \ge \Delta(M(x,y,t),M(y,z,s))$ for all $x,y,z \in X$ and for all $t,s \in [0,\infty)$.

We will refer to the fuzzy metric spaces in the sense of Kramosil and Michalek as KM-fuzzy metric spaces. If, in the above definition, condition (FM-5) is replaced by the condition

(FM-5A) $M(x,z,\max\{t,s\}) \geq \Delta(M(x,y,t),M(y,z,s))$ for all $x,y,z \in X$ and for all $t,s \in [0,\infty)$,

then (X,M,Δ) is called a strong metric space. It is easy to check that (FM-5A) implies (FM-5), that is, every strong fuzzy metric space is it self a fuzzy metric space.

Definition 1.3. [7] A triple (X,M,Δ) is called a fuzzy metric space (briefly, a FM-space) if X is an arbitrary (non-empty) set, Δ is a continuous t-norm and M is a fuzzy set on $X \times X \times [0,\infty)$ such that the following axioms hold:

(FM-1) $M(x,y,t) > 0$ for all $x,y,z \in X$,
(FM-2) $M(x,y,t) = 1$ for every $t > 0$ if and only if $x = y$,
(FM-3) $M(x,y,t) = M(y,x,t)$ for all $x,y \in X$ and $t > 0$,
(FM-4) $M(x,y,.) : [0,\infty) \to [0,1]$ is left continuous for all $x,y \in X$,
(FM-5) $M(x,z,t+s) \geq \Delta(M(x,y,t),M(y,z,s))$ for all $x,y,z \in X$ and for all $t,s \in [0,\infty)$.

We will refer to the fuzzy metric spaces in the sense of George and Veeramani as GV-fuzzy metric spaces.

Example 1.4. ([7]) (1) Let (X,d) be a metric space. Define a t-norm by $\Delta(a,b) = ab$, and set

$$M_d(x,y,t) = \frac{t}{t+d(x,y)}, \quad \text{for all } x,y \in x \text{ and } t > 0.$$

Then (X,M_d,Δ) is a strong fuzzy metric space; M_d is called the standard fuzzy metric induced by d. It is interesting to note that the topology induced by the M_d and the corresponding metric d coincide.

(2) Let (X,d) be a metric space. Define a t-norm by $\Delta(a,b) = ab$, and set

$$M(x,y,t) = exp\left[-\frac{d(x,y)}{t}\right], \quad \text{for all } x,y \in x \text{ and } t > 0.$$

Then (X,M,Δ) is a strong fuzzy metric space.

Lemma 1.5. [9] *Let (X,M,Δ) be a FM-spaces. Then $M(x,y,t)$ is non-decreasing with respect to t, for all $x,y \in X$.*

Proof. Suppose that $M(x,y,t) > M(x,y,s)$ for some $0 < t < s$. Then

$$\begin{aligned}\Delta(M(y,y,s-t),M(x,y,t)) &\geq M(x,y,s)\\ &< M(x,y,t).\end{aligned}$$

By (FM-2), we have $M(y,y,s-t) = 1$. Thus

$$M(x,y,t) < M(x,y,s) < M(x,y,t),$$

which is a contradiction.

Lemma 1.6. [18] *Let (X,M,Δ) be a FM-spaces. If there exists $k \in (0,1)$ such that*

$$M(x,y,kt) \geq M(x,y,t)$$

for all $x,y \in X$ and $t > 0$, then $x = y$.

Definition 1.7. Let (X,M,Δ) be a fuzzy metric space with a continuous t-norm Δ.

(1) A sequence $\{x_n\}$ in X is said to be convergent to x in X if for every $\varepsilon > 0$ and $\lambda > 0$ there exists a positive integer N such that $M(x_n,x,\varepsilon) > 1-\lambda$ whenever $n \geq N$.
(2) A sequence $\{x_n\}$ in X is said to be convergent to x in X if for every $\varepsilon > 0$ and $\lambda > 0$ there exists a positive integer N such that $M(x_n,x_m,\varepsilon) > 1-\lambda$ whenever $n,m \geq N$.
(3) (X,M,Δ) is complete if every Cauchy sequence in X is convergent to some point in X.

The (ε,λ)-topology τ in a fuzzy metric space is introduced by the family of neighborhoods $\mathscr{N}_x$ of a point $x \in X$ given by

$$\mathscr{N}_x = \{\mathscr{N}_x(\varepsilon,\lambda) : \varepsilon > 0, \lambda \in (0,1)\},$$

where

$$\mathscr{N}_x(\varepsilon,\lambda) = \{y \in X : M(x,y,\varepsilon) > 1-\lambda\}.$$

The (ε,λ)-topology τ is a Hausdorff and first countable [7]. In this topology the function T is continuous at $x \in X$ if and only if for every sequence $x_n \to x$ it holds that $Tx_n \to Tx$.

Definition 1.8. Let (X,M,Δ) be a fuzzy metric space with a continuous t-norm Δ. Let A be a subset of X. The closure of the set A is the smallest closed set containing A, denoted by $\overline{A}$.

Definition 1.9. Let (X,M,Δ) be a fuzzy metric space with a continuous t-norm Δ. Let $r \in (0,1)$, $t>0$ and $x \in X$. The set $N_x[\varepsilon,\lambda] = \{y \in X : M(x,y,\varepsilon) \geq 1-\lambda\}$ is called closed (ε,λ)-neighborhood of a point x in X.

Definition 1.10. A subset K of a fuzzy metric space is called compact if following statement holds:

$$K \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha \Longrightarrow K \subseteq \bigcup_{i=1}^{n} U_{\alpha_i} \text{ for some } \alpha_1,\cdots,\alpha_n \in \Lambda$$

for every collection $\{U_\alpha : \alpha \in \Lambda\}$ of open sets $U_\alpha \subseteq X$.

Lemma 1.11. *Let (X,M,Δ) be a fuzzy metric space with a continuous t-norm Δ and $K \subseteq X$. Then, K is compact if and only if for every collection of closed sets $\{F_\alpha\}$ such that $F_\alpha \subseteq K$ it holds that*

$$\bigcap_{\alpha\in\Lambda} F_\alpha \Longrightarrow \bigcap_{i=1}^{n} F_{\alpha_i} = \emptyset \textit{ for some } \alpha_1,\cdots,\alpha_n \in \Lambda\,.$$

Lemma 1.12. *Let (X,M,Δ) be a fuzzy metric space with a continuous t-norm Δ and $K \subseteq X$. Then $x \in \overline{K}$ if and only if there exists a sequence $\{x_n\}$ in K such that $x_n \to x$.*

Definition 1.13. Let (X,M,Δ) be a fuzzy metric space with a continuous t-norm Δ and $A \subseteq X$. The fuzzy diameter of A is given by

$$\delta_A(t) = \inf_{x,y\in A} \sup_{s<t} M(x,y,s).$$

The diameter of the set A is defined as

$$\delta_A = \sup_{t>0} \inf_{x,y\in A} \sup_{s<t} M(x,y,s).$$

If there exists a number $\lambda \in (0,1)$ such that $\delta_A = 1-\lambda$, then the set A is called fuzzy semi-bounded. If $\delta_A = 1$, then A is called fuzzy bounded.

Lemma 1.14. *Let (X,M,Δ) be a fuzzy metric space with a continuous t-norm Δ and $A \subseteq X$. A set A is fuzzy bounded if and only if for each $\lambda \in (0,1)$ there exists $t>0$ such that $M(x,y,t) > 1-\lambda$ for all $x,y \in A$.*

Proof. The proof follows immediately from the definition of $\sup A$ and $\inf A$ of non-empty sets.

It is not difficult to see that every metrically bounded set is also fuzzy bounded if it is considered in the induced FM-space.

Theorem 1.15. *Every compact subset A of a fuzzy metric space (X,M,Δ) with continuous t-norm Δ is fuzzy semi-bounded.*

Proof. Let A be a compact subset of X. Let fix $\varepsilon > 0$ and $\lambda > 0$. Now, we will consider an (ε,λ)-cover $\{N_x(\varepsilon,\lambda) : x \in A\}$. Since A is compact, there exist $x_1, x_2, \cdots, x_n \in A$ such that $A \subseteq \bigcup_{i=1}^{n} N_{x_i}(\varepsilon,\lambda)$. Let $x,y \in A$. Then there exists $i \in \{1,\cdots n\}$ such that $x \in N_{x_i}(\varepsilon,\lambda)$ and exists $j \in \{1,\cdots n\}$ such that $y \in N_{x_j}(\varepsilon,\lambda)$. Thus we have $M(x,x_i,\varepsilon) > 1-\lambda$ and $M(y,x_j,\varepsilon) > 1-\lambda$. Now, let $m = \min\{M(x_i,x_j,\varepsilon) : 1 \le i,j \le n\}$. It is obvious that $m > 0$ and we have

$$M(x,y,\varepsilon) \ge \Delta(M(x,x_i,\varepsilon), M(x_i,x_j,\varepsilon), M(x_j,y,\varepsilon)) \ge \Delta(1-\lambda, m, 1-\lambda) > 1-\delta,$$

for some $0 < \delta < 1$. If we take $\varepsilon_1 = 3\varepsilon$, we have $M(x,y,\varepsilon) > 1-\delta$ for all $x,y \in A$. Hence we obtain that A is fuzzy semi-bounded set.

Lemma 1.16. *In a fuzzy metric space (X,M,Δ) with a continuous t-norm Δ, every compact set is closed and bounded.*

2. Convex structure, normal structure and strictly convex structure on FM-spaces

Takahashi [22] introduced the notion of metric spaces with a convex structure. This class of metric spaces includes normed linear spaces and metric spaces of the hyperbolic type.

Definition 2.1. Let (X,d) be a metric space. We say that a metric space possesses a Takahashi convex structure if there exists a function $W : X^2 \times [0,1] \to X$ which satisfies

$$d(z, W(x,y,\mu)) \le \mu d(z,x) + (1-\mu) d(z,y),$$

for all $x,y,z \in X$ and arbitrary $\mu \in [0,1]$. A metric space (X,d) with Takahashi's structure is called convex metric space.

In this section, we introduce a generaliztion of Takahashi's definition to the case of a fuzzy metric space.

Definition 2.2. Let (X,M,Δ) be a fuzzy metric space with continuous t-norm Δ. A mapping $S: X\times X\times[0,1]\to X$ is said to be a convex structure on X if for every $(x,y)\in X\times X$ holds $S(x,y,0)=y$, $S(x,y,1)=1$ and for all $x,y,z\in X$, $\mu\in[0,1]$ and $t>0$

$$M(S(x,y,\mu),z,2t)\geq\Delta\left(M\left(x,z,\frac{t}{\mu}\right),M\left(x,z,\frac{t}{1-\mu}\right)\right). \tag{2.1}$$

It is easy to see that every metric space (X,d) with a convex structure S can be consider as a fuzzy metric space $(X,M,\Delta_{\min})$ (the associated fuzzy metric space) with the same function S. A fuzzy metric space (X,M,Δ) with a convex structure S is called a convex fuzzy metric space.

Here we give some terminology will be used in the sequel.

Definition 2.3. A point $x\in A$ is called diametral if $\inf_{y\in A}\sup_{s<t}M(x,y,s)=\delta_A(t)$ holds for all $t>0$.

Definition 2.4. Let (X,M,Δ) be a fuzzy metric space with continuous t-norm Δ and a convex structure $S(x,y,\mu)$. A subset $A\subseteq X$ is said to be a convex if for every $x,y\in A$ and $\mu\in[0,1]$ it follows that $S(x,y,\mu)\in A$.

Lemma 2.5. *Let (X,M,Δ) be a fuzzy metric space with continuous t-norm Δ and let $\{K_\alpha\}_{\alpha\in\Lambda}$ be a family of convex subsets of X. Then the intersection $K=\bigcap_{\alpha\in\Lambda}K_\alpha$ is a convex set.*

Proof. If $x,y\in K$, then $x,y\in K_\alpha$ for every $\alpha\in\Lambda$. It follows that $S(x,y,\mu)\in K_\alpha$ for every $\alpha\in\Lambda$, i.e., $S(x,y,\mu)\in K$, which means that K is convex.

Definition 2.6. A convex fuzzy metric space (X,M,Δ) is said to have the property (C) if every decreasing sequence of nonempty fuzzy bounded closed subsets of X has nonempty intersection.

Definition 2.7. A convex fuzzy metric space (X,M,Δ) is said to be uniformly convex if for any $\varepsilon>0$ there exists $\alpha=\alpha(\varepsilon)$ such that, for all $r>0$ and $x,y,z\in X$ with $M(z,x,t)\geq r$, $M(z,y,t)\geq r$ and $M(x,y,t)\leq r\varepsilon$,

$$M(z,S(x,y,\frac{1}{2}),t)\geq r(1-\alpha)>r.$$

Example 2.8. Let (X,d) be a uniformly convex metric space. We define

$$M(x,y,t)=\begin{cases}1, & \text{if } d(x,y)<t\ ;\\ \frac{t}{d(x,y)}, & \text{if } 0<t\leq d(x,y);\\ 0, & \text{if } t\leq 0.\end{cases}$$

An easy verification shows that (X,M,Δ) is a uniformly convex fuzzy metric space and $M(x,y,.)$ is continuous.

Theorem 2.9. *Let (X,M,Δ) be a complete and uniformly convex fuzzy metric space. Then (X,M,Δ) has the property (C).*

Proof. Let A_n be a decreasing sequence of non-empty bounded closed subsets of X. If $\delta(A_n) > 0$ for every positive integer n, then there exist $x,y \in A_n$ such that $M(x,y,t) \leq \frac{\delta(A_n)}{2}$. Since $M(z,x,t) \geq \delta(A_n), M(z,y,t) \geq \delta(A_n)$ for all $z \in A_n$ and the space is uniformly convex, there exists $\alpha > 0$ such that

$$M(z,S(x,y,1/2),t) \geq \delta(A_n)(1-\alpha) > \delta(A_n)$$

for all $z \in A_n$, $t > 0$ and hence we obtain $w_n^1 \in A_n$ such that

$$M(z,w_n^1,t) \geq \delta(A_n)(1-\alpha)$$

for all $z \in A_n$ and $t > 0$. Let

$$A_n^1 = \{w_n^1, w_{n+1}^1, \cdots\}.$$

Then it is obvious that $A_n^1 \neq \emptyset$ and $A_n^1 \supset A_{n+1}^1$ for every n. Suppose $\delta(A_n^1) > 0$ for every n. Then there exist $x,y \in A_n^1$ such that $M(x,y,t) \leq \delta(A_n^1)/2$. Put

$$B_n^1 = \bigcap_{k=0}^{\infty} B[w_{n+k}^1, \delta(A_n^1)].$$

Then $B_n^1 \supset \overline{co}(A_n^1)$ and $M(z,x,t) \geq \delta(A_n^1), M(z,y,t) \geq \delta(A_n^1)$ for every $z \in \overline{co}(A_n^1)$ and $t > 0$. Since X is uniformly convex, there exists $w_n^2 \in \overline{co}(A_n^1) \subset A_n$ such that

$$M(z,w_n^2,t) \geq \delta(A_n)(1-\alpha)^2$$

for all $z \in \overline{co}(A_n^1)$ and $t > 0$. By the same method, we obtain $\overline{co}(A_n^2), \overline{co}(A_n^3), \cdots$ and $w_n^3, w_n^4, \cdots$. It follows that

$$A_n \supset \overline{co}(A_n^1) \supset \overline{co}(A_n^2) \supset \cdots \quad \text{and} \quad \delta(\overline{co}(A_n^m)) \to 0$$

as $m \to \infty$. Since X is complete, there exists $w_n \in X$ such that

$$\bigcap_{m=1}^{\infty} \overline{co}(A_n^m) = \{w_n\}$$

for every n. From

$$\bigcap_{m=1}^{\infty} \overline{co}(A_n^m) \supset \bigcap_{m=1}^{\infty} \overline{co}(A_{n+1}^m),$$

we obtain $w_1 = w_2 = w_3 = \cdots$. Therefore, there exists w with $w \in A_n$ for all n and hence $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$.

Definition 2.10. A convex fuzzy metric space (X, M, Δ) with a convex structure $S : X \times X \times [0,1] \to X$ and continuous t-norm Δ will be called strictly convex if, for arbitrary $x, y \in X$ and $\mu \in (0,1)$ the element $z = S(x,y,\mu)$ is the unique element which satisfies

$$M\left(x, y, \frac{t}{\mu}\right) = M(x,y,t), \quad M\left(x, y, \frac{t}{1-\mu}\right) = M(x,y,t), \tag{2.2}$$

for all $t > 0$.

Lemma 2.11. *Let (X, M, Δ) be a fuzzy metric space with a convex structure $S(x,y,\mu)$ and continuous t-norm Δ. Suppose that for every $\mu \in (0,1)$, $t > 0$ and $x, y, z \in X$ hold*

$$M(S(x,y,\mu), z, t) > \min\{M(z,x,t), M(z,y,t)\}. \tag{2.3}$$

If there exists $z \in X$ such that

$$M(S(x,y,\mu), z, t) = \min\{M(z,x,t), M(z,y,t)\} \tag{2.4}$$

is satisfied, for all $t > 0$, then $S(x,y,\mu) \in \{x,y\}$.

Proof. Let us assume that (2.4) holds for some $z \in X$ and for all $t > 0$. Since (2.3) holds, it follows that $\mu = 0$ or $\mu = 1$ and, consequently we have that $S(x,y,0) = y$ or $S(x,y,1) = x$, which proves this lemma.

Lemma 2.12. *Let (X, M, Δ) be a fuzzy metric space with a convex structure $S(x,y,\mu)$ and continuous t-norm Δ. Then for arbitrary $x, y \in X$, $x \neq y$ there exists $\mu \in (0,1)$ such that $S(x,y,\mu) \notin \{x,y\}$.*

Proof. Suppose that for every $\mu \in (0,1)$, it holds that $S(x,y,\mu) \in \{x,y\}$. From (2.2) it follows that $M(x,y,t) = 1$ for all $t > 0$ which means that $x = y$ and so the proof is completed.

Definition 2.13. A fuzzy metric space (X,M,Δ) with continuous t-norm Δ possesses a normal structure if, for every closed , fuzzy semi-bounded and convex set $Y \subset X$, which consists of at least two different points, there exists a point $x \in Y$ which is non-diametral, i.e., there exists $t_0 > 0$ such that

$$\delta_Y(t_0) < \inf_{y\in Y} \sup_{s<t_0} M(x,y,s)$$

holds.

It is obvious that compact and convex sets in convex metric space possess a normal structure (see [22]).

Definition 2.14. Let (X,M,Δ) be a convex fuzzy metric space with continuous t-norm Δ and $Y \subseteq X$. The closed convex shell of a set Y denoted by $cov(Y)$, is the intersection of all closed, convex sets that contain Y.

It is easy to see that the set $cov(Y)$ exists, since the collection of closed, convex sets that contain Y is non-empty, because the fact that X belongs to this collection. From Lemma 2.5, it follows that this intersection is convex set. Also, this intersection is closed as an intersection of closed sets.

Definition 2.15. Let (X,M,Δ) be a fuzzy metric space with continuous t-norm Δ and let f be a self-mapping on X. We say that f is a non-expansive mapping if

$$M(fx,fy,t) \geq M(x,y,t) \tag{2.5}$$

holds for all $x,y \in X$ and $t > 0$.

3. Common fixed point theorems

Lemma 3.1. *Let (X,M,Δ) be a strictly convex fuzzy metric space with continuous t-norm Δ and a convex structure $S(x,y,\mu)$ satisfying (2.3) and let $K \subseteq X$ be non-empty, convex and compact subset of X. Then K possesses a normal structure.*

Proof. Suppose that K does not possess a normal structure. Then there exists a closed, fuzzy semi-bounded and convex subset $Y \subset K$, which contains at least two different points such that

Y does not contain a non-diametral point, i.e.,

$$\inf_{y\in Y}\sup_{s<t} M(x,y,s) = \delta_Y(t)$$

for every $x \in Y$. Since X is strictly convex and condition (2.3) satisfied, then the assertions of Lemma 2.11 and Lemma 2.12 hold. Let x_1 and x_2 be arbitrary points in Y. It follows from Lemma 2.12 that there exists $\mu_0 \in (0,1)$ such that $S(x_1,x_2,\mu_0) \notin \{x_1,x_2\}$. Since Y is convex set, it follows that $S(x1,x_2,\mu_0) \in Y$. Y is a closed subset of the compact set K, so Y is compact, too. Since $\delta_Y(t) = \inf_{y\in Y}\sup_{s<t} M(y,S(x_1,x_2,\mu),s)$ is left-continuous function on the compact set Y for arbitrary $t > 0$, there exists $x_3,x_4 \in Y$ such that $\sup_{s<t} M(x_3,S(x_1,x_2,\mu_0),s) = \delta_Y(t)$ holds. From Lemma 2.11 and the fact that $M(x,y,.)$ is non-decreasing left-continuous function, it follows that

$$\begin{aligned}
\delta_Y(t) &= \sup_{s<t} M(x_3,S(x_1,x_2,\mu_0),s) = M(x_3,S(x_1,x_2,\mu_0),t) \\
&> \min\{M(x_3,x_1,t),M(x_3,x_2,t)\} \qquad (3.1)\\
&= \min\{\sup_{s<t} M(x_3,x_1,s),\sup_{s<t} M(x_3,x_2,s)\} \geq \delta_Y(t).
\end{aligned}$$

From the last inequality, it follows that $\delta_Y(t) > \delta_Y(t)$ which is a contradiction. This completes the proof.

Lemma 3.2. *Let (X,M,Δ) be a convex fuzzy metric space with a convex structure $S(x,y,\mu)$ satisfying the condition (2.3). Then the closed (ε,λ)neigborhoods $N_x[\varepsilon,\lambda]$ are convex sets.*

Proof. Let $a,b \in N_x[\varepsilon,\lambda]$ be arbitrary points. This implies that $M(a,x,\varepsilon) \geq 1-\lambda$ and $M(b,x,\varepsilon) \geq 1-\lambda$ for all $\varepsilon > 0$. We shall prove that $M(S(a,b,\mu),x,\varepsilon) \geq 1-\lambda$ for all $\varepsilon > 0$, i.e., $S(a,b,\mu) \in N_x[\varepsilon,\lambda]$. Indeed, for $\mu \in (0,1)$, from (2.3), we have that

$$M(S(a,b,\mu),x,\varepsilon) > \min\{M(a,x,\varepsilon),M(b,x,\varepsilon)\} \geq \min\{1-\lambda,1-\lambda\} = 1-\lambda.$$

For $\mu = 0$ or $\mu = 1$. It follows that $S(a,b,0) = b$ or $S(a,b,1) = a$ belongs to $N_x[\varepsilon,\lambda]$.

Now we give the main theorem in this paper.

Theorem 3.3. *Let (X,M,Δ) be a convex fuzzy metric space with a convex structure $S(x,y,\mu)$ satisfying the condition (2.3) and let $K \subseteq X$ be a non-empty, convex and compact subset of X. Let f and g be a self-mappings on K, $g(K)\cap f(K) \subseteq K$, satisfying the conditions*

$$M(f(x),g(y),t) \geq M(x,y,t) \tag{3.2}$$

for all $x, y \in K$, $x \neq y$ *and for every* $t > 0$. *Then* f *and* g *have at least one common fixed point on* K.

Proof. First, note that the collection $\mathfrak{C}$ of all non-empty, closed, convex sets $K_\alpha \subseteq K$ such that $g(K_\alpha) \cap f(K_\alpha) \subseteq K_\alpha$ is non-empty, because $K \subseteq \mathfrak{C}$. Indeed, K is closed set because the fact that it is compact set in a Hausdorff pace and it is satisfied that $g(K) \cap f(K) \subseteq K$. If we order this collection with inclusion, then $(\mathfrak{C}, \subseteq)$ is a partially ordered set. Let $K_\alpha : \alpha \in \Lambda$ be an arbitrary chain of this family. Then the set $\bigcap\limits_{\alpha \in \Lambda} K_\alpha$ is nonempty, closed, convex subset of K, which is a lower bound of this chain. Indeed, let us assume that $\bigcap\limits_{\alpha \in \Lambda} K_\alpha = \emptyset$. Then, from Lemma 1.11, it follows that there exists a finite sub-collection $K_{\alpha_1} \supseteq K_{\alpha_2} \supseteq \cdots \supseteq K_{\alpha_n}$ of the chain $\{K_\alpha : \alpha \in \Lambda$ which has an empty intersection, which is impossible, since this intersection is $K_{\alpha_n} \neq \emptyset$. By Zorn's Lemma, it follows that there exists a minimal element K_0 of the collection $\mathfrak{C}$ such that $g(K_0) \cap f(K_0) \subseteq K_0$. We will prove that K_0 consists of only one point and since $g \cap f : K_0 \to K_0$ this will mean that g and f have a common fixed point.

Let us assume that K_0 contains at least two different points.It follows from Lemma 4.16 that K possesses a normal structure. It follows From Theorem 1.15 that K_0 is fuzzy semi-bounded set. Since K_0 is closed and convex, it follows that there exists some non-diametral point $x_0 \in K_0$, i.e., there exists $t_0 > 0$ such that the following inequality holds:

$$\inf_{y \in K_0} \sup_{s<t} M(x_0, y, s) > \delta_{K_0}(t_0). \tag{3.3}$$

Denote $1 - \rho := \inf\limits_{y \in K_0} \sup\limits_{s<t} M(x_0, y, s)$. Let us denote with K_1 the closed convex shell of the set $g(K_0) \cap f(K_0)$. Since $g(K_0) \cap f(K_0) \subseteq K_0$, it holds that

$$K_1 = cov(g(K_0) \cap f(K_0)) = \overline{cov(g(K_0) \cap f(K_0))} \subseteq \overline{cov(K_0)} = \overline{K_0} = K_0.$$

Therefore, $K_1 \subseteq K_0$ and it follows that

$$g(K_1) \cap f(K_1) \subseteq g(K_0) \cap f(K_0) \subseteq \overline{cov(g(K_0) \cap f(K_0))} = K_1,$$

i.e., $g(K_1) \cap f(K_1) \subseteq K_1$. This means that $K_1 \in \mathfrak{C}$, and since K_0 is the minimal element, we have that $K_1 = K_0$. If inequality (3.3) holds, i.e., if $1 - \rho > \delta_{K_0}(t_0)$, let us define sets

$$A := \left(\bigcap_{y \in K_0} N_y[\rho, t_0] \right) \bigcap K_0 \text{ and } A_1 := \left(\bigcap_{y \in g(K_0) \cap f(K_0)} N_y[\rho, t_0] \right) \bigcap K_0.$$

The set A is nonempty since $x_0 \in A$. Indeed , from inequality (3.3), it follows that $M(x_0, y, t_0) \geq 1-\rho$. From the previous we conclude that $x_0 \in N_y[\rho, t_0]$ for all $y \in K_0$. Consequently, x_0 belongs to A. We will show that $A = A_1$. Since $g(K_0) \cap f(K_0) \subseteq K_0$, we have $A \subset A_1$.

Now, let $z \in A_1$, we will prove that $z \in A$. Since $z \in A_1$, for arbitrary $y \in g(K_0) \cap f(K_0)$, it holds that $M(y, z, t_0) \geq 1-\rho$, i.e., $y \in N_y[\rho, t_0]$. Since y is an arbitrary point from $g(K_0) \cap f(K_0)$, it follows that $g(K_0) \cap f(K_0) \subseteq N_y[\rho, t_0]$. Because of the fact that $N_z[\rho, t_0]$ is a closed and convex set which contains $g(K_0) \cap f(K_0)$, we conclude that

$$K_1 \subseteq \overline{cov(g(K_0) \cap f(K_0))} \subseteq N_z[\rho, t_0]$$

holds. Since $K_0 = K_1$, it follows that $K_0 \subseteq N_z[\rho, t_0]$. From last we have that for every $y \in K_0$, it holds that $z \in N_y[\rho, t_0]$, which means that $A_1 \subseteq A$, i.e., $A = A_1$.

We will show that $A \in \mathfrak{C}$. Note that A is closed as an intersection of closed sets. From Lemma 2.5 and Lemma 3.2 that A is a convex set. Let us prove that $g(A) \cap f(A) \subseteq A$. Let $z \in A$ and $y \in g(K_0) \cap f(K_0)$. Then there exists $x \in K_0$ such that $y = f(x)$ and $y = f(x)$. Applying inequality (3.2) for $t = t_0$, we have

$$M(f(z), y, t_0) = M(f(z), g(x), t_0) \geq M(z, x, t_0) \geq 1-\rho.$$

This means that $f(z) \in A$. Since z is an arbitrary point from A, we obtain $f(A) \subseteq A$.

On the other hand, we have

$$M(g(z), y, t_0) = M(g(z), f(x), t_0) \geq M(z, x, t_0) \geq 1-\rho.$$

This means that $g(z) \in A$. Since z is an arbitrary point from A, we obtain $g(A) \subseteq A$. Finally, we obtain $g(A) \cap f(A) \subseteq A$.

Since $A \subseteq K_0$ and K_0 is the minimal element of the collection $\mathfrak{C}$, it follows that $A = K_0$. Now, we have that $\delta_A(t_0) \geq 1-\rho \geq \delta_{K_0}(t_0)$. This is a contradiction with $A = K_0$, i.e., the assumption that K_0 contains at least two different points is wrong, which means that K_0 contains only one point which is a common fixed point of the mappings g and f. This achieves the proof.

Corollary 3.4. *Let (X, M, Δ) be a strictly convex fuzzy metric space with a convex structure $S(x, y, \mu)$ satisfying the condition (2.3) and let $K \subseteq X$ be a non-empty, convex and compact subset of X. Let f be a non-expansive self-mappings on K. Then f has at least one fixed point on K.*

Proof. Putting, in the Theorem 3.3, $f = g$, we have that mapping f is a self-mapping on K and in this case, from conditions (3.2) and (3.3) we obtain that mapping f is non-expansive on K. Hence the conclusion holds.

4. Proximity points on star-shaped sets

Let A and B be two nonempty subsets of X and $T : A \to B$. Suppose that $d(A,B) = \inf\{d(a,b) : a \in A, b \in B\}$ is the measure of a distance between two sets A and B. A point z is called a best proximity point of T if $d(z,Tz) = d(A,B)$, which is a solution of the following optimization problem

$$f(x) = d(x,Tx) \leftarrow \min$$

subject to the constraint

$$x \in A.$$

Definition 4.1. Suppose that A and B are nonempty subsets of a fuzzy metric space (X,M,Δ). Then the fuzzy distance of A and B of A,B is the mapping $M(A,B,t)$ defined on $[0,\infty)$ by

$$M(A,B,t) = \sup_{x\in A, y\in B} M(x,y,t)$$

for all $t \geq 0$.

Definition 4.2. Let (X,M,Δ) be a fuzzy metric space. For subsets A and B of X, define

$$A_0 = \{x \in A : \exists y \in B \text{ such that } \forall t \geq 0, M(x,y,t) = M(A,B,t)\},$$

$$B_0 = \{y \in B : \exists x \in A \text{ such that } \forall t \geq 0, M(x,y,t) = M(A,B,t)\}.$$

Clearly, if A_0 (or B_0) is a nonempty subset, then A and B are nonempty subsets.

Definition 4.3. Let (X,M,Δ) be a fuzzy metric space, and (A,B) be a pair of nonempty subsets of X. A mapping $T : A \to B$ is said to be the proximal contraction (proximal nonexpansive) if there exists a real number $0 < c \leq 1$ such that

$$M(u,Tx,t) = M(A,B,t) = M(v,Ty,t) \Longrightarrow M(u,v,t) \geq M(x,y,t/c)$$

for all $u,v,x,y \in A$ and $t > 0$.

Example 4.4. Let $X=[0,2]$ and $T:X\to X$ be the mapping defined by $Tx=\frac{1}{8}x$. If $M(x,y,t)=\frac{t}{t+|x-y|}$, then it is easy to check that $M(X,X,t)=1=M(v,Ty,t)$, then for $c=\frac{1}{8}$, we have $M(u,v,t)=M(x,y,t/c)$, where $x,y,u,v\in X$. Therefore, T is a proximal contraction.

Definition 4.5. Let (X,M,Δ) be a fuzzy metric space, $A\subseteq X$, and $T:A\to A$ be a mapping. The mapping T is said to be an isometry if $M(Tx,Ty,t)=M(x,y,t)$ for all $x,y\in X$ and $t\geq 0$.

Definition 4.6. Let (X,M,Δ) be a fuzzy metric space, and $A,B\subseteq X$. A mapping $T:A\to B$ is said to be continuous at $x\in A$ if every sequence $\{x_n\}$ in A that converges to x, the sequence $\{Tx_n\}$ in B converges to Tx.

Remark 4.7. If T is an isometry mapping on a subset A of a fuzzy metric space (X,M,Δ), then T is continuous mapping Because

$$M(Tx_n,Tx,t)=M(x_n,x,t)\to 1$$

for all $t>0$. Also, it is easy to see that T is an injective mapping.

Remark 4.8. Let A,B,C be a nonempty subsets of a fuzzy metric space (X,M,Δ) such that Δ is continuous t-norm and $x\in A$. If two mappings $T:A\to B$ and $S:A\to C$ are continuous at x, then $T+S$ is continuous at x because

$$M((T+S)x_n,(T+S)x,t)\geq\Delta(M(Tx_n,Tx,t/2),M(Sx_n,Sx,t/2))\to 1$$

for all $t>0$.

Definition 4.9. [21] Let X be a vector space, and A be a nonempty subset of X. Then the subset A is called a p-star-shaped set if there exists a point $p\in A$ such that $\mu p+(1-\mu)x\in A$ for all $x\in A$, $\mu\in[0,1]$, and p is called the center of A.

Remark 4.10. Note that, each convex set C is a p-star-shaped set for each $p\in C$. Let (X,M,Δ_m) be a fuzzy metric space, A be a p-star-shaped set, B be a q-star-shaped set, and $M(p,q,t)=M(A,B,t)$. If $x\in A_0$, then there exists a point $y\in B$ such that $M(p,q,t)=M(A,B,t)$ for all

$t > 0$. So we have

$$\begin{aligned} M(p,q,t) &= M(\mu p + (1-\mu)x, \mu q + (1-\mu)y, t) \\ &\geq \Delta_m(M(\mu p, \mu q, \mu t), M((1-\mu)x, (1-\mu)y, (1-\mu)t) \\ &= \Delta_m(M(p,q,t), M(x,y,t)) \\ &= \Delta_m(M(A,B,t), M(A,B,t)) = M(A,B,t) \end{aligned}$$

for all $t > 0$. Therefore, $M(\mu p + (1-\mu)x, \mu q + (1-\mu)y, t) = M(A,B,t)$, which means that A_0 is a p-star-shaped set and, similarly, that B_0 is a q-star-shaped set.

Definition 4.11. Let (X, M, Δ) be a fuzzy metric space. A pair (A,B) of nonempty subsets of X is said to have

(a) P-property if A_0 is nonempty and

$$M(u,x,t) = M(A,B,t) = M(v,y,t) \Longrightarrow M(u,v,t) = M(x,y,t),$$

(b) weak P-property if A_0 is nonempty and

$$M(u,x,t) = M(A,B,t) = M(v,y,t) \Longrightarrow M(u,v,t) \geq M(x,y,t)$$

for all $u, v \in A_0$, $x, y \in B_0$, and $t > 0$.

Example 4.12. Let $X = \mathbb{R}^2$ and define

$$M((x,y),(u,v),t) = \frac{t}{t + \sqrt{(x-u)^2 + (y-v)^2}}.$$

Clearly, (X, M, Δ_m) is a complete fuzzy metric space. Let

$$A = \{(0, 1/n) : n \in \mathbb{N}\} \cup \{(0,0)\},$$

$$B = \{(1, 1/n) : n \in \mathbb{N}\} \cup \{(1,0)\}.$$

Then it is easy to check that $A_0 = A, B_0 = B$, and $M(A,B,t) = \frac{t}{t+1}$. If

$$M((0,x),(1,y),t) = M(A,B,t) = \frac{t}{t+1} = M((0,u),(1,v),t),$$

then $x = y$ and $u = v$, so that

$$M((0,x),(0,u),t) = \frac{t}{t+|x-u|} = \frac{t}{t+|y-v|} = M((1,y),(1,v),t).$$

Therefore, the pair (A,B) has the P-property.

Example 4.13. Let $X=\mathbb{R}^2$ and define

$$M((x,y),(u,v),t)=\frac{t}{t+\sqrt{(x-u)^2+(y-v)^2}}.$$

Let $A=\{(0,0)\}$ and $B=\{(x,y)\in X : y=1+\sqrt{1-x^2}\}$. Clearly, $A_0=\{(0,0)\}$ and $B_0=\{(-1,1),(1,1)\}$. If

$$M((0,0),(x,y),t)=M(A,B,t)=\frac{t}{t+\sqrt{2}}=M((0,0),(u,v),t),$$

then

$$1=M((0,0),(0,0),t)\geq M((x,y),(u,v),t),$$

where $(x,y),(u,v)\in B_0$. Therefore, the pair (A,B) has the weak P-property.

Remark 4.14. It is easy to check that the P-property is stronger than the weak P-property. If a pair (A,B) has the weak P-property and $T:A\to B$ is a nonexpansive mapping, then for all $u,v,x,y\in A$, we have

$$M(u,Tx,t)=M(A,B,t)=M(v,Ty,t)\Longrightarrow M(u,v,t)\geq M(Tx,Ty,t)\geq M(x,y,t).$$

That is, T is a proximal nonexpansive mapping. Similarly, if a pair (A,B) has the weak P-property and $T:A\to B$ is a contraction mapping, then T is a proximal contraction pairs (A,B) and (B,A) have the weak P-property.

Definition 4.15. [21] Let X and Y be vector spaces. A mapping $T:X\to Y$ is affine if

$$T(\sum_{k=1}^{n}\mu_k x_k)=\sum_{k=1}^{n}\mu_k T(x_k)$$

for all $n\in\mathbb{N}$, $x_1,\cdots,x_n$, and $\mu_1,\cdots,\mu_n\in\mathbb{R}$ such that $\sum_{k=1}^{n}\mu_k=1$.

Lemma 4.16. *Let (X,M,Δ) be a complete fuzzy metric space such that Δ is a t-norm of H-type, and $A,B\subseteq X$ be such that A_0 is a nonempty closed set. If $T:A\to B$ is a proximal contraction mapping such that $T(A_0)\subseteq B_0$, then there exists a unique $x\in A_0$ such that $M(x,Tx,t)=M(A,B,t)$ for all $t>0$.*

Proof. Since is nonempty and $T(A_0)\subseteq B_0$, there exists $x_0,x_1\in A_0$ such that $M(x_1,Tx_0,t)=M(A,B,t)$. Since $Tx_1\in B_0$, there exists $x_2\in A_0$ such that $M(x_2,Tx_1,t)=M(A,B,t)$. Continuing this process, we obtain a sequence $\{x_n\} \subseteq A_0$ such that $M(x_{n+1}, Tx_n, t) = M(A, B, t)$ for all $n \in \mathbb{N}$ and $t > 0$. Since for all $n \in \mathbb{N}$ and $t > 0$,

$$M(x_n, Tx_{n-1}, t) = M(A, B, t) = M(x_{n+1}, Tx_n, t)$$

and T is a proximal contraction, we have

$$M(x_{n+1}, x_n, t) \geq M(x_n, x_{n-1}, t/c) \ (0 < c < 1, t > 0).$$

Therefore, by [11], $\{x_n\}$ is a Cauchy sequence and so converges to some $x \in A_0$. again by the assumption $T(A_0) \subseteq B_0$. $Tx \in B_0$. Then there exists an element $u \in A_0$ such that $M(u, Tx, t) = M(A, B, t)$ for all $t > 0$. Since for all $n \in \mathbb{N}$,

$$M(u, Tx, t) = M(A, B, t) = M(x_{n+1}, Tx_n, t) \ (t > 0),$$

by the hypothesis we have

$$M(u, x_{n+1}, t) \geq M(x, x_n, t/c) \geq M(x, x_n, t) \ (t > 0).$$

Letting $n \to \infty$ shows that $x_n \to u$ and thus $x = u$, so $M(x, Tx, t) = M(A, B, t)$. If there exists an other element y such that $M(y, Ty, t) = M(A, B, t)$, then by the hypothesis we have $M(x, y, t) \geq M(x, y, t/c)$, which means that $x = y$.

Proposition 4.17. *Let Let (X, M, Δ) be a fuzzy metric space, and $A, B \subseteq X$ be such that A_0 is a nonempty set. Suppose that $T : A \to B$ is a proximal contraction mapping such that $T(A_0) \subseteq B_0$ and $g : A \to A$ is an isometry mapping such that $A_0 \subseteq g(A_0)$. Denote $G = g(A)$ and*

$$G_0 = \{z \in G; \exists B \text{ such that } \forall t \geq 0, M(x, y, t) = M(G, B, t)\}.$$

Then Tg^{-1} is a proximal contraction, and $G_0 = A_0$.

Proof. Since $G \subseteq A$, $M(G, B, t) \leq M(A, B, t)$ for all $t > 0$. Assume that $x \in A_0 \subseteq g(A_0)$. Then $x = g(z)$ for some $z \in A_0$, and so there exists $y \in B$ such that $M(A, B, t) = M(g(z), y, t) \leq M(G, B, t)$ for all $t > 0$. Thus, $M(A, B, t) = M(G, B, t)$ for all $t > 0$. Now we show that Tg^{-1} is a proximal contraction. To this end, suppose that $u, v, x, y \in G$ are such that

$$M(u, Tg^{-1}x, t) = M(G, B, t) = M(A, B, t) = M(v, Tg^{-1}y, t) \ (t > 0).$$

By the hypothesis, we have

$$M(u, v, t) \geq M(g^{-1}x, g^{-1}y, t/c) = M(gg^{-1}x, gg^{-1}y, t/c) = M(x, y, t/c) \ (t > 0)$$

for some $c \in (0,1)$. Therefore, Tg^{-1} is a proximal contraction. If $x \in G_0$, then $x \in G \subseteq A$, and there exists $y \in B$ such that $M(x,y,t) = M(G,B,t) = M(A,B,t)$ for all $t > 0$, so that $x \in A_0$. If $x \in A_0 \subseteq A$, there exists $y \in B$ such that $M(x,y,t) = M(A,B,t) = M(G,B,t)$ for all $t > 0$. On the other hand, by the hypothesis $x \in G$, and hence $G_0 = A_0$.

Corollary 4.18. *Let Let (X,M,Δ) be a fuzzy metric space, and $A,B \subseteq X$ be such that A_0 is a nonempty set. Suppose that $T : A \to B$ is a proximal contraction mapping such that $T(A_0) \subseteq B_0$ and $g : A \to A$ is an isometry mapping such that $A_0 \subseteq g(A_0)$. Then there exists a unique $x \in A_0$ such that $M(gx,Tx,t) = M(A,B,t)$.*

Proof. By Proposition 4.17, $Tg^{-1} : G = g(A) \to B$ is a proximal contraction, $Tg^{-1}(G_0) = Tg^{-1}(A_0) \subseteq T(A_0) \subseteq B_0$. Now by Lemma 4.16 there exists a unique $z \in A_0$ such that $M(z,Tg^{-1}z,t) = M(A,B,t)$. Since $A_0 \subseteq g(A_0)$, there exists $x \in A_0$ such that $z = g(x)$, so that $M(g(x),Tx,t) = M(A,B,t)$. Note that g is an injective mapping, therefore, by Lemma 4.16, x is unique, and hence the result follows.

Example 4.19. Let $X = \mathbb{R}^2$, $A = \{(0,y) : y \in \mathbb{R}\}$ and $B = \{(1,y) : y \in \mathbb{R}\}$. Suppose that $T : A \to B$ is defined by $T(0,y) = (1,y/4)$, $g : A \to A$ is defined by $g(0,y) = (0,-y)$, and $M((x,z),(y,w),t) = \frac{t}{t+|x-y|+|z-w|}$. It is easy to see that (X,M,Δ_m) is a complete fuzzy metric space, $M(A,B,t) = \frac{t}{t+1}$, $A_0 = A, B_0 = B$, $T(A_0) \subseteq B_0$, and

$$M(g(0,x),g(0,y),t) = M((0,-x),(0,-y),t) = \frac{t}{t+|x-y|} = M((0,x),(0,y),t).$$

If $(0,u),(0,x),(0,v),(0,y) \in A$ are such that

$$\frac{t}{t+1+|u-x/4|} = M((0,u),T(0,x),t) = M(A,B,t) = M((0,v),T(0,y),t) = \frac{t}{t+1+|v-y/4|},$$

then $u = x/4$ and $v = y/4$. It follows that

$$M((0,u),(0,v),t) = M((0,x/4),(0,y/4),t) = \frac{t}{t+\frac{1}{4}|x-y|} = M((0,x),(0,y),4t).$$

Therefore, all hypothesis of Corollary 4.18 are satisfied. Therefore, we have

$$M((0,0),T(0,0),t) = M((0,0),(1,0),t) = \frac{t}{t+1} = M(A,B,t).$$

Definition 4.20. Suppose that A is a nonempty subset of a fuzzy metric space (X,M,Δ). Then the fuzzy diameter of A is the mapping D_A defined on $[0,\infty)$ by $D_A(\infty) = 1$ and $D_A(x) = \lim_{t\to x^-} \phi_A(t)$, where $\phi_A(t) = \inf\{M(a,b,t) : a,b \in A\}$.

A nonempty set A in fuzzy metric space is bounded if $\lim_{x\to\infty} D_A(x) = 1$. It is easy to see that $M(a,b,t) \geq D_A(t)$ for all $a,b \in A$ and $t \geq 0$.

Theorem 4.21. *Let (X,M,Δ_m) be a complete fuzzy metric space, $A,B \subseteq X$ be such that A is a convex set, A_0 be a nonempty compact set, and B be a bounded convex set. Suppose that $T : A \to B$ is a continuous affine and proximal nonexpansive mapping such that $T(A_0) \subseteq B_0$ and $g : A \to A$ is an isometry mapping such that $A_0 \subseteq g(A_0)$. Then there exists an element $x \in A_0$ such that $M(gx,Tx,t) = M(A,B,t)$ for all $t > 0$.*

Proof. Fix $T_k : A \to B$ by

$$T_k x = (1-k)Tz + kTx.$$

We show that T_k is a proximal contraction. Let $u,v,x,y \in A$ be such that

$$M(u,T_k x,t) = M(A,B,t) = M(v,T_k y,t)\ (t > 0).$$

Since T is an affine mapping, we have

$$M(u,T((1-k)x+kx),t) = M(A,B,t) = M(v,T((1-k)z+ky),t)\ (t > 0).$$

Using the hypothesis, we have

$$M(u,v,t) \geq M((1-k)x+kx,(1-k)z+ky,t) = M(x,y,t) = M(x,y,t/k)\ (t > 0).$$

Hence, T_k is a proximal contraction. Let $x \in A_0$, so that $Tx \in B_0$ and $Tz \in B_0$. Therefore, there exists $u,v \in A_0$ such that

$$M(u,Tx,t) = M(A,B,t) = M(v,Tz,t)\ (t > 0).$$

Put $y = ku+(1-k)v \in A$. Then

$$\begin{aligned}
M(y,T_k x,t) &= M(ku+(1-k)v,(1-k)Tz+kTx,t)\\
&\geq \Delta_m(M(ku,kTx,kt),M((1-k)v,(1-k)Tz,(1-k)t))\\
&\geq \Delta_m(M(u,Tx,t),M(v,Tz,t)) = \Delta_m(M(A,B,t),M(A,B,t)) = M(A,B,t)\ (t > 0),
\end{aligned}$$

and thus $T_k(A_0) \subseteq B_0$. By Corollary 4.18, there exists a unique $x_k \in A_0$ such that $M(gx_k, T_k x_k, t) = M(A,B,t)$ for all $t > 0$. Fix $s \in (0,1)$. Then

$$\begin{aligned}
M(gx_k, Tx_k, t) &\geq \Delta_m(M(gx_k, T_k x_k, st), M(T_k x_k, Tx_k, (1-s)t) \\
&= \Delta_m(M(A,B,st), M((1-s)Tz, (1-s)Tx_k, (1-s)t) \\
&= \Delta_m(M(A,B,st), M(Tz, Tx_k, \frac{(1-s)t}{1-k}) \\
&\geq \Delta_m\left(M(A,B,st), D_B\left(\frac{(1-s)t}{1-k}\right)\right) \ (t > 0).
\end{aligned}$$

Now letting $k \to 1$, we obtain

$$\lim_{k \to 1} M(gx_k, Tx_k, t) \geq \Delta_m(M(A,B,st), 1) = M(A,B,st)$$

for all $k \in (0,1)$ and $t > 0$. Then letting $s \to 1$, we have

$$\lim_{k \to 1} M(gx_k, Tx_k, t) = M(A,B,t) \ (t > 0).$$

So we can create a sequence $\{x_n\}$ in A_0 such that

$$M(gx_n, Tx_n, t) \to M(A,B,t) \ (t > 0).$$

Since A_0 is compact, the sequence $\{x_n\}$ has a subsequence $\{x_{n_m}\}$ such that $x_{n_m} \to x \in A_0$. By Remark 4.7, g is continuous, and so $g - T$ is continuous mapping by Remark 4.8. Indeed, since Δ_m is a continuous t-norm, $M(x,y,.)$ is continuous, and we have

$$M(gx, Tx, t) = \lim_{m \to \infty} M(gx_{n_m}, Tx_{n_m}, t) = M(A,B,t),$$

as required.

Example 4.22. Let $X = \mathbb{R}$, $A = [0,2]$ and $B = [3,5]$. For every $x \in X$, define $M(x,y,t) = \frac{t}{t+|x-y|}$. It is easy to see that (X, M, Δ) is a complete fuzzy metric space, $M(A,B,t) = \frac{t}{t+1}$, $A_0 = \{2\}$ and $B_0 = \{3\}$. For every $x \in A$, define $T : A \to B$ by $Tx = 5 - x$ and let g be the identity mapping. Clearly, T is a continuous affine and approximal nonexpansive mapping, and $T(A_0) = \{T(2)\} = \{3\}$. Therefore, all hypotheses of Theorem 4.21 are satisfied, and also we have

$$M(2, T2, t) = M(2,3,t) = \frac{t}{t+1} = M(A,B,t).$$

Theorem 4.23. *Let (X,M,Δ) be a complete fuzzy metric space such that Δ is a t-norm of H-type, and (A,B) be a pair of subsets of X with the weak P-property such that A_0 is a nonempty closed set. If $T : A \to B$ is a contraction mapping such that $T(A_0) \subseteq B_0$, then there exists a unique x in A such that $M(x,Tx,t) = M(A,B,t)$ for all $t > 0$.*

Proof. It is a direct consequence of Remark 4.14 and Lemma 4.16. ■

The following example shows that the weak P-property of the pair (A,B) cannot be removed from Theorem 4.23.

Example 4.24. Let $X = \mathbb{R}$, $A = \{-10,10\}$, $B = \{-2,2\}$ and $M(x,y,t) = \frac{t}{t+|x-y|}$. Clearly, (X,M,Δ_m) is a complete fuzzy metric space. Then $A_0 = A, B_0 = B$, and $M(A,B,t) = \frac{t}{t+8}$. Let $T : A \to B$ be a mapping given by $T(-10) = 2$ and $T(10) = -2$. It is easy to see that for $c = 1/5$, T is a contraction mapping with $T(A_0) \subseteq B_0$. The mapping T does not have any best proximity point because $M(x,Tx,t) = \frac{t}{t+12} < \frac{t}{t+8} = M(A,B,t)$ for all $x \in A$. It should be noted that the pair (A,B) does not have the weak P-property.

Corollary 4.25. *Let (X,M,Δ) be a complete fuzzy metric space such that Δ is a t-norm of H-type. Then every contraction self-mapping from each nonempty closed subset of X has a unique fixed point.*

Definition 4.26. Let (X,M,Δ) be a fuzzy metric space. A pair (A,B) of nonempty subsets of X is called a semi-sharp proximal pair if there exists at most one $(x_0,y_0) \in A_0 \times B_0$ such that $M(x,y_0,t) = M(A,B,t) = M(x_0,y,t)$ for all $(x,y) \in A \times B$.

It is easy to check that if a pair (A,B) has the P-property, then the pair (A,B) is a semi-sharp proximal pair. Clearly, a semi-sharp proximal pair (A,B) does not necessarily have the P-property.

Example 4.27. Let $X = \mathbb{R}$, $A = \{-10,10\}$, $B = \{-2,2\}$ and $M(x,y,t) = \frac{t}{t+|x-y|}$. It is easy to verify that $M(A,B,t) = \frac{t}{t+8}$, $A = A_0, B_0 = B$, and the pair (A,B) is a semi-sharp proximal pair but does not have the P-property.

Theorem 4.28. *Let (X,M,Δ) be a complete fuzzy metric space, and (A,B) be a semi-sharp proximal pair of X such that A is a p-star-shaped set, A_0be a nonempty compact set, B be a q–star-shaped set, and let $M(p,q,t) = M(A,B,t)$ for all $t > 0$. If $T : A \to B$ is a proximal* *nonexpansive mapping such that $T(A_0) \subseteq B_0$, then there exists an element $x \in A_0$ such that $M(x,Tx,t) = M(A,B,t)$ for all $t > 0$.*

Proof. For each integer $k \geq 1$, define $T_k(x) = (1-\frac{1}{k})Tx + \frac{1}{k}q$ $(X \in A_0)$. Then by the hypothesis we have $T_k(A_0) \subseteq B_0$. Next, we show that for each k, T_k is a proximal contraction with $c = 1-\frac{1}{k} < 1$. To do this, suppose that $x,y,u,v,s,r \in A_0$ and $t > 0$ are such that

$$M(u,T_kx,t) = M(v,T_ky,t) = M(A_0,B_0,t) = M(A,B,t) = M(s,Tx,t) = M(r,Ty,t).$$

Now we define

$$u' = \left(1-\frac{1}{k}\right)s + \frac{1}{k}p \in A_0,\ v' = \left(1-\frac{1}{k}\right)r + \frac{1}{k}p \in A_0.$$

It follows that

$$\begin{aligned} M(A,B,t) &\geq M(u',T_kx,t) = M(\left(1-\frac{1}{k}\right)s + \frac{1}{k}p, \left(1-\frac{1}{k}\right)Tx + \frac{1}{k}q,t) \\ &\geq \Delta_m\left(M(\left(1-\frac{1}{k}\right)s, \left(1-\frac{1}{k}\right)Tx, \left(1-\frac{1}{k}\right)t), M(\frac{1}{k}p,\frac{1}{k}q,\frac{1}{k}t)\right) \\ &= \Delta_m(M(s,Tx,t),M(p,q,t)) \\ &= \Delta_m(M(A,B,t),M(A,B,t)) = M(A,B,t). \end{aligned}$$

Hence, $M(u',T_kx,t) = M(A,B,t)$. Since $M(u,T_kx,t) = M(A,B,t)$ and (A,B) is a semi-sharp proximal pair, we have $u' = u$. By the same method we also have $v' = v$. Since T is a proximal nonexpansive mapping, we have

$$M(u,v,t) = M(u',v',t) = M((1-1/k)s,(1-1/k)r,t) = M(s,r,\frac{t}{1-1/k}) \geq M(x,y,\frac{t}{1-1/k}).$$

Therefore, T_k is a proximal contraction with $c = 1-1/k < 1$. By Lemma 4.16, for each $k \geq 1$, there exists a unique $u_k \in A_0$ such that $M(u_k,T_ku_k,t) = M(A_0,B_0,t) = M(A,B,t)$. Since A_0 is compact and $\{u_k\} \subseteq A_0$, without loss of generality, we can assume that $\{u_k\}$ is a convergent sequence and $u_k \to x \in A_0$.

For each $k \geq 1$, since $T(u_k) \in T(A_0) \subseteq B_0$,there exists $v_k \in A_0$ such that $M(v_k,Tu_k,t) =$

$M(A,B,t)$. So we have

$$\begin{aligned} M(A,B,t) &\geq M\left((1-\frac{1}{k})v_k+\frac{1}{k}p, T_k u_k, t\right) \\ &= M\left((1-\frac{1}{k})v_k+\frac{1}{k}p, (1-\frac{1}{k})Tu_k+\frac{1}{k}q, t\right) \\ &\geq \Delta_m\left(M((1-\frac{1}{k})v_k, (1-\frac{1}{k})Tu_k, (1-\frac{1}{k})t), M(\frac{1}{k}p, \frac{1}{k}q, \frac{1}{k}t\right) \\ &= \Delta_m(M(v_k, Tu_k, t), M(p,q,t)) = \Delta_m(M(A,B,t), M(A,B,t)) = M(A,B,t). \end{aligned}$$

Thus, $M(A,B,t) = M\left((1-\frac{1}{k})v_k+\frac{1}{k}p, T_k u_k, t\right)$. Since (A,B) is a semi-sharp proximal pair and $M(A,B,t) = M(u_k, T_k u_k, t)$, we have $u_k = (1-\frac{1}{k})v_k + \frac{1}{k}p$, and so

$$M(u_k, v_k, t) = M(\frac{1}{k}v_k, \frac{1}{k}p, t) = M(v_k, p, kt).$$

Since A_0 is compact and $\{z_k\} \subseteq A_0$, without loss of generality, we can assume that $\{z_k\}$ is a convergent sequence and $z_k \to z \in A_0$. For every $j \leq k$, we have

$$M(u_k, z_k, t) = M(p, z_k, t) \geq M(z_k, p, jt) \geq \Delta_m(M(z_k, z, jt/2), M(z, p, jt/2)).$$

Letting $k \to \infty$, we have

$$\lim_{k\to\infty} M(u_k, z_k, t) \geq M(z, p, jt/2)\ (\forall j \geq 1).$$

Letting $j \to \infty$, we have

$$\lim_{k\to\infty} M(u_k, z_k, t) \geq \lim_{j\to\infty} M(z, p, jt/2) = 1.$$

Therefore, $M(u_k, z_k, t) \to 1$, so that $z = \lim_{k\to\infty} u_k = \lim_{k\to\infty} z_k = x$. Since $Tx \in B_0$, there must exists $u \in A_0$ such that $M(A,B,t) = M(u, Tx, t)$. Since we know that $M(A,B,t) = M(z_k, Tu_k, t)$ and T is a proximal nonexpansive mapping, it follows that $M(z_k, u, t) \geq M(u_k, x, t) \to 1$. This implies that $u = \lim_{k\to\infty} z_k = x$. Then we have $M(A,B,t) = M(x, Tx, t)$.

Example 4.29. Let $X = \mathbb{R}$, $A = [0,1]$, and $B = [2,3]$. For every $x \in X$, define $M(x,y,t) = \frac{t}{t+|x-y|}$. it is easy to see that that (X, M, Δ_m) is a complete fuzzy metric space. A is 1-star-shaped set, B is 2-star-shaped set,

$$M(A,B,t) = \sup_{x\in A, y\in B} M(x,y,t) = \frac{t}{t1},\ A_0 = \{1\},\ B_0 = \{2\},$$

and

$$M(1,2,t)=\frac{t}{t+1}=M(A,B,t),$$

Also, (A,B) is a semi-sharp proximal pair. Now for each $x\in A$. Define $T:A\to B$ by $Tx=3-x$. If $u,v,x,y\in A$, then

$$\frac{t}{t+|u-3+x|}=M(u,Tx,t)=M(A,B,t)=M(v,Ty,t)=\frac{t}{t+|v-3+y|}.$$

It follows that $u=x=1$ and $v=y=1$. Thus

$$M(u,v,t)=M(x,y,t).$$

So, T is a proximal non-expansive, and $T(A_0)=B_0$. Therefore, all the hypotheses of Theorem 4.28 are satisfied, and we also have

$$M(1,T1,t)=M(1,2,t)=\frac{t}{t+1}=M(A,B,t).$$

Theorem 4.30. *Let (X,M,Δ_m) be a complete fuzzy metric space, (A,B) be a semisharp proximal pair of X with the weak P-property such that A is a p-star-shaped set, A_0 be a nonempty compact set, B be a q-star-shaped set, and let $M(p,q,t)=M(A,B,t)$ for all $t>0$. If $T:A\to B$ is a nonexpansive mapping such that $T(A_0)\subseteq B_0$, then T has a best proximity point in A_0.*

Proof. It is a direct consequence of Remark 4.14 and Theorem 4.28.

Example 4.31. Let $X=\mathbb{R}^2$, $A=\{(x,0):x\in[0,1]\}$, $B_1=\{(x,y):x+y=1,x\in[-1,0]\}$, $B_2=\{(x,1):x\in[0,1]\}$, $B=B_1\cup B_2$, and $M(x,y,t)=\frac{t}{t+|x|+|y|}$. It is easy to see that (X,M,Δ_m) is a complete fuzzy metric space, $M(A,B,t)=\frac{t}{t+1}$, B is not convex but is a $(0,1)$-star-shaped set , and A is $(0,0)$-star-shaped set. Clearly, $A_0=A$ and $B_0=B$. So

$$M((0,0),(0,1),t)=\frac{t}{t+|0|+|1|}=\frac{t}{t+1}=M(A,B,t),$$

and (A,B) is a semisharp proximal pair. Suppose that $T:A\to B$ is defined by

$$T(x,0)=\begin{cases}(0,1), & \text{if } x=0;\\ (\sin x,1), & \text{if } x\neq 0.\end{cases}$$

and $(u,0),(v,0),(x,0),(y,0)\in A$ are such that

$$M((u,0),T(x,0),t)=M(A,B,t)=\frac{t}{t+1}=M((v,0),T(y,0),t).$$

If $x = y = 0$, then $u = v = 0$. Hence,

$$M((u,0),T(x,0),t) = M((0,0),(0,0),t) = 1 = M((x,0),(y,0),t).$$

If $x, y \neq 0$, then $u = \sin x, v = \sin y$. Therefore,

$$M((u,0),(v,0),t) = M((\sin x,0),(\sin y,0),t) = \frac{t}{t + |\sin x - \sin y|} \geq \frac{t}{t + |x-y|} = M((x,0),(y,0),t).$$

If $x = 0$ and $y \neq 0$, then $u = 0$ and $v = \sin y$, and hence

$$M((u,0),(v,0),t) = M((0,0),(\sin y,0),t) = \frac{t}{t + |\sin y|} \geq \frac{t}{t + |y|} \geq M((x,0),(0,0),t).$$

If $x \neq 0$ and $y = 0$, then $u = \sin x$ and $v = 0$. Hence,

$$M((u,0),(v,0),t) = M((\sin x,0),(0,0),t) = \frac{t}{t + |\sin x|} \geq \frac{t}{t + |x|} \geq M((x,0),(0,0),t).$$

Therefore, T is proximal nonexpansive, and $T(A_0) \subseteq B_2 = B_0$. So all the hypotheses of Theorem 4.30 are satisfied, and we also have

$$M((0,0),T(0,0),t) = M((0,0),(1,0),t) = \frac{t}{t+1} = M(A,B,t).$$

Proposition 4.32. *Let (X, M, Δ) be a fuzzy metric space, and $A, B \subseteq X$ be such that A_0 is a nonempty set. Suppose that $T : A \to B$ is a proximal nonexpansive mapping such that $T(A_0) \subseteq B_0$ and $g : A \to A$ is an isometry mapping such that $A_0 \subseteq g(A_0)$. Denote $G = g(A)$ and*

$$G_0 = \{z \in G : \exists y \in B \, \text{such that} \, \forall t \geq 0, M(z,y,t) = M(G,B,t)\}.$$

Then Tg^{-1} is a proximal nonexpansive, and $G_0 = A_0$.

The following theorem is an immediate consequence of Theorem 4.30 and Proposition 4.32.

Theorem 4.33. *Let (X, M, Δ_m) be a complete fuzzy metric space, (A, B) be a semisharp proximal pair of X such that A is a p-star-shaped set, A_0 be a nonempty compact set, B be a q-star-shaped set, and let $M(p,q,t) = M(A,B,t)$ for all $t > 0$. If $T : A \to B$ is a proximal nonexpansive mapping such that $T(A_0) \subseteq B_0$ and $g : A \to A$ is an isometry mapping such that $A_0 \subseteq g(A_0)$, then there exists an element $x \in A_0$ such that $M(g(x), Tx, t) = M(A,B,t)$ for all $t > 0$.*

Corollary 4.34. *Let (X, M, Δ_m) be a complete fuzzy metric space, (A, B) be a pair of convex subsets of X with the P-property such that A_0 is a nonempty compact set. If $T : A \to B$ is a nonexpansive mapping such that $T(A_0) \subseteq B_0$ and $g : A \to A$ is an isometry mapping such that*

$A_0 \subseteq g(A_0)$, then there exists an element $x \in A_0$ such that $M(g(x),Tx,t) = M(A,B,t)$ for all $t > 0$.

In Corollary 4.34, if $g(x) = x$, then we have the following corollary.

Corollary 4.35. *Let (X,M,Δ_m) be a complete fuzzy metric space, (A,B) be a pair of convex subsets of X with the P-property such that A_0 is a nonempty compact set. If $T : A \to B$ is a nonexpansive mapping such that $T(A_0) \subseteq B_0$, then T has a best proximity point.*

In Corollary 4.35, if $A = B$, then we have the following collaray.

Corollary 4.36. *If A is a nonempty, compact, and convex subset of a complete fuzzy metric space (X,M,Δ_m) and $T : A \to A$ is a nonexpansive mapping, then T has a fixed point.*

Example 4.37. Let $X = \mathbb{R}$, $A = [0,1]$, $B = [15/8,2]$, and $M(x,y,t) = \frac{t}{t+|x-y|}$. Clearly, (X,M,Δ_m) is a complete metric space, $M(A,B,t) = \frac{t}{t+7/8}$, the pair (A,B) has the P-property, $A_0 = \{1\}$, and $B = \frac{15}{8}$. If $Tx = -\frac{1}{8}x+2$, then $T(A_0) = \{T(1)\} = \{\frac{15}{8}\} = B_0$. Let $x,y \in A$. Then we have

$$M(Tx,Ty,t) = M(\frac{1}{8}x,\frac{1}{8}y,t) = M(x,y,8t) \geq M(x,y,t).$$

Therefore, all the hypotheses of Corollary 4.35 are satisfied. Hence T has a best proximity point, and we also have

$$M(1,T1,t) = M(1,\frac{15}{8},t) = \frac{t}{t+7/8} = M(A,B,t).$$

REFERENCES

[1] M.S. Brodskii, D.P. Milman, On the center of a convex set, Doki. Akad. Nauk. SSSR, 59 (1948), 837–840.

[2] M.S. Browder, Non-expansive nonlinear operators in a Banach space, Proc. Nat. Acad. Sci. USA 54 (1965), 1041–1044.

[3] S.S. Chang, Fixed point theorem in problabilistic metric spaces with applications, Math. Sci. Sinica (Series A) 26 (1983), 1144–1155.

[4] M.S.El Naschie, A review of E-infinity theory and the mass spectrum of high energy particle physics, Chaos Solitons Fractals, 19 (2004), 209–236.

[5] R.J. Egbert, Products and quotients of probabilistic metric spaces, Pacific J. Math. 24 (1968), 437–455.

[6] K. Fan, Existensions of two fixed point theorems of F. E. Browder, Math. Zeit. 112 (1969), 234–240.

[7] A. George and P. Veeramani, On some results in fuzzy metric spaces, J. Fuzzy Sets Sys. 64 (1994), 395–399.

[8] D. Göhde, Zum prinzip der Konttraktiven Abbildung, Math. Nachr. 30 (1965), 251–258.

[9] M. Grabiec, Fixed points in fuzzy metric spaces, Fuzzy Set Sys. 27 (1998), 385–390.

[10] O. Hadžić, Common fixed point theorems in probabilistic metric spaces with convex structure, Zb. rad. Prirod-Mat.Fak.ser. Math. 18 (1987),165–178.

[11] O.Hadžić, E. Pap, Fixed point theory in probabilistic metric space, Kluwer Acadmic, Dordrecht, 2001.

[12] S.N. Ješić, Convex structure, normal structure and a fixed point theorem in intuitionistic fuzzy metric space, Chaos, Solitions & Fractals, 41 (2008), 292–301.

[13] S.N. Ješić, R. M. Nikolic and A. Babačev, A common fixed point theorem in strictly convex Menger PM-spaces, Filomat 28 (2016), 735–743.

[14] W.A. Kirk, A fixed point theorem for mappings which do not increase distances, Amer. Math. Monthly 72 (1965), 1004–1006.

[15] K. Menger, Statistical metrics, Proc. Natl. Acad. Sci. USA. 28 (1942) 535–537.

[16] M. Koireng, L. Ningombam, Y. Rohen, Common Fixed Points of Compatible Mappings of Type (R), General Math. Notes 10 (2012), 58–62.

[17] O. Kramosil, J.Michalek, Fuzzy metric and Statistical metric spaces, Kybernetika, 11 (1975), 326–334.

[18] S.N. Mishra, N. Mishra, S.L.Singh, Common fixed point of maps in fuzzy metric space, Int. J. Math. Math. Sci. 17 (1994), 253–258.

[19] V.M. Sehgal, A.T. Bharucha-Reid, Fixed points of contraction mappings in PM-spaces, Math. System Theory, 16 (1972), 97–102.

[20] B. Schweizer, A. Sklar, Probabilistical Metric Spaces, Dover Publications, New York, 2005.

[21] H. Shayanpour, M. Shams, A. Nematizadeh, Some results on best proximity point on star-shaped sets in probabilistic Banach (Menger) spaces, Fixed point Theory Appl. 2016 (2016), Article ID 13.

[22] W. Takahashi, A convexity in metric space and nonexpansive mappings, Kodai Math. Sem. Rep. 22 (1970), 142–149.

[23] L.A. Zadeh, Fuzzy Sets, Information and Control 8 (1965), 338–353.