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FIXED POINT THEOREMS FOR MONOTONE OPERATORS IN ORDERED BANACH SPACES WITH h-ORDERING DIFFERENCES

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Abstract. In this paper, we use the partial order theory, h-ordering differences and altering distance functions to study monotone operators. The existence and uniqueness of fixed points without assuming the operator to be compact and continuous are obtained.

Keywords. Fixed point; Monotone operator; $h$ordering difference; Altering distance function; Normal cone.

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1. Introduction and preliminaries

In the past several decades, monotone operators in ordered Banach spaces have been studied extensively, and many new fixed point theorems have been established; see [1-19] and the references therein. Most of these new results have been proved based on the cone theory and monotone iterative technique.

The purpose of this paper is to present some new fixed point theorems for monotone operators in ordered Banach spaces. We use the partial order theory, $h-$ordering differences and

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altering distance functions to study monotone operators. We also get the existence and uniqueness of fixed points without assuming the operator to be compact and continuous. Our results compliment the theory of monotone operators in ordered Banach spaces.

For the discussion of the following section, we state here some definitions and notations. For convenience of readers, we refer to [1,2,19] for details.

Suppose that $(E, \|\cdot\|)$ is a real Banach space which is partially ordered by a cone $P \subset E, i.e., x \leq y$ if and only if $y - x \in P$. By θ we denote the zero element of E. P is called normal if there exists a constant $N > 0$ such that, for all $x, y \in E,\ \theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$; in this case N is called the normality constant of P. If $x_1, x_2 \in E$, the set $[x_1, x_2] = \{x \in E | x_1 \leq x \leq x_2\}$ is called the order interval between x_1 and x_2. We say that an operator $A : E \to E$ is increasing (decreasing) if $x \leq y$ implies $Ax \leq Ay$ $(Ax \geq Ay)$.

Definition 1.1. [20] Let E be a real Banach space and P is a cone in E. Suppose $h \in P$ and $\theta \leq u \leq v \leq Mh$, where $M > 0$. Set $a = \inf\{\alpha | v \leq \alpha h\}, b = \sup\{\beta | \beta h \leq u\}$. Then $d_h(u, v) := a - b$ is called $h-$ordering difference of u, v.

Definition 1.2. [21] A function $\varphi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if the following properties are satisfied:

(1) $\varphi(t) = 0$ if and only if $t = 0$.

(2) φ is continuous and strictly increasing.

Lemma 1.1. *Let E be a real Banach space and P is a cone in E. Suppose $h \in P$ and $\theta \leq u \leq v \leq Mh$, where $M > 0$. Then*

(i) $d_h(u, v) \geq 0,\ d_v(u, v) \leq 1$;

(ii) if $d_h(u, v) = 0$*, then* $u = v$;

(iii) $d_h(u, v) \leq d_h(u, w) + d_h(w, v), \forall\ w \in [u, v]$. *Moreover, " = " is satisfied iff* $d_h(w, w) = 0$*;*

(iv) if $\theta \leq u_1 \leq u \leq v$*, then* $d_h(u_1, v) \geq d_h(u, v)$;

(v) $d_h(ku, kv) = kd_h(u, v),\ d_v(kv, v) = 1 - k,\ \forall\ k \in [0, 1]$.

Proof. Let $a = \inf\{\alpha | v \leq \alpha h\}$ and $b = \sup\{\beta | \beta h \leq u\}$.

(i) Since $\theta \le u \le v, h \in P$, we have $\theta \le bh \le u \le v \le ah$, and then $0 \le b \le a$. By Definition 1.1, we get $d_h(u,v) = a-b \ge 0$. Set $a_1 = \inf\{\alpha | v \le \alpha v\}$, $b_1 = \sup\{\beta | \beta v \le u\}$. From $\theta \le u \le v$, we have $0 \le b_1 \le a_1 = 1$. Hence, $d_v(u,v) = a_1 - b_1 \le 1$.

(ii) If $d_h(u,v) = 0$, then $a = b$. It follows from $\theta \le bh \le u \le v \le ah$ that $u = v$.

(iii) From $\theta \le u \le w \le v \le Mh$, we know that $d_h(u,w), d_h(w,v)$ exist. Let $c = \inf\{\alpha | w \le \alpha h\}$, $d = \sup\{\beta | \beta h \le w\}$. Since $w \le ch, v \le ah$ and $w \le v$, we can easily prove $c \le a$. Similarly, $b \le d$. Hence, $0 \le b \le d \le c \le a$ and

$$d_h(u,v) = a-b \le a-b+(c-d) = c-b+a-d = d_h(u,w)+d_h(w,v).$$

If $d_h(u,v) = d_h(u,w)+d_h(w,v)$, then $a-b = c-b+a-d$ and thus $c-d = 0$. Therefore, $d_h(w,w) = 0$. Conversely, if $d_h(w,w) = 0$, then $c-d = 0$ and

$$d_h(u,v) = a-b = c-b+a-d = d_h(u,w)+d_h(w,v).$$

(iv) Letting $b_2 = \sup\{\beta | \beta h \le u_1\}$, we get $b_2 \le b$. So $a-b_2 \ge a-b$, that is, $d_h(u_1,v) \ge d_h(u,v)$.

(v) From Definition 1.1, we can easily obtain $d_h(ku,kv) = ka-kb = kd_h(u,v)$, $d_v(kv,v) = 1-k$.

Remark 1.1. Lemma 1.1 was given in [20]. Here we present its proof only for completeness.

Lemma 1.2. *Let E be a real Banach space and P is a cone in E. Suppose $h \in P$ and $\theta \le u \le v \le Mh$, where $M > 0$. Then*
(i) if $u \le v_1 \le v$, then $d_h(u,v_1) \le d_h(u,v)$;
(ii) suppose that $\theta \le x_n \le x_0 \le v$ with $x_n \to x_0$ $(n \to \infty)$, then $d_h(x_n,x_0) \to 0$ $(n \to \infty)$. Conversely, if $d_h(x_n,x_0) \to 0$, then $x_n \to x_0$ $(n \to \infty)$.

Proof. (i) Let $a = \inf\{\alpha | v \le \alpha h\}$, $b = \sup\{\beta | \beta h \le u\}$ and $a^{'} = \inf\{\alpha | v_1 \le \alpha h\}$. Then $d_h(u,v_1) = a^{'} - b$, $d_h(u,v) = a-b$. From $v_1 \le v$, we have $a^{'} \le a$ and then $d_h(u,v_1) \le d_h(u,v)$.

(ii) Let $a=\inf\{\alpha|x_0\leq \alpha h\},\ b_n=\sup\{\beta_n|\beta_n h\leq x_n\}$. Then

$$\begin{aligned} d_h(x_n,x_0) &= a-b_n=\inf\{\alpha|x_0\leq \alpha h\}-\sup\{\beta_n|\beta_n h\leq x_n\} \\ &= \inf\{\alpha|x_0\leq \alpha h\}+\inf\{-\beta_n|-x_n\leq -\beta_n h\} \\ &= \inf\{\alpha-\beta_n|\theta\leq x_0-x_n\leq (\alpha-\beta_n)h\}. \end{aligned}$$

It follows from $x_n\to x_0\ (n\to\infty)$ that $\lim\limits_{n\to\infty}(\alpha-\beta_n)h\geq\theta$. So $\lim\limits_{n\to\infty}d_h(x_n,x_0)=\lim\limits_{n\to\infty}\inf\{\alpha-\beta_n|(\alpha-\beta_n)h\geq\theta\}=0$. That is, $d_h(x_n,x_0)\to 0\ (n\to\infty)$. Conversely, if $d_h(x_n,x_0)\to 0\ (n\to\infty)$, we have $a-b_n\to 0\ (n\to\infty)$. From $x_0\leq ah, x_n\geq b_nh$, we obtain $\theta\leq x_0-x_n\leq ah-b_nh=(a-b_n)h$. Therefore, $x_0-x_n\to\theta\ (n\to\infty)$.

2. Main results

In this section, using the partial order theory, $h-$ordering differences and altering distance functions, we present the existence and uniqueness of fixed points for monotone operators in ordered Banach spaces.

Theorem 2.1. *Let E be a real Banach space and P be a normal cone in $E, \theta\leq u_0\leq v_0$. $A:[u_0,v_0]\to E$ is an increasing operator which satisfies:*
(i) $u_0\leq Au_0,\ Av_0\leq v_0$;
(ii) for $x,y\in[u_0,v_0]$ with $x\leq y$, there exist an altering distance function φ and a constant $k\in(0,1)$ such that $\varphi(d_{v_0}(Ax,Ay))\leq k\varphi(d_{v_0}(x,y))$.
Then A has a unique fixed point x^* in $[u_0,v_0]$. Moreover, constructing successively the sequence $x_n=Ax_{n-1},\ n=1,2,\ldots$ for any initial value $x_0\in[u_0,v_0]$, we have $\|x_n-x^*\|\to 0$ as $n\to\infty$.

Proof. Let

$$u_n=Au_{n-1}, v_n=Av_{n-1}, n=1,2,\ldots. \tag{2.1}$$

From condition (i) and the monotonicity of A, we have

$$u_0\leq u_1\leq u_2\leq\cdots\leq u_n\leq\cdots\leq v_n\leq\cdots\leq v_2\leq v_1\leq v_0. \tag{2.2}$$

We define $\tau_n = d_{v_0}(u_n, v_n), n = 0, 1, 2, \ldots$. By (i) of Lemma 1.1, we know $\tau_n \in [0, 1]$. Next we show that $\{\tau_n\}$ is monotone. From (2.2), (iv) of Lemma 1.1 and (i) of Lemma 1.2, we obtain

$$\tau_n = d_{v_0}(u_n, v_n) \leq d_{v_0}(u_{n-1}, v_n) \leq d_{v_0}(u_{n-1}, v_{n-1}) = \tau_{n-1}.$$

That is, $\{\tau_n\}$ is a decreasing sequence. So $\{\tau_n\}$ has a limit and we put $\lim_{n\to\infty} \tau_n = \tau$. By condition (ii), we have

$$\begin{aligned}
\varphi(\tau_n) &= \varphi(d_{v_0}(u_n, v_n)) = \varphi(d_{v_0}(Au_{n-1}, Av_{n-1})) \\
&\leq k\varphi(d_{v_0}(u_{n-1}, v_{n-1})) = k\varphi(d_{v_0}(Au_{n-2}, Av_{n-2})) \\
&\leq k^2\varphi(d_{v_0}(u_{n-2}, v_{n-2})) = k^2\varphi(d_{v_0}(Au_{n-3}, Av_{n-3})) \\
&\leq \cdots \leq k^n\varphi(d_{v_0}(u_0, v_0)) = k^n\varphi(\tau_0).
\end{aligned}$$

Let $n \to \infty$, since φ is continuous, we get $\varphi(\tau) \leq 0$. From the definition of φ, we have $\varphi(\tau) = 0$ and then $\tau = 0$. That is, $d_{v_0}(u_n, v_n) \to 0$ as $n \to \infty$. Now we prove that u_n, v_n are two Cauchy sequences. Let $a_n = \inf\{\alpha | v_n \leq \alpha v_0\},\ b_n = \sup\{\beta | \beta v_0 \leq u_n\}$. Then we have

$$a_n - b_n \to 0\ (n \to \infty),\ b_n v_0 \leq u_n \leq v_n \leq a_n v_0. \tag{2.3}$$

By (2.2), (2.3), we get

$$\theta \leq u_{n+p} - u_n \leq v_n - u_n,\ \theta \leq v_n - v_{n+p} \leq v_n - u_n,\ \theta \leq v_n - u_n \leq (a_n - b_n)v_0.$$

Since P is normal, we have

$$\| v_n - u_n \| \leq N(a_n - b_n) \| v_0 \| \to 0 \text{ (as } n \to \infty).$$

Further,

$$\| u_{n+p} - u_n \| \leq N \| v_n - u_n \| \to 0 \text{ (as } n \to \infty),$$

$$\| v_n - v_{n+p} \| \leq N \| v_n - u_n \| \to 0 \text{ (as } n \to \infty).$$

Here N is the normality constant.

So we can claim that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences. Because E is complete, there exist $u^*, v^* \in E$ such that $u_n \to u^*,\ v_n \to v^*$ as $n \to \infty$. By (2.2), we know that

$$u_0 \leq u_n \leq u^* \leq v^* \leq v_n \leq v_0,$$

and then $\theta \leq v^* - u^* \leq v_n - u_n$. Further,

$$\| v^* - u^* \| \leq N \| v_n - u_n \| \to 0 \text{ (as } n \to \infty),$$

and thus $v^* = u^*$. Set $x^* := u^* = v^*$.

Next we prove x^* is a fixed point of A. By condition (ii), we have

$$\varphi(d_{v_0}(u_{n+1}, Ax^*)) = \varphi(d_{v_0}(Au_n, Ax^*)) \leq k\varphi(d_{v_0}(u_n, x^*)).$$

From (ii) of Lemma 1.2 and Definition 1.2, we obtain $\varphi(d_{v_0}(u_n, x^*)) \to 0$ as $n \to \infty$. Hence, $\varphi(d_{v_0}(u_{n+1}, Ax^*)) \to 0$ as $n \to \infty$. Also from Definition 1.2, we get $d_{v_0}(u_{n+1}, Ax^*) \to 0$ as $n \to \infty$. By (ii) of Lemma 1.2, $u_{n+1} \to Ax^*$ as $n \to \infty$. Therefore, $Ax^* = x^*$.

In the following, we prove that x^* is the unique fixed point of A in $[u_0, v_0]$. Suppose that there is $y^* \in [u_0, v_0]$ such that $Ay^* = y^*$. Then we have $u_0 \leq y^* \leq v_0$. By induction method and the monotonicity of A, we have

$$u_n = Au_{n-1} \leq y^* = Ay^* \leq Av_{n-1} = v_n, \ \ n = 1, 2, \ldots.$$

Then from the normality of P, we have $y^* = x^*$.

Moreover, constructing successively the sequence $x_n = Ax_{n-1}, \ n = 1, 2, \ldots$ for any initial value $x_0 \in [u_0, v_0]$, from the proof of uniqueness for fixed points, we have $\| x_n - x^* \| \to 0$ as $n \to \infty$.

Remark 2.1. (1) From the proof of Theorem 2.1, we can see that we do not use the monotonicity of φ. However, we can show that φ can not be decreasing. In fact, if we assume that φ is decreasing, then $\varphi(\tau_n) \geq \varphi(\tau_{n-1})$ and $\varphi(\tau_n) \leq k\varphi(\tau_{n-1})$, we get $k \geq 1$, this is a contradiction.
(2) If we take a special altering distance function $\varphi(t) = t$, then we have the following corollary.

Corollary 2.1. (See Theorem 1 in [20]) *Let E be a real Banach space and P be a normal cone in $E, \theta \leq u_0 \leq v_0$. $A : [u_0, v_0] \to E$ is an increasing operator which satisfies:*
(i) $u_0 \leq Au_0, \ Av_0 \leq v_0$;
(ii) for $x, y \in [u_0, v_0]$ with $x \leq y$, there exists a constant $k \in (0, 1)$ such that $d_{v_0}(Ax, Ay) \leq kd_{v_0}(x, y)$.
Then, A has a unique fixed point x^* in $[u_0, v_0]$. Moreover, constructing successively the sequence $x_n = Ax_{n-1}, \ n = 1, 2, \ldots$ for any initial value $x_0 \in [u_0, v_0]$, we have $\| x_n - x^* \| \to 0$ as $n \to \infty$.

Remark 2.2. In [20], the proof of Corollary 2.1 was not complete, because the author has not showed that x^* is a fixed point of A. Here we first establish Lemma 1.2 and then we can prove the conclusion by using Lemma 1.2.

Theorem 2.2. *Let E be a real Banach space and P be a normal cone in $E, \theta \leq u_0 \leq v_0$. $A : [u_0, v_0] \to E$ is a decreasing operator which satisfies:*
(i) $u_0 \leq Av_0,\ Au_0 \leq v_0$;
(ii) for $x, y \in [u_0, v_0]$ with $x \leq y$, there exist an altering distance function φ and a constant $k \in (0,1)$ such that $\varphi(d_{v_0}(Ay, Ax)) \leq k\varphi(d_{v_0}(x, y))$.
Then, A has a unique fixed point x^* in $[u_0, v_0]$. Moreover, constructing successively the sequence $x_n = Ax_{n-1},\ n = 1, 2, \ldots$ for any initial value $x_0 \in [u_0, v_0]$, we have $\| x_n - x^* \| \to 0$ as $n \to \infty$.

Proof. Let $B = A^2$. Then $B : [u_0, v_0] \to E$ is an increasing operator and

$$Bu_0 = A(Au_0) \geq Av_0 \geq u_0,\ Bv_0 = A(Av_0) \leq Au_0 \leq v_0.$$

Further, for $x, y \in [u_0, v_0]$ with $x \leq y$, we have

$$\varphi(d_{v_0}(Bx, By)) = \varphi(d_{v_0}(A(Ax), A(Ay))) \leq k\varphi(d_{v_0}(Ay, Ax)) \leq k^2\varphi(d_{v_0}(x, y)).$$

Hence, B satisfies all the conditions of Theorem 2.1. So operator B has a unique fixed point x^* in $[u_0, v_0]$. Moreover, constructing successively the sequence $z_n = Bz_{n-1},\ n = 1, 2, \ldots$ for any initial value $z_0 \in [u_0, v_0]$, we have $z_n \to x^*$ as $n \to \infty$.

Next we show that x^* is the unique fixed point of A in $[u_0, v_0]$. First, we know $u_0 \leq Av_0 \leq Ax^* \leq Au_0 \leq v_0$. So

$$B(Ax^*) = A^2(Ax^*) = A(A^2x^*) = Ax^*.$$

That is, Ax^* is a fixed point of B in $[u_0, v_0]$. By the uniqueness of fixed points for B, we get $Ax^* = x^*$. So x^* is a fixed point of A in $[u_0, v_0]$. On the other hand, suppose y^* is another fixed point of A in $[u_0, v_0]$. Then

$$A^2(y^*) = A(Ay^*) = Ay^* = y^*,$$

and thus y^* is a fixed point of B in $[u_0, v_0]$. By the uniqueness of fixed points for B, we get $y^* = x^*$. This proves the uniqueness of fixed points for operator A.

Finally, take $z_0 = x_0$ or $z_0 = Ax_0$, we obtain $x_{2n} = A^{2n}x_0 = Bz_{n-1} \to x^*$ or $x_{2n+1} = A^{2n+1}x_0 = A^{2n}z_0 = Bz_{n-1} \to x^*$ as $n \to \infty$.

Corollary 2.2. *Let E be a real Banach space and P be a normal cone in $E, \theta \leq u_0 \leq v_0$. $A : [u_0, v_0] \to E$ is a decreasing operator which satisfies:*
(i) $u_0 \leq Av_0,\ Au_0 \leq v_0$;
(ii) for $x, y \in [u_0, v_0]$ with $x \leq y$, there exists a constant $k \in (0,1)$ such that $d_{v_0}(Ay, Ax) \leq kd_{v_0}(x,y)$.
Then, A has a unique fixed point x^* in $[u_0, v_0]$. Moreover, constructing successively the sequence $x_n = Ax_{n-1},\ n = 1, 2, \ldots$ for any initial value $x_0 \in [u_0, v_0]$, we have $\| x_n - x^* \| \to 0$ as $n \to \infty$.

Theorem 2.3. *Let E be a real Banach space and P be a normal cone in $E, \theta \leq u_0 \leq v_0$. $A : [u_0, v_0] \to E$ is an increasing operator which satisfies:*
(i) $u_0 \leq Au_0,\ Av_0 \leq v_0$;
(ii) for $x \in [u_0, v_0]$, there exists a constant $k \in (0,1)$ such that $d_{v_0}(Ax, v_0) \leq kd_{v_0}(x, v_0)$.
Then, A has a unique fixed point v_0 in $[u_0, v_0]$. Moreover, constructing successively the sequence $x_n = Ax_{n-1},\ n = 1, 2, \ldots$ for any initial value $x_0 \in [u_0, v_0]$, we have $\| x_n - v_0 \| \to 0$ as $n \to \infty$.

Proof. Let $u_n = Au_{n-1}, v_n = Av_{n-1}, n = 1, 2, \ldots$. From the condition (i) and the monotonicity of A, we have

$$u_0 \leq u_1 \leq u_2 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_2 \leq v_1 \leq v_0.$$

Further, from condition (ii), we have

$$d_{v_0}(u_n, v_0) = d_{v_0}(Au_{n_1}, v_0) \leq kd_{v_0}(u_{n_1}, v_0) \leq \cdots \leq k^n d_{v_0}(u_0, v_0) \leq k^n.$$

Letting $n \to \infty$, we obtain $d_{v_0}(u_n, v_0) \to 0$. By (ii) of Lemma 1.2, we have $u_n \to v_0$ as $n \to \infty$. From Lemma 1.2, we have

$$d_{v_0}(u_n, Av_0) \leq d_{v_0}(u_n, v_0) \to 0,\ n \to \infty.$$

Also by (ii) of Lemma 1.2, we have $u_n \to Av_0$ as $n \to \infty$. Therefore, $Av_0 = v_0$. That is, v_0 is a fixed point of A. Similar to the proof of Theorem 2.1, v_0 is the unique fixed point of A in $[u_0, v_0]$. Moreover, constructing successively the sequence $x_n = Ax_{n-1},\ n = 1, 2, \ldots$ for any initial value $x_0 \in [u_0, v_0]$, we have $\| x_n - v_0 \| \to 0$ as $n \to \infty$.

Theorem 2.4. *Let E be a real Banach space and P be a normal cone in $E, \theta \leq u_0 \leq v_0$. $A : [u_0, v_0] \to E$ is a decreasing operator which satisfies:*

(i) $u_0 \leq Av_0,\ Au_0 \leq v_0$;

(ii) for $x \in [u_0, v_0]$, there exists a constant $k \in (0,1)$ such that $d_{v_0}(Ax, v_0) \leq kd_{v_0}(x, v_0)$.

Then, A has a unique fixed point v_0 in $[u_0, v_0]$. Moreover, constructing successively the sequence $x_n = Ax_{n-1},\ n = 1, 2, \ldots$ for any initial value $x_0 \in [u_0, v_0]$, we have $\| x_n - v_0 \| \to 0$ as $n \to \infty$.

Proof. Let $B = A^2$. Then $B : [u_0, v_0] \to E$ is an increasing operator and

$$Bu_0 = A(Au_0) \geq Av_0 \geq u_0,\ Bv_0 = A(Av_0) \leq Au_0 \leq v_0.$$

Further, for $x \in [u_0, v_0]$, we have

$$d_{v_0}(Bx, v_0) = d_{v_0}(A(Ax), v_0) \leq kd_{v_0}(Ay, v_0) \leq k^2\varphi(d_{v_0}(x, v_0)).$$

Hence, B satisfies all the conditions of Theorem 2.3. So operator B has a unique fixed point v_0 in $[u_0, v_0]$. Moreover, constructing successively the sequence $z_n = Bz_{n-1},\ n = 1, 2, \ldots$ for any initial value $z_0 \in [u_0, v_0]$, we have $z_n \to v_0$ as $n \to \infty$. The rest proof is similar to Theorem 2.2.

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