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NONLOCAL PROBLEM FOR A THIRD ORDER PARTIAL DIFFERENTIAL EQUATION OF MIXED TYPE WITH AN INTEGRAL TWO-SPACE-VARIABLES CONDITION

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Abstract. We study a mixed problem with two-space-variables condition for a class of third-order partial differential equation of mixed type. We prove the existence and uniqueness of the solution. The proof is based on a priori estimate "energy inequality" and on the density of the range of the operator generated by the considered problem.

Keywords. Partial differential equation; Energy inequality; Mixed problem; Integral condition; Integral boundary two-space-variables condition.

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1. Introduction

Boundary-value problems for parabolic equations with integral boundary conditions have been extensively investigated by Batten [1], Bouziani and Benouar [2], Cannon [3, 4], Ionkin [5], Kamynin [6], Kartynnik [7], Shi [8], Yurchuk [9], and many researchers. We remark that integral boundary conditions for evolution problems have various applications in chemical engineering, thermoelasticity, underground water flow and population dynamics; see, for example, [5,10] and the references therein.

The first who drew attention to these problems with an integral one-space-variables condition is Cannon [3], which gold of the study of heat conduction in a bar heated thin, has demonstrated

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by using the potential method, and the importance of the problems with integral conditions has been pointed out by Samarskii [11]. The existence and uniqueness of the classical solution of mixed problem combining a Dirichlet and integral condition for the equation of heat. Always using the potential method, established in Kamynin [6] the existence and uniqueness of the solution of a similar problem with a more general representation, specially a third order partial differential equation of mixed type with integral condition has been investigated in Denche and Marhoune [12].

Recently, many authors modified the integral condition by taking:

$$\int_0^\alpha u(x,t)\,dx+\int_\beta^1 u(x,t)\,dx=0,\ 0<\alpha<\beta<1,\ \alpha+\beta=1,\ t\in(0,T),$$

which is considered a generalization of integral boundary conditions.

Motivated by the research going on this direction, the present paper is devoted to the study of a problem with a boundary integral two-space-variables condition for a third order partial differential equation of mixed type, which is considered a supplement to the work published in Oussaeif-Bouziani [13,14].

2. Setting of the problem

In the rectangle $\Omega=(0,1)\times(0,T)$, with $T<\infty$, we consider the equation

$$\mathscr{L}u=\frac{\partial^2 u}{\partial t^2}-\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial^2 u}{\partial x\partial t}\right)=f(x,t), \tag{2.1}$$

where the coefficient $a(x,t)$ is bounded with belonging to $C^2\left(\overline{\Omega}\right)$ such that

$$0<c_0\le a(x,t)\le c_1,\ 0<\frac{\partial a(x,t)}{\partial t}\le c_2. \tag{H}$$

In the rest of the paper, we use k, c_i, $i=1,\dots,6$ to denote strictly positive constants. We adjoin to (2.1) the initial condition

$$\ell u=\frac{\partial u}{\partial t}(x,0)=\psi(x),\ \ x\in(0,1), \tag{2.2}$$

$$\frac{\partial u}{\partial x}(0,t)=\frac{\partial u}{\partial x}(1,t)=0,\ \ t\in(0,T) \tag{2.3}$$

and the integral condition

$$\int_0^\alpha u(x,t)\,dx+\int_\beta^1 u(x,t)\,dx=0,\ 0<\alpha<\ \beta,\ \alpha+\beta=1,\ t\in(0,T), \tag{2.4}$$

where ψ is a known function.

We shall assume that the functions ψ and f satisfy the compatibility conditions with (2.4), i.e.,

$$\int_0^\alpha \psi(x,t)\,dx + \int_\beta^1 \psi(x,t)\,dx = 0, \tag{2.5}$$

$$\int_0^\alpha f(x,t)\,dx + \int_\beta^1 f(x,t)\,dx = 0. \tag{2.6}$$

The presence of integral terms in boundary conditions can, in general, greatly complicate the application of standard functional or numerical techniques, specially the integral two-space-variables condition. Then to avoid this difficulty, we introduce a technique for transfer this problem to another classically less complicated and does not contain integral conditions. For that, we establish the following lemma.

Lemma 2.1. *Problem* $(2.1)-(2.4)$ *is equivalent to the following problem* (PR) :

$$(PR):\begin{cases} \mathscr{L}u = \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial^2 u}{\partial x \partial t}\right) = f(x,t), \\ 0 < c_0 \leq a(x,t) \leq c_1,\ 0 < \frac{\partial a(x,t)}{\partial t} \leq c_2, \\ \ell u = \frac{\partial u}{\partial t}(x,0) = \psi(x),\ \ x \in (0,1), \\ \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0, \\ a(\alpha,t)\frac{\partial^2 u}{\partial x \partial t}(\alpha,t) - a(\beta,t)\frac{\partial^2 u}{\partial x \partial t}(\beta,t) = 0. \end{cases}$$

Proof. Letting $u(x,t)$ be a solution of $(2.1)-(2.4)$, we prove that

$$a(\alpha,t)\frac{\partial^2 u}{\partial x \partial t}(\alpha,t) - a(\beta,t)\frac{\partial^2 u}{\partial x \partial t}(\beta,t) = 0.$$

So, integrating equation (2.1) with respect to x over $(0,\alpha)$ and $(\beta,1)$, and taking into account of (2.5), we obtain

$$\int_0^\alpha \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial^2 u}{\partial x \partial t}\right) dx + \int_\beta^1 \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial^2 u}{\partial x \partial t}\right) dx = 0.$$

Using (2.3), we obtain

$$a(\alpha,t)\frac{\partial^2 u}{\partial x \partial t}(\alpha,t) - a(\beta,t)\frac{\partial^2 u}{\partial x \partial t}(\beta,t) = 0. \tag{2.7}$$

Let $u(x,t)$ be a solution of (PR). We next prove that $\int_0^\alpha u(x,t)\,dx + \int_\beta^1 u(x,t)\,dx = 0$. So, by integrating equation (2.1) with respect to x over $(0,\alpha)$ and over $(\beta,1)$, and taking into account that $a(\alpha,t)\frac{\partial^2 u}{\partial x \partial t}(\alpha,t) - a(\beta,t)\frac{\partial^2 u}{\partial x \partial t}(\beta,t) = 0$, and using conditions (2.3), and taking into account of (2.5), we get $\int_0^\alpha u(x,t)\,dx + \int_\beta^1 u(x,t)\,dx = 0$.

3. A priori estimate and its consequences

In this paper, we prove the existence and the uniqueness for solution of the problem $(2.1)-(2.4)$ as a solution of the operator equation

$$Lu=\mathscr{F}, \tag{3.1}$$

where $L=(\mathscr{L},\ell)$, with domain of difinition B consisting of functions $u\in L^2(\Omega)$ such that $\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial t^2}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x\partial t}\in L^2(\Omega)$ and u satisfies conditions $(2.2)-(2.4)$; the operator L is considered from B to F, where B is the Banach space consisting of all functions $u(x,t)$ having a finite norm

$$\begin{aligned}\|u\|_B^2 \;=\; & \int_\Omega\left(\frac{\partial^2 u}{\partial t^2}\right)^2 dxdt+\sup_{0\le\tau\le T}\int_0^1\left(\frac{\partial^2 u(x,\tau)}{\partial x\partial t}\right)^2 dx\\ & +\int_0^\tau\left(\left(\int_\alpha^x\frac{\partial^2 u}{\partial t^2}d\xi\right)^2+\left(\int_x^\beta\frac{\partial^2 u}{\partial t^2}d\xi\right)^2\right)dt\\ & +\int_0^\tau\left(\left(\int_0^x\frac{\partial^2 u}{\partial t^2}d\xi\right)^2+\left(\int_x^1\frac{\partial^2 u}{\partial t^2}d\xi\right)^2\right)dt,\end{aligned}$$

and F is the Hilbert space consisting of all elements $\mathscr{F}=(f,\psi)$ for which the following norm

$$\|\mathscr{F}\|_F^2=\int_\Omega f^2dxdt+\int_0^1\left(\frac{\partial\psi}{\partial x}\right)^2dx$$

is finite.

Theorem 3.1. *For any function $u\in B$ we have the inequality*

$$\|u\|_B\le c\,\|Lu\|_F\,, \tag{3.2}$$

where c is a positive constant independent of u.

Proof. Multiplying the equation (2.1) by the following Mu :

$$Mu=\begin{cases} Mu_1=(2-x)\frac{\partial^2 u}{\partial t^2}+\int_0^x\frac{\partial^2 u}{\partial t^2}d\xi, & 0\le x\le\alpha,\\ Mu_2=(2-x)\frac{\partial^2 u}{\partial t^2}+\frac12\int_\alpha^x\frac{\partial^2 u}{\partial t^2}d\xi-\frac12\int_x^\beta\frac{\partial^2 u}{\partial t^2}d\xi, & \alpha\le x\le\beta,\\ Mu_3=(2-x)\frac{\partial^2 u}{\partial t^2}-\int_x^1\frac{\partial^2 u}{\partial t^2}d\xi, & \beta\le x\le 1,\end{cases}$$

and integrating over Ω^τ,where $\Omega^\tau=(0,1)\times(0,\tau)$.

1) By integrating over $\Omega_\alpha^\tau = \Omega_\alpha = (0,\alpha)\times(0,\tau)$, we have

$$\begin{aligned}\int_{\Omega_\alpha} \mathscr{L}u.Mu_1 dxdt &= \int_{\Omega_\alpha}\left[(2-x)\frac{\partial^2 u}{\partial t^2}+\left(\int_0^x \frac{\partial^2 u}{\partial t^2}d\xi\right)\right]\frac{\partial^2 u}{\partial t^2}dxdt \\ &\quad -\int_{\Omega_\alpha}\left[(2-x)\frac{\partial^2 u}{\partial t^2}+\left(\int_0^x \frac{\partial^2 u}{\partial t^2}d\xi\right)\right]\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial^2 u}{\partial x\partial t}\right)dxdt \\ &= \int_{\Omega_\alpha}\left[(2-x)\frac{\partial^2 u}{\partial t^2}+\left(\int_0^x \frac{\partial^2 u}{\partial t^2}d\xi\right)\right]fdxdt.\end{aligned} \tag{3.3}$$

Employing integration by parts in (3.3), and taking into accont the boundary conditions (2.3), we obtain

$$\begin{aligned}&\int_{\Omega_\alpha}\left[(2-x)\frac{\partial^2 u}{\partial t^2}+\left(\int_0^x \frac{\partial^2 u}{\partial t^2}d\xi\right)\right]\frac{\partial^2 u}{\partial t^2}dxdt \\ &= \int_{\Omega_\alpha}(2-x)\left(\frac{\partial^2 u}{\partial t^2}\right)^2 dxdt + \frac{1}{2}\int_0^\tau\left(\int_0^x \frac{\partial^2 u}{\partial t^2}d\xi\right)^2 dt,\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}&-\int_{\Omega_\alpha}\left[(2-x)\frac{\partial^2 u}{\partial t^2}+\left(\int_0^x \frac{\partial^2 u}{\partial t^2}d\xi\right)\right]\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial^2 u}{\partial x\partial t}\right)dxdt \\ &= -a(\alpha,t)\frac{\partial^2 u}{\partial x\partial t}(\alpha,t)\int_0^\alpha \frac{\partial^2 u}{\partial t^2}d\xi - (2-\alpha)a(\alpha,t)\frac{\partial^2 u}{\partial x\partial t}(\alpha,t)\frac{\partial^2 u}{\partial t^2}(\alpha,t) \\ &\quad + \int_{\Omega_\alpha}(2-x)a(x,t)\frac{\partial^2 u}{\partial x\partial t}\frac{\partial^3 u}{\partial x\partial t^2}dxdt \\ &= \quad -a(\alpha,t)\frac{\partial^2 u}{\partial x\partial t}(\alpha,t)\int_0^\alpha \frac{\partial^2 u}{\partial t^2}d\xi - (2-\alpha)a(\alpha,t)\frac{\partial^2 u}{\partial x\partial t}(\alpha,t)\frac{\partial^2 u}{\partial t^2}(\alpha,t) \\ &\quad + \frac{1}{2}\int_0^\alpha (2-x)a(x,\tau)\left(\frac{\partial^2 u(x,\tau)}{\partial x\partial t}\right)^2 dx \\ &\quad - \frac{1}{2}\int_0^\alpha (2-x)a(x,0)\left(\frac{\partial \psi}{\partial x}\right)^2 dx - \frac{1}{2}\int_{\Omega_\alpha}(2-x)\frac{\partial a(x,t)}{\partial t}\left(\frac{\partial^2 u}{\partial x\partial t}\right)^2 dxdt.\end{aligned} \tag{3.5}$$

By using the Cauchy's inequality, we get

$$\begin{aligned}&\int_{\Omega_\alpha}\left[(2-x)\frac{\partial^2 u}{\partial t^2}+\left(\int_0^x \frac{\partial^2 u}{\partial t^2}d\xi\right)\right]fdxdt \\ &\le \frac{1}{2}\int_{\Omega_\alpha}(2-x)^2 f^2 dxdt + \frac{1}{2}\int_{\Omega_\alpha}\left(\frac{\partial^2 u}{\partial t^2}\right)^2 dxdt + \frac{1}{2}\int_{\Omega_\alpha} f^2 dxdt + \frac{1}{2}\int_{\Omega_\alpha}\left(\int_0^x \frac{\partial^2 u}{\partial t^2}d\xi\right)^2 dxdt.\end{aligned} \tag{3.6}$$

Substituting (3.4)-(3.6) into (3.3), we obtain

$$\begin{aligned}
&\int_{\Omega_\alpha}(2-x)\left(\frac{\partial^2 u}{\partial t^2}\right)^2 dxdt+\frac{1}{2}\int_0^\tau\left(\int_0^x \frac{\partial^2 u}{\partial t^2}d\xi\right)^2 dt\\
&\quad+\frac{1}{2}\int_0^\alpha (2-x)\,a(x,\tau)\left(\frac{\partial^2 u(x,\tau)}{\partial x\partial t}\right)^2 dx\\
&\quad-a(\alpha,t)\frac{\partial^2 u}{\partial x\partial t}(\alpha,t)\int_0^\alpha \frac{\partial^2 u}{\partial t^2}d\xi-(2-\alpha)\,a(\alpha,t)\frac{\partial^2 u}{\partial x\partial t}(\alpha,t)\,\frac{\partial^2 u}{\partial t^2}(\alpha,t)\\
&\leq\frac{1}{2}\int_{\Omega_\alpha}(2-x)^2 f^2 dxdt++\frac{1}{2}\int_{\Omega_\alpha}f^2 dxdt+\frac{1}{2}\int_{\Omega_\alpha}\left(\frac{\partial^2 u}{\partial t^2}\right)^2 dxdt\\
&\quad+\frac{1}{2}\int_{\Omega_\alpha}\left(\int_0^x \frac{\partial^2 u}{\partial t^2}d\xi\right)^2 dxdt+\frac{1}{2}\int_0^\alpha (2-x)\,a(x,0)\left(\frac{\partial \psi}{\partial x}\right)^2 dx\\
&\quad+\frac{1}{2}\int_{\Omega_\alpha}(2-x)\,\frac{\partial a(x,t)}{\partial t}\left(\frac{\partial^2 u}{\partial x\partial t}\right)^2 dxdt.
\end{aligned}$$

Using the Cauchy's inequality and according to conditions (H), we get

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_\alpha}\left(\frac{\partial^2 u}{\partial t^2}\right)^2 dxdt+\frac{1}{2}\int_0^\tau\left(\int_0^x \frac{\partial^2 u}{\partial t^2}d\xi\right)^2 dt\\
&\quad+\frac{c_0}{2}\int_0^\alpha\left(\frac{\partial^2 u(x,\tau)}{\partial x\partial t}\right)^2 dx-a(\alpha,t)\frac{\partial^2 u}{\partial x\partial t}(\alpha,t)\int_0^\alpha \frac{\partial^2 u}{\partial t^2}d\xi\\
&\quad-(2-\alpha)\,a(\alpha,t)\frac{\partial^2 u}{\partial x\partial t}(\alpha,t)\,\frac{\partial^2 u}{\partial t^2}(\alpha,t)\\
&\leq\frac{5}{2}\int_{\Omega_\alpha}f^2 dxdt+c_1\int_0^\alpha\left(\frac{\partial \psi}{\partial x}\right)^2 dx\\
&\quad+\frac{1}{2}\int_{\Omega_\alpha}\left(\int_0^x \frac{\partial^2 u}{\partial t^2}d\xi\right)^2 dxdt+c_2\int_{\Omega_\alpha}\left(\frac{\partial^2 u}{\partial x\partial t}\right)^2 dxdt.
\end{aligned}\tag{3.7}$$

2) By integrating over $\Omega^\tau_{\alpha,\beta}=\Omega_{\alpha,\beta}=(\alpha,\beta)\times(0,\tau)$, we have

$$\begin{aligned}
\int_{\Omega_{\alpha,\beta}}\mathscr{L}u.Mu_2dxdt&=\int_{\Omega_{\alpha,\beta}}\left[\begin{array}{c}(2-x)\,\frac{\partial^2 u}{\partial t^2}\\ +\left(\frac{1}{2}\int_\alpha^x \frac{\partial^2 u}{\partial t^2}d\xi-\frac{1}{2}\int_x^\beta \frac{\partial^2 u}{\partial t^2}d\xi\right)\end{array}\right]\frac{\partial^2 u}{\partial t^2}dxdt\\
&\quad-\int_{\Omega_{\alpha,\beta}}\left[\begin{array}{c}(2-x)\,\frac{\partial^2 u}{\partial t^2}\\ +\left(\frac{1}{2}\int_\alpha^x \frac{\partial^2 u}{\partial t^2}d\xi-\frac{1}{2}\int_x^\beta \frac{\partial^2 u}{\partial t^2}d\xi\right)\end{array}\right]\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial^2 u}{\partial x\partial t}\right)dxdt\\
&=\int_{\Omega_{\alpha,\beta}}\left[\begin{array}{c}(2-x)\,\frac{\partial^2 u}{\partial t^2}\\ +\left(\frac{1}{2}\int_\alpha^x \frac{\partial^2 u}{\partial t^2}d\xi-\frac{1}{2}\int_x^\beta \frac{\partial^2 u}{\partial t^2}d\xi\right)\end{array}\right]f dxdt.
\end{aligned}\tag{3.8}$$

Employing integration by parts in (3.8), and taking into accont the boundary conditions in (PR), we obtain

$$\begin{aligned}
&\int_{\Omega_{\alpha,\beta}}\left[\begin{array}{c}(2-x)\frac{\partial^2 u}{\partial t^2}\\ +\left(\frac{1}{2}\int_\alpha^x\frac{\partial^2 u}{\partial t^2}d\xi-\frac{1}{2}\int_x^\beta\frac{\partial^2 u}{\partial t^2}d\xi\right)\end{array}\right]\frac{\partial^2 u}{\partial t^2}dxdt\\
&=\int_{\Omega_{\alpha,\beta}}(2-x)\left(\frac{\partial^2 u}{\partial t^2}\right)^2dxdt+\frac{1}{4}\int_0^\tau\left(\int_\alpha^x\frac{\partial^2 u}{\partial t^2}d\xi\right)^2dt+\frac{1}{4}\int_0^\tau\left(\int_x^\beta\frac{\partial^2 u}{\partial t^2}d\xi\right)^2dt,
\end{aligned}\tag{3.9}$$

and

$$\begin{aligned}
&-\int_{\Omega_{\alpha,\beta}}\left[\begin{array}{c}(2-x)\frac{\partial^2 u}{\partial t^2}\\ +\left(\frac{1}{2}\int_\alpha^x\frac{\partial^2 u}{\partial t^2}d\xi-\frac{1}{2}\int_x^\beta\frac{\partial^2 u}{\partial t^2}d\xi\right)\end{array}\right]\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial^2 u}{\partial x\partial t}\right)dxdt\\
&=-(2-\beta)a(\beta,t)\frac{\partial^2 u}{\partial x\partial t}(\beta,t)\frac{\partial^2 u}{\partial t^2}(\beta,t)+(2-\alpha)a(\alpha,t)\frac{\partial^2 u}{\partial x\partial t}(\alpha,t)\frac{\partial^2 u}{\partial t^2}(\alpha,t)\\
&\quad+\int_{\Omega_{\alpha,\beta}}(2-x)a(x,t)\frac{\partial^2 u}{\partial x\partial t}\frac{\partial^3 u}{\partial x\partial t^2}dxdt\\
&=-(2-\beta)a(\beta,t)\frac{\partial^2 u}{\partial x\partial t}(\beta,t)\frac{\partial^2 u}{\partial t^2}(\beta,t)+(2-\alpha)a(\alpha,t)\frac{\partial^2 u}{\partial x\partial t}(\alpha,t)\frac{\partial^2 u}{\partial t^2}(\alpha,t)\\
&\quad+\frac{1}{2}\int_\alpha^\beta(2-x)a(x,\tau)\left(\frac{\partial^2 u(x,\tau)}{\partial x\partial t}\right)^2dx-\frac{1}{2}\int_\alpha^\beta(2-x)a(x,0)\left(\frac{\partial\psi}{\partial x}\right)^2dx\\
&\quad-\frac{1}{2}\int_{\Omega_{\alpha,\beta}}(2-x)\frac{\partial a(x,t)}{\partial t}\left(\frac{\partial^2 u}{\partial x\partial t}\right)^2dxdt.
\end{aligned}\tag{3.10}$$

By virtue of the Cauchy's inequality, we obtain

$$\begin{aligned}
&\int_{\Omega_{\alpha,\beta}}\left[\begin{array}{c}(2-x)\frac{\partial^2 u}{\partial t^2}\\ +\left(\frac{1}{2}\int_\alpha^x\frac{\partial^2 u}{\partial t^2}d\xi-\frac{1}{2}\int_x^\beta\frac{\partial^2 u}{\partial t^2}d\xi\right)\end{array}\right]f dxdt\\
&\le 4\int_{\Omega_{\alpha,\beta}}f^2dxdt+\frac{1}{2}\int_{\Omega_{\alpha,\beta}}\left(\frac{\partial^2 u}{\partial t^2}\right)^2dxdt\\
&\quad+\frac{1}{4}\int_{\Omega_{\alpha,\beta}}\left(\int_\alpha^x\frac{\partial^2 u}{\partial t^2}d\xi\right)^2dxdt+\frac{1}{4}\int_{\Omega_{\alpha,\beta}}\left(\int_x^\beta\frac{\partial^2 u}{\partial t^2}d\xi\right)^2dxdt.
\end{aligned}\tag{3.11}$$

Substituting (3.9)-(3.11) into (3.8), and according to conditions (H), we get

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_{\alpha,\beta}}\left(\frac{\partial^2 u}{\partial t^2}\right)^2 dxdt+\frac{1}{4}\int_0^\tau\left(\int_\alpha^x \frac{\partial^2 u}{\partial t^2}d\xi\right)^2 dt+\frac{1}{4}\int_0^\tau\left(\int_x^\beta \frac{\partial^2 u}{\partial t^2}d\xi\right)^2 dt\\
&\quad+\frac{c_0}{2}\int_\alpha^\beta (2-x)\left(\frac{\partial^2 u(x,\tau)}{\partial x\partial t}\right)^2 dx\\
&\quad-(2-\beta)\,a(\beta,t)\frac{\partial^2 u}{\partial x\partial t}(\beta,t)\frac{\partial^2 u}{\partial t^2}(\beta,t)+(2-\alpha)\,a(\alpha,t)\frac{\partial^2 u}{\partial x\partial t}(\alpha,t)\frac{\partial^2 u}{\partial t^2}(\alpha,t) \qquad (3.12)\\
&\le 4\int_{\Omega_{\alpha,\beta}} f^2 dxdt+\frac{1}{4}\int_{\Omega_{\alpha,\beta}}\left(\int_\alpha^x \frac{\partial^2 u}{\partial t^2}d\xi\right)^2 dxdt+\frac{1}{4}\int_{\Omega_{\alpha,\beta}}\left(\int_x^\beta \frac{\partial^2 u}{\partial t^2}d\xi\right)^2 dxdt\\
&\quad+c_1\int_\alpha^\beta\left(\frac{\partial\psi}{\partial x}\right)^2 dx+c_2\int_{\Omega_{\alpha,\beta}}\left(\frac{\partial^2 u}{\partial x\partial t}\right)^2 dxdt.
\end{aligned}$$

3) By integrating over $\Omega_\beta^\tau=\Omega_\beta=(\beta,1)\times(0,\tau)$, one has

$$\begin{aligned}
\int_{\Omega_\beta}\mathscr{L}u.Mu_3 dxdt &= \int_{\Omega_\beta}\left((2-x)\frac{\partial^2 u}{\partial t^2}-\int_x^1 \frac{\partial^2 u}{\partial t^2}d\xi\right)\frac{\partial^2 u}{\partial t^2}dxdt\\
&\quad-\int_{\Omega_\beta}\left((2-x)\frac{\partial^2 u}{\partial t^2}-\int_x^1 \frac{\partial^2 u}{\partial t^2}d\xi\right)\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right)dxdt \qquad (3.13)\\
&=\int_{\Omega_\beta}\left((2-x)\frac{\partial^2 u}{\partial t^2}-\int_x^1 \frac{\partial^2 u}{\partial t^2}d\xi\right)f dxdt.
\end{aligned}$$

Employing integration by parts in (3.13), and taking into accont the boundary conditions in (PR), we obtain

$$\begin{aligned}
&\int_{\Omega_\beta}\left[(2-x)\frac{\partial^2 u}{\partial t^2}-\int_x^1 \frac{\partial^2 u}{\partial t^2}d\xi\right]\frac{\partial^2 u}{\partial t^2}dxdt\\
&=\int_{\Omega_\beta}(2-x)\left(\frac{\partial^2 u}{\partial t^2}\right)^2 dxdt+\frac{1}{2}\int_0^\tau\left(\int_x^1 \frac{\partial^2 u}{\partial t^2}d\xi\right)^2 dt, \qquad (3.14)
\end{aligned}$$

and

$$\begin{aligned}
&-\int_{\Omega_\beta}\left[(2-x)\frac{\partial^2 u}{\partial t^2}-\int_x^1 \frac{\partial^2 u}{\partial t^2}d\xi\right]\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial^2 u}{\partial x\partial t}\right)dxdt\\
&=-a(\beta,t)\frac{\partial^2 u}{\partial x\partial t}(\beta,t)\int_\beta^1 \frac{\partial^2 u}{\partial t^2}d\xi+(2-\beta)\,a(\beta,t)\frac{\partial^2 u}{\partial x\partial t}(\beta,t)\frac{\partial^2 u}{\partial t^2}(\beta,t)\\
&\quad+\frac{1}{2}\int_\beta^1 (2-x)\,a(x,\tau)\left(\frac{\partial^2 u(x,\tau)}{\partial x\partial t}\right)^2 dx \qquad (3.15)\\
&\quad-\frac{1}{2}\int_\beta^1 (2-x)\,a(x,0)\left(\frac{\partial\psi}{\partial x}\right)^2 dx-\frac{1}{2}\int_{\Omega_\beta}(2-x)\frac{\partial a(x,t)}{\partial t}\left(\frac{\partial^2 u}{\partial x\partial t}\right)^2 dxdt.
\end{aligned}$$

Using the Cauchy's inequality, we get

$$\int_{\Omega_\beta}\left[(2-x)\frac{\partial^2 u}{\partial t^2}-\int_x^1\frac{\partial^2 u}{\partial t^2}d\xi\right]f\,dxdt \le \frac{1}{2}\int_{\Omega_\beta}(2-x)^2 f^2 dxdt+\frac{1}{2}\int_{\Omega_\beta}\left(\frac{\partial^2 u}{\partial t^2}\right)^2 dxdt$$
$$+\frac{1}{2}\int_{\Omega_\beta} f^2 dxdt+\frac{1}{2}\int_{\Omega_\beta}\left(\int_x^1\frac{\partial^2 u}{\partial t^2}d\xi\right)^2 dxdt. \tag{3.16}$$

Substituting (3.14)-(3.16) into (3.13), we obtain

$$\begin{aligned}
&\int_{\Omega_\beta}(2-x)\left(\frac{\partial^2 u}{\partial t^2}\right)^2 dxdt+\frac{1}{2}\int_0^\tau\left(\int_x^1\frac{\partial^2 u}{\partial t^2}d\xi\right)^2 dt\\
&+\frac{1}{2}\int_\beta^1(2-x)\,a(x,\tau)\left(\frac{\partial^2 u(x,\tau)}{\partial x\partial t}\right)^2 dx\\
&-a(\beta,t)\frac{\partial^2 u}{\partial x\partial t}(\beta,t)\int_\beta^1\frac{\partial^2 u}{\partial t^2}d\xi+(2-\beta)\,a(\beta,t)\frac{\partial^2 u}{\partial x\partial t}(\beta,t)\frac{\partial^2 u}{\partial t^2}(\beta,t)\\
\le\ &\frac{1}{2}\int_{\Omega_\beta}(2-x)^2 f^2 dxdt+\frac{1}{2}\int_{\Omega_\beta}\left(\frac{\partial^2 u}{\partial t^2}\right)^2 dxdt\\
&+\frac{1}{2}\int_{\Omega_\beta} f^2 dxdt+\frac{1}{2}\int_{\Omega_\alpha}\left(\int_x^1\frac{\partial^2 u}{\partial t^2}d\xi\right)^2 dxdt\\
&+\frac{1}{2}\int_\beta^1(2-x)\,a(x,0)\left(\frac{\partial\psi}{\partial x}\right)^2 dx+\frac{1}{2}\int_{\Omega_\beta}(2-x)\frac{\partial a(x,t)}{\partial t}\left(\frac{\partial^2 u}{\partial x\partial t}\right)^2 dxdt.
\end{aligned}$$

Then, using the Cauchy's inequality and according to conditions (H), we get

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_\beta}\left(\frac{\partial^2 u}{\partial t^2}\right)^2 dxdt+\frac{1}{2}\int_0^\tau\left(\int_x^1\frac{\partial^2 u}{\partial t^2}d\xi\right)^2 dt+\frac{c_0}{2}\int_\beta^1\left(\frac{\partial^2 u(x,\tau)}{\partial x\partial t}\right)^2 dx\\
&\quad-a(\beta,t)\frac{\partial^2 u}{\partial x\partial t}(\beta,t)\int_\beta^1\frac{\partial^2 u}{\partial t^2}d\xi+(2-\beta)\,a(\beta,t)\frac{\partial^2 u}{\partial x\partial t}(\beta,t)\frac{\partial^2 u}{\partial t^2}(\beta,t)\\
&\le\frac{5}{2}\int_{\Omega_\beta} f^2 dxdt+c_1\int_\beta^1\left(\frac{\partial\psi}{\partial x}\right)^2 dx\\
&\quad+\frac{1}{2}\int_{\Omega_\beta}\left(\int_x^1\frac{\partial^2 u}{\partial t^2}d\xi\right)^2 dxdt+c_2\int_{\Omega_\beta}\left(\frac{\partial^2 u}{\partial x\partial t}\right)^2 dxdt.
\end{aligned}\tag{3.17}$$

By combining $(3.7), (3.12)$ and (3.17), and using conditions in (PR), we get

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega}\left(\frac{\partial^2 u}{\partial t^2}\right)^2 dxdt+\frac{c_0}{2}\int_0^1\left(\frac{\partial^2 u(x,\tau)}{\partial x\partial t}\right)^2 dx\\
&+\frac{1}{2}\int_0^{\tau}\left(\left(\int_0^x\frac{\partial^2 u}{\partial t^2}d\xi\right)^2+\left(\int_x^1\frac{\partial^2 u}{\partial t^2}d\xi\right)^2\right)dt\\
&+\frac{1}{4}\int_0^{\tau}\left(\left(\int_\alpha^x\frac{\partial^2 u}{\partial t^2}d\xi\right)^2+\left(\int_x^\beta\frac{\partial^2 u}{\partial t^2}d\xi\right)^2\right)dt\\
\leq\ & 4\int_{\Omega}f^2dxdt+c_1\int_0^1\left(\frac{\partial\psi}{\partial x}\right)^2dx+c_2\int_{\Omega}\left(\left(\frac{\partial^2 u}{\partial x\partial t}\right)^2\right)dxdt\\
&+\frac{1}{4}\int_{\Omega_{\alpha,\beta}}\left(\left(\int_\alpha^x\frac{\partial^2 u}{\partial t^2}d\xi\right)^2+\left(\int_x^\beta\frac{\partial^2 u}{\partial t^2}d\xi\right)^2\right)dxdt\\
&+\frac{1}{2}\int_{\Omega_\alpha}\left(\int_0^x\frac{\partial^2 u}{\partial t^2}d\xi\right)^2dxdt+\frac{1}{2}\int_{\Omega_\beta}\left(\int_x^1\frac{\partial^2 u}{\partial t^2}d\xi\right)^2dxdt.
\end{aligned}$$

Applying Lemma 2.1, we find

$$\begin{aligned}
&\int_{\Omega}\left(\frac{\partial^2 u}{\partial t^2}\right)^2 dxdt+\int_0^1\left(\frac{\partial^2 u(x,\tau)}{\partial x\partial t}\right)^2 dx\\
&+\int_0^{\tau}\left(\left(\int_0^x\frac{\partial^2 u}{\partial t^2}d\xi\right)^2+\left(\int_x^1\frac{\partial^2 u}{\partial t^2}d\xi\right)^2\right)dt\\
&+\int_0^{\tau}\left(\left(\int_\alpha^x\frac{\partial^2 u}{\partial t^2}d\xi\right)^2+\left(\int_x^\beta\frac{\partial^2 u}{\partial t^2}d\xi\right)^2\right)dt\\
\leq\ & c_3\left(\int_{\Omega}f^2dxdt+\int_0^1\left(\frac{\partial\psi}{\partial x}\right)^2dx\right)\\
&+c_3\int_{\Omega_{\alpha,\beta}}\left(\left(\int_\alpha^x\frac{\partial^2 u}{\partial t^2}d\xi\right)^2+\left(\int_x^\beta\frac{\partial^2 u}{\partial t^2}d\xi\right)^2\right)dxdt\\
&+c_3\int_{\Omega_\alpha}\left(\int_0^x\frac{\partial^2 u}{\partial t^2}d\xi\right)^2dxdt+\int_{\Omega_\beta}\left(\int_x^1\frac{\partial^2 u}{\partial t^2}d\xi\right)^2dxdt,
\end{aligned}$$

where

$$c_3=\frac{\max\left(c_1,4\right)}{\min\left(\frac{1}{4},\frac{c_0}{2}\right)}\exp\left(c_2T\right),$$

It also follows from Lemma 2.1 that

$$\int_{\Omega}\left(\frac{\partial^2 u}{\partial t^2}\right)^2 dxdt+\int_0^1\left(\frac{\partial^2 u(x,\tau)}{\partial x\partial t}\right)^2 dx+\int_0^{\tau}\left(\left(\int_0^x \frac{\partial^2 u}{\partial t^2}d\xi\right)^2+\left(\int_x^1 \frac{\partial^2 u}{\partial t^2}d\xi\right)^2\right)dt$$
$$+\int_0^{\tau}\left(\left(\int_{\alpha}^x \frac{\partial^2 u}{\partial t^2}d\xi\right)^2+\left(\int_x^{\beta} \frac{\partial^2 u}{\partial t^2}d\xi\right)^2\right)dt$$
$$\leq c_4\left(\int_{\Omega} f^2 dxdt+\int_0^1\left(\frac{\partial \psi}{\partial x}\right)^2 dx\right), \tag{3.18}$$

where

$$c_4=c_3\exp(c_3\alpha)\exp(c_3\exp(c_3\alpha))\exp(c_3\exp(c_3\alpha)\exp(c_3\exp(c_3\alpha))\beta).$$

The right-hand side of (3.18) is independent of τ, hence replacing the left-hand side by its upper bound with respect to τ from 0 to T, we obtain the desired inequality, where $c=(c_4)^{\frac{1}{2}}$. This completes the proof.

Corollary 3.2. *A solution of the problem* $(2.1)-(2.4)$ *is unique if it exists, and depends continuously on* $\mathscr{F}\in F$.

4. Solvability of the problem

To show the existance of solutions, we prove that $R(L)$ is dense in F for all $u\in B$ and for arbitrary $\mathscr{F}=(f,\psi)\in F$.

Theorem 4.1. *Suppose the conditions of theorem 3.1 and the condition* (H) *are satisfied. Then the problem* $(2.1)-(2.5)$ *admits a unique solution* $u=L^{-1}\mathscr{F}$.

Proof. First we prove that $R(L)$ is dense in F for the special case where $D(L)\equiv B$ is reduced to $D_0(L)$, where $D_0(L)=\{u,\ u\in D(L):\ell u=0\}$.

Proposition 4.2. *Let the conditions of theorem 4.1 be satisfied . if, for* $\omega\in L^2(\Omega)$ *and for all* $u\in D_0(L)$, *we have*

$$\int_{\Omega}\mathscr{L}u.\omega dxdt=0, \tag{4.1}$$

then ω *vanishes almost everywhere in* Ω.

Proof. The scalar product of F is defined by

$$(Lu,\omega)_F=\int_\Omega \mathscr{L}u.\omega dxdt+\int_0^1\left(\frac{\partial \ell u}{\partial x}\right)\left(\frac{\partial \omega_0}{\partial x}\right)dx. \tag{4.2}$$

Equality (4.1) can be written as

$$\int_\Omega \frac{\partial^2 u}{\partial t^2}.\omega dxdt=\int_\Omega \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial^2 u}{\partial x\partial t}\right).\omega dxdt. \tag{4.3}$$

If we put

$$u=\Im_t^2\left(e^{kt}z\right)=\int_0^t\int_0^h e^{k\tau}z(x,\tau)\,d\tau dh,$$

where k is a constant such that $kc_0-c_2\geq 0$, and $z,\frac{\partial z}{\partial x},\frac{\partial}{\partial x}\left(a\frac{\partial \Im_t\left(e^{kt}z\right)}{\partial x}\right)\in L^2(\Omega)$, then, u satisfies the boundary conditions in (PR). As a result of (4.3), we obtain the equality

$$\int_\Omega e^{kt}z\omega dxdt=\int_\Omega \frac{\partial}{\partial x}\left(a\frac{\partial\left(\Im_t e^{k\tau}z\right)}{\partial x}\right).\omega dxdt. \tag{4.4}$$

The left-hand side of (4.4) shows that the mapping

$$L^2(\Omega)\ni z\rightarrow\int_\Omega \frac{\partial}{\partial x}\left(a\frac{\partial\left(\Im_t e^{k\tau}z\right)}{\partial x}\right).\omega dt$$

is a continuous linear functional of z. From the right-hand side of (4.4) there follows that is true if the function ω has following properties:

$$\Im_t^\star(a\Im_x\omega),\frac{\partial}{\partial x}\left(\Im_t^\star(a\Im_x\omega)\right)\in L^2(\Omega).$$

In terms of the given function ω, and from the equality (4.4), we give the function ω in terms of z as follows:

$$\omega=\begin{cases}\omega_1=(2-x)z-\int_x^\alpha zd\xi, & 0\leq x\leq\alpha,\\ \omega_2=(2-x)z+\int_\alpha^x zd\xi, & \alpha\leq x\leq\beta,\\ \omega_3=(2-x)z+\int_\beta^x zd\xi, & \beta\leq x\leq 1.\end{cases} \tag{4.5}$$

z satisfies the same conditions of the function u in (PR), and satisfies the conditions

$$\frac{\partial z}{\partial x}(\alpha,t)=\frac{\partial z}{\partial x}(\beta,t).$$

Replacing ω in (4.4) by its representation (4.5) and integrating by parts each term of (4.4) with the use of conditions of z, we obtain **1)** on the interval $\Omega_\alpha=(0,\alpha)\times(0,\tau)$, we obtain

$$\int_{\Omega_\alpha} e^{kt}z\omega_1 dxdt=\int_{\Omega_\alpha}\frac{\partial}{\partial x}\left(a\frac{\partial\left(\Im_t e^{k\tau}z\right)}{\partial x}\right).\omega_1 dxdt. \tag{4.6}$$

Integrating by parts each term of (4.6) with respect to x and t by taking the conditions of the function z yields

$$
\begin{aligned}
&\int_{\Omega_\alpha} \frac{\partial}{\partial x}\left(a\frac{\partial\left(\Im_t e^{kt}z\right)}{\partial x}\right)\left((2-x)z-\int_x^\alpha z d\xi\right)dxdt \\
= & -\int_{\Omega_\alpha} a(x,t)(2-x)\frac{\partial \Im_t\left(e^{kt}z\right)}{\partial x}\frac{\partial z}{\partial x}\,dxdt \\
= & -\frac{1}{2}\int_0^\alpha e^{-kt}(2-x)a(x,t)\left(\frac{\partial \Im_t\left(e^{kt}z\right)}{\partial x}\right)^2\Bigg|_{t=0}^{t=T} dx \\
& -\frac{1}{2}\int_{\Omega_\alpha} e^{-kt}(2-x)\left(ka(x,t)-\frac{\partial a(x,t)}{\partial t}\right)\left(\frac{\partial \Im_t\left(e^{kt}z\right)}{\partial x}\right)^2 dxdt.
\end{aligned}
$$

By conditions of z, we obtain

$$
-\frac{1}{2}(kc_0-c_2)\int_{\Omega_\alpha} e^{-kt}(2-x)\left(\frac{\partial \Im_t\left(e^{kt}z\right)}{\partial x}\right)^2 dxdt \leq 0 \tag{4.7}
$$

and

$$
\int_{\Omega_\alpha} e^{kt}z\omega_1 dxdt = \int_{\Omega_\alpha} e^{kt}(2-x)z^2 dxdt + \int_0^\tau e^{kt}\left(\int_x^\alpha z d\xi\right)^2 dt. \tag{4.8}
$$

2) On the interval $\Omega_{\alpha,\beta} = (\alpha,\beta)\times(0,\tau)$, we obtain

$$
\int_{\Omega_{\alpha,\beta}} e^{kt}z\omega_2 dxdt = \int_{\Omega_{\alpha,\beta}} \frac{\partial}{\partial x}\left(a\frac{\partial\left(\Im_t e^{k\tau}z\right)}{\partial x}\right).\omega_2 dxdt. \tag{4.9}
$$

Integrating by parts each term of (4.9) with respect to x and t by taking the conditions of the function z yields

$$
\begin{aligned}
&\int_{\Omega_{\alpha,\beta}} \frac{\partial}{\partial x}\left(a\frac{\partial\left(\Im_t e^{kt}z\right)}{\partial x}\right)\left((2-x)z-\int_\alpha^x z d\xi\right)dxdt \\
= & -\int_{\Omega_{\alpha,\beta}} a(x,t)(2-x)\frac{\partial \Im_t\left(e^{kt}z\right)}{\partial x}\frac{\partial z}{\partial x}\,dxdt \\
= & -\frac{1}{2}\int_\alpha^\beta e^{-kt}(2-x)a(x,t)\left(\frac{\partial \Im_t\left(e^{kt}z\right)}{\partial x}\right)^2\Bigg|_{t=0}^{t=T} dx \\
& -\frac{1}{2}\int_{\Omega_{\alpha,\beta}} e^{-kt}(2-x)\left(ka(x,t)-\frac{\partial a(x,t)}{\partial t}\right)\left(\frac{\partial \Im_t\left(e^{kt}z\right)}{\partial x}\right)^2 dxdt.
\end{aligned}
$$

By conditions of z, we obtain

$$-\frac{1}{2}(kc_0-c_2)\int_{\Omega_{\alpha,\beta}} e^{-kt}(2-x)\left(\frac{\partial \Im_t\left(e^{kt}z\right)}{\partial x}\right)^2 dxdt \leq 0 \tag{4.10}$$

and

$$\int_{\Omega_{\alpha,\beta}} e^{kt} z\omega_2 dxdt = \int_{\Omega_{\alpha,\beta}} e^{kt}(2-x)z^2 dxdt + \int_0^\tau e^{kt}\left(\int_\alpha^x z d\xi\right)^2 dt. \tag{4.11}$$

3) On the interval $\Omega_\beta = (\beta, 1)\times(0,\tau)$, we obtain

$$\int_{\Omega_\beta} e^{kt} z\omega_3 dxdt = \int_{\Omega_\beta} \frac{\partial}{\partial x}\left(a\frac{\partial\left(\Im_t e^{k\tau}z\right)}{\partial x}\right).\omega_3 dxdt. \tag{4.12}$$

Integrating by parts each term of (4.12) with respect to x and t by taking the conditions of the function z yields

$$\begin{aligned}
&\int_{\Omega_\beta} \frac{\partial}{\partial x}\left(a\frac{\partial\left(\Im_t e^{kt}z\right)}{\partial x}\right)\left((2-x)z + \int_\beta^x z d\xi\right) dxdt \\
= & -\int_{\Omega_\beta} a(x,t)(2-x)\frac{\partial \Im_t\left(e^{kt}z\right)}{\partial x}\frac{\partial z}{\partial x}\, dxdt \\
= & -\frac{1}{2}\int_\beta^1 e^{-kt}(2-x)a(x,t)\left(\frac{\partial \Im_t\left(e^{kt}z\right)}{\partial x}\right)^2\Bigg|_{t=0}^{t=T} dx \\
& -\frac{1}{2}\int_{\Omega_\beta} e^{-kt}(2-x)\left(ka(x,t) - \frac{\partial a(x,t)}{\partial t}\right)\left(\frac{\partial \Im_t\left(e^{kt}z\right)}{\partial x}\right)^2 dxdt
\end{aligned}$$

By conditions of z, we obtain

$$-\frac{1}{2}(kc_0-c_2)\int_{\Omega_\beta} e^{-kt}(2-x)\left(\frac{\partial \Im_t\left(e^{kt}z\right)}{\partial x}\right)^2 dxdt \leq 0 \tag{4.13}$$

and

$$\int_{\Omega_\beta} e^{kt} z\omega_3 dxdt = \int_{\Omega_\beta} e^{kt}(2-x)z^2 dxdt + \int_0^\tau e^{kt}\left(\int_\beta^x z d\xi\right)^2 dt. \tag{4.14}$$

Putting and using the results of (4.7),(4.8) and (4.10),(4.11) and (4.13),(4.14) into (4.4), we obtain

$$\begin{aligned}
&\int_{\Omega} e^{kt}(2-x)z^2 dxdt + \int_0^{\tau} e^{kt}\left[\left(\int_x^{\alpha} zd\xi\right)^2 + \left(\int_{\beta}^{x} zd\xi\right)^2 + \left(\int_{\alpha}^{x} zd\xi\right)^2\right] dt \\
&\leq -\frac{1}{2}(kc_0 - c_2)\int_{\Omega} e^{-kt}(2-x)\left(\frac{\partial \Im_t\left(e^{k\tau}z\right)}{\partial x}\right)^2 dxdt \\
&\leq 0.
\end{aligned} \tag{4.15}$$

Thus $z=0$ in Ω. Hence $\omega=0$ in Ω. This proves Proposition 4.2.

We return to the proof of Theorem 4.1, we have already noted that it is sufficient to prove that the set $R(L)$ dense in F. Suppose that, for some $W=(\omega,\omega_0)\in R(L)^{\perp}$ and for all $u\in D(L)\equiv B$,

$$(Lu,\omega)_F = \int_{\Omega}\mathscr{L}u.\omega dxdt + \int_0^1 \left(\frac{\partial \ell u}{\partial x}\right)\left(\frac{\partial \omega_0}{\partial x}\right) dx = 0. \tag{4.16}$$

holds. Then we must prove that $W=0$. Putting $u\in D_0(L)$ in (4.16), we have

$$\int_{\Omega}\mathscr{L}u.\omega \; dxdt = 0, \quad u\in D_0(L).$$

Hence Proposition 4.2 implies that $\omega=0$. Thus (4.16) takes the form

$$\int_0^1 \left(\frac{\partial \ell u}{\partial x}\right)\left(\frac{\partial \omega_0}{\partial x}\right) dx = 0, \quad u\in D(L). \tag{4.17}$$

Since the range of the trace operator ℓ is independnt and the range of value ℓ is everywhere dense in the Hilbert space F with the norm

$$\left(\int_0^1 \left[\left(\frac{\partial \ell u}{\partial x}\right)^2\right]\right)^{\frac{1}{2}}.$$

Equality (4.17) implies that $\omega_0=0$ (we recall satisfies a compatibility conditions). Hence $W=0$ implies $\left(\overline{R(L)}=F\right)$. Therefore, the proof of Theorem 4.1 is complete.

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