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MORE ON OSTROWSKI AND OSTROWSKI-GRÜSS TYPE INEQULITIES

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Abstract. Based on the Grüss inequality and the Ostrowski inequality, we obtain some new versions of the Ostrowski and the Ostrowski-Grüss type inequalities.

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1. Introduction

An integral inequality that establishes a connection between the integral of the product of two functions and the product of integrals stated in the following result is known as Grüss inequality [1].

Theorem 1.1. *Let* $f,g:[a,b]\to\mathbb{R}$ *be integrable functions such that* $\phi\le f(x)\le\Phi$ *and* $\gamma\le g(x)\le\Gamma$ *for all* $x\in[a,b]$, *where* ϕ,Φ,γ,Γ *are constants. Then we have*

$$(1)\qquad \left|\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{b-a}\int_a^b f(x)dx.\frac{1}{b-a}\int_a^b g(x)dx\right|\le\frac{1}{4}(\Phi-\phi)(\Gamma-\gamma),$$

where the consant $\frac{1}{4}$ *is sharp.*

Ostrowski [2] proved the following inequality.

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Theorem 1.2. *Let $f : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, be a mapping differentiable in I^o the interior of I and $a,\ b \in I^o$, $a < b$. If $\left|f^{'}(t)\right| \leq M$, for all $t \in [a,b]$, then we have*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)M, \tag{2}$$

for all $x \in [a,b]$.

Ostrowski and Ostrowski-Grüss type inequalities have been studied extensively since they have published, for example see in [3, 4, 5, 6] and references there in. Actually it plays a vital role to study the error bounds of different numerical quadrature rules for example mid point's, trapezoidal's, Simpson's and other generalized Riemann type. It also provides the error bounds of nonnegative differences of the well known Hadamard inequality.

In 1997, Dragomir *et al.* [5] improved Ostrowski inequality using Grüss inequality which is a connection between Ostrowski inequality and Grüss inequality. In [5], they gave the following Ostrowski-Grüss inequality.

Theorem 1.3. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$ and differentiable on (a,b), and its derivative satisfies the condition $\gamma \leq f'(x) \leq \Gamma$ for all $x \in [a,b]$ and γ, Γ are real constants. Then we have the inequality*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \left(\frac{f(b)-f(a)}{b-a}\right)\left(x - \frac{a+b}{2}\right)\right| \leq \frac{1}{4}(b-a)(\Gamma-\gamma), \tag{3}$$

for all $x \in [a,b]$.

In 2001, Barnett *et al.* [4] pointed out a similar result to the above for twice differentiable mappings.

Theorem 1.4. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$ and twice differentiable on (a,b), and assume that the second order derivative f'' satisfies the condition $\gamma \leq f''(x) \leq \Gamma$ for all $x \in [a,b]$ and γ, Γ are real constants. Then we have*

$$\begin{aligned}&\left|f(x) - \left(x - \frac{a+b}{2}\right)f'(x) + \left[\frac{(b-a)^2}{24} + \frac{1}{2}\left(x - \frac{a+b}{2}\right)^2\right]\left(\frac{f'(b)-f'(a)}{b-a}\right)\right.\\ &\left. - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{1}{8}(\Gamma-\gamma)\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^2,\end{aligned} \tag{4}$$

for all $x \in [a,b]$.

In 2001, Cheng gave generalized form of Ostrowski-Grüss type integral inequality [6] as follows.

Theorem 1.5. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$ and twice differentiable on (a,b), and assume that the second order derivative f'' satisfies the condition $\gamma \le f''(x) \le \Gamma$ for all $x \in [a,b]$ and γ, Γ are real constants. Then we have*

$$\left| f(x) - \frac{2}{3}\left(x - \frac{a+b}{2}\right) f'(x) + \frac{(x-b)^2 f'(b) - (x-a)^2 f'(a)}{6(b-a)} - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \frac{1}{18\sqrt{3}(b-a)}\left((x-a)^3 + (b-x)^3\right)(\Gamma - \gamma), \tag{5}$$

for all $x \in [a,b]$.

In [8], Farid *et al.* gave versions of Theorem 1.3 and Theorem 1.4 on two coordinates as follows.

Theorem 1.6. *et $f : [a,b] \times [c,d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a continuous function, also let its partial derivatives exist and satisfy the condition $\gamma_1 \le \frac{\partial f}{\partial x} \le \Gamma_1$ for all $x \in [a,b]$, $\gamma_2 \le \frac{\partial f}{\partial y} \le \Gamma_2$ for all $y \in [c,d]$ and γ_1, γ_2, Γ_1, Γ_2 are real constants. Then we have*

$$\left| \int_a^b \frac{f(x,c)+f(x,d)}{2}dx + \int_c^d \frac{f(a,y)+f(b,y)}{2}dy - \left(\frac{1}{b-a} + \frac{1}{d-c}\right) \int_a^b \int_c^d f(x,y)dxdy \right| \le \frac{(b-a)(d-c)}{4}\left[(\Gamma_2 - \gamma_2) + (\Gamma_1 - \gamma_1)\right]. \tag{6}$$

Theorem 1.7. *Let $f : [a,b] \times [c,d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a continuous function and let its second order partial derivatives exist and satisfy the condition $\gamma_1 \le \frac{\partial^2 f}{\partial x^2} \le \Gamma_1$ for all $x \in [a,b]$, $\gamma_2 \le \frac{\partial^2 f}{\partial y^2} \le \Gamma_2$ for all $y \in [c,d]$ and γ_1, γ_2, Γ_1, Γ_2 are real constants. Then we have*

$$\left| \frac{1}{2}\left[\int_a^b (f(x,c)+f(x,d))\,dx + \int_c^d (f(a,y)+f(b,y))\,dy\right] + \frac{1}{12} \left[(b-a)\int_c^d \left(\frac{\partial f(a,y)}{\partial x} - \frac{\partial f(b,y)}{\partial x}\right)dy + (d-c)\int_a^b \left(\frac{\partial f(x,c)}{\partial y} - \frac{\partial f(x,d)}{\partial y}\right)dx\right] - \left(\frac{1}{b-a} + \frac{1}{d-c}\right)\int_a^b \int_c^d f(x,y)dydx \right| \le \frac{1}{8}(d-c)(b-a)\left((\Gamma_2 - \gamma_2)(b-a) + (\Gamma_1 - \gamma_1)(d-c)\right). \tag{7}$$

In this paper, we are interested to present generalized Ostrowski-Grüss type inequality given in Theorem 1.5 for a function of two variables defined on a rectangle from the plane. Also we extend this result for a function of several variables. At the end we extend improvements of Ostrowski-Grüss type inequalities given in Theorem 1.6 and Theorem 1.7.

2. Generalized Ostrowski-Grüss type integral inequalities

In the following we establish generalized Ostrowski-Grüss type integral inequality given in Theorem 1.5 for a function of two variables defined on a rectangle from the plane.

Theorem 2.1. *Let $f : [a,b] \times [c,d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a continuous function and let its second order partial derivatives satisfy the condition $\gamma_1 \leq \frac{\partial f}{\partial x} \leq \Gamma_1$ for all $x \in [a,b]$ and $\gamma_2 \leq \frac{\partial f}{\partial y} \leq \Gamma_2$ for all $y \in [c,d]$, where $\gamma_1, \gamma_2, \Gamma_1, \Gamma_2$ are real constants. Then we have*

$$\begin{aligned}(8)\quad &\left|\frac{1}{2}\left[\int_a^b (f(x,c)+f(x,d))\,dx + \int_c^d (f(a,y)+f(b,y))\,dy\right] + \frac{1}{12}\right. \\ &\left[(b-a)\int_c^d \left(\frac{\partial f(a,y)}{\partial x} - \frac{\partial f(b,y)}{\partial x}\right)dy + (d-c)\int_a^b \left(\frac{\partial f(x,c)}{\partial y} - \frac{\partial f(x,d)}{\partial y}\right)dx\right] \\ &\left. -\left(\frac{1}{d-c}+\frac{1}{b-a}\right)\int_a^b\int_c^d f(x,y)dydx\right| \leq \frac{1}{18\sqrt{3}}(d-c)(b-a)[(b-a)(\Gamma_2-\gamma_2) \\ &\qquad + (d-c)(\Gamma_1-\gamma_1)].\end{aligned}$$

Proof. Applying (5) for mapping f_y at $x = b$, we have

$$\left|f(b,y) - \frac{(b-a)\frac{\partial f(b,y)}{\partial x}}{3} - \frac{(b-a)\frac{\partial f(a,y)}{\partial x}}{6} - \frac{1}{b-a}\int_a^b f(t,y)dt\right| \leq \frac{1}{18\sqrt{3}}(b-a)^2(\Gamma_2-\gamma_2).$$

On integrating over $[c,d]$, we get

$$\begin{aligned}(9)\quad &\left|\int_c^d f(b,y)dy - \frac{(b-a)}{3}\int_c^d \frac{\partial f(b,y)}{\partial x}dy - \frac{(b-a)}{6}\int_c^d \frac{\partial f(a,y)}{\partial x}dy\right. \\ &\left. -\frac{1}{b-a}\int_c^d\int_a^b f(x,y)dxdy\right| \leq \frac{1}{18\sqrt{3}}(b-a)^2(d-c)(\Gamma_2-\gamma_2).\end{aligned}$$

Applying (5) for mapping f_y at $x = a$, then integrating over $[c,d]$ we have

$$\begin{aligned}(10)\quad &\left|\int_c^d f(a,y)dy + \frac{(b-a)}{3}\int_c^d \frac{\partial f(a,y)}{\partial x}dy + \frac{(b-a)}{6}\int_c^d \frac{\partial f(b,y)}{\partial x}dy\right. \\ &\left. -\frac{1}{b-a}\int_c^d\int_a^b f(x,y)dxdy\right| \leq \frac{1}{18\sqrt{3}}(b-a)^2(d-c)(\Gamma_2-\gamma_2).\end{aligned}$$

Adding (9) and (10), we get

$$\left|\frac{1}{2}\left(\int_c^d f(a,y)dy+\int_c^d f(b,y)dy\right)+\frac{(b-a)}{12}\int_c^d\left(\frac{\partial f(a,y)}{\partial x}-\frac{\partial f(b,y)}{\partial x}\right)dy\right.$$
$$\left.-\frac{1}{b-a}\int_c^d\int_a^b f(x,y)dxdy\right|\leq\frac{1}{18\sqrt{3}}(b-a)^2(d-c)(\Gamma_2-\gamma_2). \tag{11}$$

Similarly applying (5) for mapping f_x at $y=c$, then integrating over $[a,b]$ and repeating the same process at $y=d$, after that adding two results, we have

$$\left|\frac{1}{2}\left(\int_a^b f(x,c)dx+\int_a^b f(x,d)dx\right)+\frac{(d-c)}{12}\int_a^b\left(\frac{\partial f(x,c)}{\partial y}-\frac{\partial f(x,d)}{\partial y}\right)dx\right.$$
$$\left.-\frac{1}{d-c}\int_c^d\int_a^b f(x,y)dxdy\right|\leq\frac{1}{18\sqrt{3}}(b-a)(d-c)^2(\Gamma_1-\gamma_1). \tag{12}$$

Adding (11) and (12), we get (8). This completes the proof.

3. Extensions of Ostrowski-Grüss type inequalities

In [7], the definition of convex function on n-coordinates is given. Motivated by this concept, we give extensions of Ostrowski-Grüss type inequalities using partial mappings on n-coordinates and their related partial derivatives in this section. In this whole section, we follow notation $\Delta^n=\prod_{i=1}^n I_i$ for $a_i,b_i\in\mathbb{R}$ and $i=1,2,...,n$.

Theorem 3.1. *Let $f:\Delta^n\to\mathbb{R}$ be a continuous function and let its partial derivatives satisfy the condition $\gamma_i\leq\frac{\partial^2 f}{\partial x_i^2}\leq\Gamma_i$ for $x_i\in[a_i,b_i]$ where γ_i, Γ_i are real constants and $i=1,2,3,...,n$. Then we have*

$$\left|\frac{1}{2}\sum_{k=1}^n\int_{a_k}^{b_k}\left(f_{x_n}^{k+1}(a_{k+1})+f_{x_n}^{k+1}(b_{k+1})\right)dx_k+\frac{1}{12}\sum_{k=1}^n(b_{k+1}-a_{k+1})\right.$$
$$\int_{a_k}^{b_k}\left(\frac{\partial f_{x_n}^{k+1}(a_{k+1})}{\partial x_{k+1}}-\frac{\partial f_{x_n}^{k+1}(b_{k+1})}{\partial x_{k+1}}\right)dx_k-\sum_{k=1}^n\frac{1}{b_{k+1}-a_{k+1}}$$
$$\left.\int_{a_{k+1}}^{b_{k+1}}\int_{a_k}^{b_k}f(\mathbf{x})dx_kdx_{k+1}\right|\leq\frac{1}{18\sqrt{3}}\sum_{k=1}^n(b_k-a_k)(b_{k+1}-a_{k+1})^2(\Gamma_{k+1}-\gamma_{k+1}), \tag{13}$$

with $n+1\to 1$.

Proof. Applying (5) for mapping $f_{x_n}^{i+1}:[a_{i+1},b_{i+1}]\to\mathbb{R}$ at $x=b_{i+1}$, where $i=1,2,...,n$, we have

$$\left|f_{x_n}^{i+1}(b_{i+1})-\frac{(b_{i+1}-a_{i+1})\frac{\partial f_{x_n}^{i+1}(b_{i+1})}{\partial x_{i+1}}}{3}-\frac{(b_{i+1}-a_{i+1})\frac{\partial f_{x_n}^{i+1}(a_{i+1})}{\partial x_{i+1}}}{6}-\frac{1}{b_{i+1}-a_{i+1}}\right.$$
$$\left.\int_{a_{i+1}}^{b_{i+1}} f_{x_n}^{i+1}(x_{i+1})dx_{i+1}\right|\le\frac{1}{18\sqrt{3}}(b_{i+1}-a_{i+1})^2(\Gamma_{i+1}-\gamma_{i+1}).$$

On integrating over $[a_i,b_i]$, we get

$$\left|\int_{a_i}^{b_i} f_{x_n}^{i+1}(b_{i+1})dx_i-\frac{(b_{i+1}-a_{i+1})}{6}\int_{a_i}^{b_i}\left(\frac{\partial f_{x_n}^{i+1}(a_{i+1})}{\partial x_{i+1}}+2\frac{\partial f_{x_n}^{i+1}(b_{i+1})}{\partial x_{i+1}}\right)dx_i\right.$$
$$\left.-\frac{1}{b_{i+1}-a_{i+1}}\int_{a_i}^{b_i}\int_{a_{i+1}}^{b_{i+1}} f(\mathbf{x})dx_{i+1}dx_i\right|\le\frac{1}{18\sqrt{3}}(b_i-a_i)(b_{i+1}-a_{i+1})^2(\Gamma_{i+1}-\gamma_{i+1}).\tag{14}$$

Now applying (5) for mapping $f_{x_n}^{i+1}:[a_{i+1},b_{i+1}]\to\mathbb{R}$ at $x=a_{i+1}$, then integrating over $[a_i,b_i]$, we get

$$\left|\int_{a_i}^{b_i} f_{x_n}^{i+1}(a_{i+1})dx_i+\frac{(b_{i+1}-a_{i+1})}{6}\int_{a_i}^{b_i}\left(\frac{\partial f_{x_n}^{i+1}(b_{i+1})}{\partial x_{i+1}}+2\frac{\partial f_{x_n}^{i+1}(a_{i+1})}{\partial x_{i+1}}\right)dx_i\right.$$
$$\left.-\frac{1}{b_{i+1}-a_{i+1}}\int_{a_i}^{b_i}\int_{a_{i+1}}^{b_{i+1}} f(\mathbf{x})dx_{i+1}dx_i\right|\le\frac{1}{18\sqrt{3}}(b_i-a_i)(b_{i+1}-a_{i+1})^2(\Gamma_{i+1}-\gamma_{i+1}).\tag{15}$$

Adding (14) and (15), we get

$$\left|\frac{1}{2}\int_{a_i}^{b_i}(f_{x_n}^{i+1}(a_{i+1})+f_{x_n}^{i+1}(b_{i+1}))dx_i+\frac{(b_{i+1}-a_{i+1})}{12}\right.$$
$$\left.\int_{a_i}^{b_i}\left(\frac{\partial f_{x_n}^{i+1}(a_{i+1})}{\partial x_{i+1}}-\frac{\partial f_{x_n}^{i+1}(b_{i+1})}{\partial x_{i+1}}\right)dx_i-\frac{1}{b_{i+1}-a_{i+1}}\int_{a_i}^{b_i}\int_{a_{i+1}}^{b_{i+1}} f(\mathbf{x})dx_{i+1}dx_i\right|$$
$$\le\frac{1}{18\sqrt{3}}(b_i-a_i)(b_{i+1}-a_{i+1})^2(\Gamma_{i+1}-\gamma_{i+1}).$$

Summing over i from 1 to n we get (13). This completes the proof.

Remark 3.2. For $n=2$ in Theorem 3.1 and $a_1\to a$, $b_1\to b$, $a_2\to c$, $b_2\to d$, we get inequality (8).

In next two theorems we give extensions of Ostrowski-Grüss type inequalities (6) and (7) on coordinates.

Theorem 3.3. *Let $f : \Delta^n \to \mathbb{R}$ be a continuous function and let its partial derivatives satisfy the condition $\gamma_i \leq \frac{\partial f}{\partial x_i} \leq \Gamma_i$ for $x_i \in [a_i, b_i]$ where γ_i, Γ_i are real constants and $i = 1, 2, 3, ..., n$. Then we have*

$$\left| \frac{1}{2} \sum_{k=1}^{n} \int_{a_k}^{b_k} \left(f_{x_n}^{k+1}(a_{k+1}) + f_{x_n}^{k+1}(b_{k+1}) \right) dx_k - \sum_{k=1}^{n} \frac{1}{b_{k+1} - a_{k+1}} \int_{a_{k+1}}^{b_{k+1}} \int_{a_k}^{b_k} f(\mathbf{x}) dx_k dx_{k+1} \right| \leq \frac{1}{4} \sum_{k=1}^{n} (b_k - a_k)(b_{k+1} - a_{k+1})(\Gamma_{k+1} - \gamma_{k+1}), \tag{16}$$

with $n + 1 \to 1$.

Proof. Applying (3) for mapping $f_{x_n}^{i+1} : [a_{i+1}, b_{i+1}] \to \mathbb{R}$ at $x = b_{i+1}$, where $i = 1, 2, ..., n$, we have

$$\left| f_{x_n}^{i+1}(b_{i+1}) - \frac{1}{b_{i+1} - a_{i+1}} \int_{a_{i+1}}^{b_{i+1}} f_{x_n}^{i+1}(x_{i+1}) dx_{i+1} - \frac{f_{x_n}^{i+1}(b_{i+1}) - f_{x_n}^{i+1}(a_{i+1})}{2} \right| \leq \frac{1}{4}(b_{i+1} - a_{i+1})(\Gamma_{i+1} - \gamma_{i+1}).$$

On integrating over $[a_i, b_i]$, we get

$$\left| \int_{a_i}^{b_i} f_{x_n}^{i+1}(b_{i+1}) dx_i - \frac{1}{b_{i+1} - a_{i+1}} \int_{a_i}^{b_i} \int_{a_{i+1}}^{b_{i+1}} f(\mathbf{x}) dx_{i+1} dx_i - \frac{1}{2} \int_{a_i}^{b_i} \left(f_{x_n}^{i+1}(b_{i+1}) - f_{x_n}^{i+1}(a_{i+1}) \right) dx_i \right| \leq \frac{1}{4}(b_i - a_i)(b_{i+1} - a_{i+1})(\Gamma_{i+1} - \gamma_{i+1}). \tag{17}$$

Now applying (3) for mapping $f_{x_n}^{i+1} : [a_{i+1}, b_{i+1}] \to \mathbb{R}$ at $x = a_{i+1}$, where $i = 1, 2, ..., n$, then integrating over $[a_i, b_i]$, we have

$$\left| \int_{a_i}^{b_i} f_{x_n}^{i+1}(a_{i+1}) dx_i - \frac{1}{b_{i+1} - a_{i+1}} \int_{a_i}^{b_i} \int_{a_{i+1}}^{b_{i+1}} f(\mathbf{x}) dx_{i+1} dx_i + \frac{1}{2} \int_{a_i}^{b_i} \left(f_{x_n}^{i+1}(b_{i+1}) - f_{x_n}^{i+1}(a_{i+1}) \right) dx_i \right| \leq \frac{1}{4}(b_i - a_i)(b_{i+1} - a_{i+1})(\Gamma_{i+1} - \gamma_{i+1}). \tag{18}$$

Adding (17) and (18), we get

$$\left| \frac{1}{2} \int_{a_i}^{b_i} (f_{x_n}^{i+1}(a_{i+1}) + f_{x_n}^{i+1}(b_{i+1})) dx_i - \frac{1}{b_{i+1} - a_{i+1}} \int_{a_i}^{b_i} \int_{a_{i+1}}^{b_{i+1}} f(\mathbf{x}) dx_{i+1} dx_i \right| \leq \frac{1}{4}(b_i - a_i)(b_{i+1} - a_{i+1})(\Gamma_{i+1} - \gamma_{i+1}).$$

Summing over i from 1 to n we get (16). This completes the proof.

Remark 3.4. For $n=2$ in Theorem 3.3 and $a_1 \to a$, $b_1 \to b$, $a_2 \to c$, $b_2 \to d$, we get inequality (6).

Theorem 3.5. *Let $f:\Delta^n \to \mathbb{R}$ be a continuous function and let its second order partial derivatives satisfy the condition $\gamma_i \leq \frac{\partial^2 f}{\partial x_i^2} \leq \Gamma_i$ for $x_i \in [a_i,b_i]$ where γ_i, Γ_i are real constants and $i=1,2,3,...,n$. Then we have*

$$\left|\frac{1}{2}\sum_{k=1}^{n}\int_{a_k}^{b_k}\left(f_{x_n}^{k+1}(a_{k+1})+f_{x_n}^{k+1}(b_{k+1})\right)dx_k+\sum_{k=1}^{n}\frac{(b_{k+1}-a_{k+1})}{12}\right.$$
$$\int_{a_k}^{b_k}\left(\frac{\partial f_{x_n}^{k+1}(a_{k+1})}{\partial x_k}-\frac{\partial f_{x_n}^{k+1}(b_{k+1})}{\partial x_k}\right)dx_k-\sum_{k=1}^{n}\frac{1}{b_{k+1}-a_{k+1}}$$
$$\left.\int_{a_{k+1}}^{b_{k+1}}\int_{a_k}^{b_k}f(\mathbf{x})dx_k dx_{k+1}\right|\leq\frac{1}{8}\sum_{k=1}^{n}(b_k-a_k)(b_{k+1}-a_{k+1})^2(\Gamma_{k+1}-\gamma_{k+1}), \tag{19}$$

with $n+1\to 1$.

Proof. Applying (4) for mapping $f_{x_n}^{i+1}:[a_{i+1},b_{i+1}]\to\mathbb{R}$ at $x=b_{i+1}$, where $i=1,2,...,n$, we have

$$\left|f_{x_n}^{i+1}(b_{i+1})-\left(\frac{b_{i+1}-a_{i+1}}{2}\right)\frac{\partial f_{x_n}^{i+1}(b_{i+1})}{\partial x_i}+\frac{b_{i+1}-a_{i+1}}{6}\right.$$
$$\left.\left(\frac{\partial f_{x_n}^{i+1}(b_{i+1})}{\partial x_i}-\frac{\partial f_{x_n}^{i+1}(a_{i+1})}{\partial x_i}\right)-\frac{1}{b_{i+1}-a_{i+1}}\int_{a_{i+1}}^{b_{i+1}}f_{x_n}^{i+1}(x_{i+1})dx_{i+1}\right|$$
$$\leq\frac{1}{8}(b_{i+1}-a_{i+1})^2(\Gamma_{i+1}-\gamma_{i+1}).$$

On integrating over $[a_i,b_i]$, we get

$$\left|\int_{a_i}^{b_i}f_{x_n}^{i+1}(b_{i+1})dx_i-\frac{(b_{i+1}-a_{i+1})}{6}\int_{a_i}^{b_i}\left(\frac{\partial f_{x_n}^{i+1}(a_{i+1})}{\partial x_i}+2\frac{\partial f_{x_n}^{i+1}(b_{i+1})}{\partial x_i}\right)dx_i\right.$$
$$\left.-\frac{1}{b_{i+1}-a_{i+1}}\int_{a_i}^{b_i}\int_{a_{i+1}}^{b_{i+1}}f(\mathbf{x})dx_{i+1}dx_i\right|\leq\frac{1}{8}(b_{i+1}-a_{i+1})^2(b_i-a_i)(\Gamma_{i+1}-\gamma_{i+1}). \tag{20}$$

Now applying (4) for mapping $f_{x_n}^{i+1}:[a_{i+1},b_{i+1}]\to\mathbb{R}$ at $x=a_{i+1}$, where $i=1,2,...,n$, then integrating over $[a_i,b_i]$, we have

$$\left|\int_{a_i}^{b_i}f_{x_n}^{i+1}(a_{i+1})dx_i+\frac{(b_{i+1}-a_{i+1})}{6}\int_{a_i}^{b_i}\left(\frac{\partial f_{x_n}^{i+1}(b_{i+1})}{\partial x_i}+2\frac{\partial f_{x_n}^{i+1}(a_{i+1})}{\partial x_i}\right)dx_i\right.$$
$$\left.-\frac{1}{b_{i+1}-a_{i+1}}\int_{a_i}^{b_i}\int_{a_{i+1}}^{b_{i+1}}f(\mathbf{x})dx_{i+1}dx_i\right|\leq\frac{1}{8}(b_{i+1}-a_{i+1})^2(b_i-a_i)(\Gamma_{i+1}-\gamma_{i+1}). \tag{21}$$

Adding (20) and (21), we get

$$\left|\frac{1}{2}\int_{a_i}^{b_i}(f_{x_n}^{i+1}(a_{i+1})+f_{x_n}^{i+1}(b_{i+1}))dx_i+\frac{(b_{i+1}-a_{i+1})}{12}\right.$$
$$\left.\int_{a_i}^{b_i}\left(\frac{\partial f_{x_n}^{i+1}(a_{i+1})}{\partial x_i}-\frac{\partial f_{x_n}^{i+1}(b_{i+1})}{\partial x_i}\right)dx_i-\frac{1}{b_{i+1}-a_{i+1}}\int_{a_i}^{b_i}\int_{a_{i+1}}^{b_{i+1}}f(\mathbf{x})dx_{i+1}dx_i\right|$$
$$\leq\frac{1}{8}(b_{i+1}-a_{i+1})^2(b_i-a_i)(\Gamma_{i+1}-\gamma_{i+1}).$$

Summing over i from 1 to n we get (19). This completes the proof.

Remark 3.6. For $n=2$ in Theorem 3.5 and $a_1\to a$, $b_1\to b$, $a_2\to c$, $b_2\to d$, we get inequality (7).

REFERENCES

[1] G. Grüss, Über das maximum des absoluten betrages von $\frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx$, Math. Z. 39 (1935), 215-226.

[2] A. Ostrowski, bed die absolutabweichung einer dierentierbaren funktion vonihren integralmittelwert, Comment. Math. Helv. 10 (1938), 226-227.

[3] G. Farid, S. Rafique, A.U. Rehman, Extensions of Ostrowski-type inequalities on coordinates, Submitted.

[4] N. Barnett, P. Cerone, S. S. Dragomir, J. Roumeliotis, A. Sofo, A survey on Ostrowski type inequalities for twice differentiable mappings and applications In: Inequality Theory and Applications, Nova Publ. Huntington., (2001).

[5] S. S. Dragomir, S. Wang, An inequality of Ostrowski-Grüss type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules, Comput. Math. Appl. 33 (1997), 15-22.

[6] X. L. Cheng, Improvement of some Ostrowski-Grüss type inequalities, Comput. Math. Appl. 42 (2001), 109-114.

[7] G. Farid, A. U. Rehman, Generalization of the Fejér-Hadamard's inequality for convex functions on coordinates, Commun. Korean Math. Soc. 31 (2016), 53-64.

[8] G. Farid, A. U. Rehman, On Chebyshev functional and Ostrowski-Grüss type inequalities for two coordinates, Int. J. Anal. App. 12 (2016), 180-187.