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COUPLED FIXED POINT THEOREMS INVOLVING RATIONAL EXPRESSIONS IN PARTIALLY ORDERED CONE METRIC SPACES

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Abstract. The aim of this paper is to prove some unique coupled fixed point theorems involving rational expressiones in a partially ordered cone metric space. An example is also provided to support our main results.

Keywords. Coupled fixed point; Mixed monotone property; Partially ordered cone metric space.

1. Introduction

The Banach contraction principle, which is the most famous fixed point theorem, plays a very important role in nonlinear analysis. This celebrated principle has been generalized by many authors in various ways. In 1975, Dass and Gupta [1] established an extension of the Banach contraction principle through rational expressions. After then, Jaggi [2] generalized unique fixed point theorems which satisfy a contractive condition of the rational type. Subsequently, many authors investigated these problems in which the rational type is involved, see, [3, 4] and the references therein. Recently, Muhammad Arshad *et al.* [5] introduced the almost Jaggi contraction in partially ordered metric spaces and proved fixed point theorems.

In 2006, Bhaskar and Lakshmikantham [6] introduced the concept of mixed monotone mappings and studied some coupled fixed point results in partially ordered metric spaces. They

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also applied their results to a first order differential equation with periodic boundary conditions. Subsequently, various authors generalized and studied coupled fixed point theorems in different ways; see [3, 7, 8, 9, 10] and the references therein.

Recently, Huang and Zhang [11] generalized the concept of a metric spaces, replacing the set of the real numbers by an ordered Banach space and obtained some fixed point theorems for mappings which satisfy different contractive conditions. After that, many authors generalized their fixed point theorems in different directions; see [12, 13, 14, 15] and the references therein.

The purpose of this paper is to present some unique coupled fixed point theorems having mixed monotone property involving rational expressions in a partially ordered cone metric s-pace.

2. Preliminaries

In this section, we give some definitions which are useful for main result in this paper.

Let E be a real Banach space and let P be a subset of E. P is called a cone if and only if

(a) P is closed, nonempty and $P \neq \{0\}$;
(b) if a, b are nonnegative real numbers and $x, y \in P$, then $ax + by \in P$;
(c) $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, the partial ordering $\leq$ with respect to P is defined by $x \leq y$ if and only if $y - x \in P$. The notation $x \ll y$ stands for $y - x \in intP$, where $intP$ denotes the interior of P. We use $x < y$ to indicate that $x \leq y$ and $x \neq y$.

The cone P is said to be normal if there exists a constant $M > 0$ such that for every $x, y \in E$ if $0 \leq x \leq y$ then $\| x \| \leq M \| y \|$. The least positive number satisfying this inequality is called the normal constant of P. The cone P is called regular if every increasing (decreasing) and bounded above (below) sequence is convergent in E. It is known that every regular cone is normal.

Definition 2.1. [11] Let X be a nonempty set and let E be a Banach space equipped with the partial ordering $\leq$ with respect to the cone $P \subseteq E$. Suppose that the mapping $d : X \times X \to E$ satisfies the following conditions:

(d_1) $0 \leq d(x,y)$ $\forall x, y \in X$ and $d(x,y) = 0 \iff x = y$;

(d_2) $d(x,y) = d(y,x)$ $\forall x, y \in X$;

(d_3) $d(x,y) \leq d(x,z) + d(z,y)$ $\forall x,y,z \in X$

Then d is called a cone metric on X, and (X,d) is called a cone metric space.

Example 2.2. [11] *Let $E = R^2, P = \{(x,y) \in E | x,y \geq 0\} \subseteq R^2, X = R$ and $d : X \times X \to E$ such that $d(x,y) = (|x-y|, \alpha|x-y|)$, where $\alpha \geq 0$ is a constant. Then (X,d) is a cone metric space.*

Definition 2.3. [11] Let (X,d) be a cone metric space, $x \in X$ and $\{x_n\}_{n\geq 1}$ be a sequence in X. Then

(a) $\{x_n\}_{n\geq 1}$ converges to x, denoted by $\lim_{n\to\infty} x_n = x$, if for every $c \in E$ with $0 \ll c$ there exists a natural number N such that $d(x_n,x) \ll c$ for all $n \geq N$;

(b) $\{x_n\}_{n\geq 1}$ is a Cauchy sequence if for every $c \in E$ with $0 \ll c$ there exists a natural number N such that $d(x_n,x_m) \ll c$ for all $n,m \geq N$;

(c) (X,d) is a complete cone metric space, if every Cauchy sequence is convergent.

Definition 2.4. [6] An element $(x,y) \in X \times X$ is said to be coupled fixed point of the mapping $F : X \times X \to X$ if $F(x,y) = x$ and $F(y,x) = y$.

Definition 2.5. [6] Let $(X,\leq)$ be a partially ordered set and $F : X \times X \to X$. We say that F has the mixed monotone property if $F(x,y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x,y \in X$,

$$x_1,x_2 \in X, x_1 \leq x_2 \implies F(x_1,y) \leq F(x_2,y)$$

and

$$y_1,y_2 \in X, y_1 \leq y_2 \implies F(x,y_1) \geq F(x,y_2).$$

The following result of Bhaskar and Lakshmikantham in [6] were extended to class of cone metric spaces in [14].

Definition 2.6. [14] Let (X,d) be a cone metric space. An element $(x,y) \in X \times X$ is said to be coupled fixed point of the mapping $F : X \times X \to X$ if $F(x,y) = x, F(y,x) = y$.

3. Main results

Theorem 3.1. *Let $(X,\leq)$ be a partially ordered set and suppose that there exists a cone metric d in X such that cone metric space (X,d) is complete. Let $F : X \times X \to X$ be a mapping having* *the mixed monotone property on X such that*

$$
\begin{aligned}
d(F(x,y),F(u,v)) \leq \alpha(max\{\frac{d(x,F(x,y))d(u,F(u,v))}{d(x,u)},d(x,u)\}) \\
+\beta(max\{\frac{d(x,F(x,y))d(u,F(u,v))}{d(x,u)},d(x,u)\}) \\
+\gamma(M((x,y),(u,v))),
\end{aligned} \tag{3.1}
$$

where

$$
\begin{aligned}
M((x,y),(u,v)) = min\big(d(x,F(x,y))\frac{2+d(u,F(u,v))+d(v,F(v,u))}{2+d(x,u)+d(y,v)}, \\
d(u,F(u,v))\frac{2+d(x,F(x,y))+d(y,F(y,x))}{2+d(x,u)+d(y,v)}\big),
\end{aligned}
$$

$\forall x,y,u,v \in X$ with $x \geq u, y \leq v$ and there exist non-negative real numbers $\alpha,\beta.\gamma \in [0,1)$ and with $\alpha+\beta+\gamma<1$. Suppose either

1) F is continuous or

2) X has the following properties,

(a) if a non-decreasing sequence$\{x_n\}$ in X converges to some point $x \in X$, then $x_n \leq x$, $\forall n$,

(b) if a non-increasing sequence $\{y_n\}$ in X converges to some point $y \in X$, then $y_n \geq y$, $\forall n$.

Then F has a coupled fixed point.

Proof. Choose $x_0, y_0 \in X$ and set $x_1 = F(x_0,y_0)$ and $y_1 = F(y_0,x_0)$. Repeating this process, set $x_{n+1} = F(x_n,y_n)$ and $y_{n+1} = F(y_n,x_n)$. By (3.1), we have

$$
\begin{aligned}
d(x_n,x_{n+1}) &= d(F(x_{n-1},y_{n-1}),F(x_n,y_n)) \\
&\leq \alpha(max\{\frac{d(x_{n-1},F(x_{n-1},y_{n-1}))d(x_n,F(x_n,y_n))}{d(x_{n-1},x_n)},d(x_{n-1},x_n)\}) \\
&\quad +\beta(max\{\frac{d(x_{n-1},F(x_{n-1},y_{n-1}))d(x_n,F(x_n,y_n))}{d(x_{n-1},x_n)},d(x_{n-1},x_n)\}) \\
&\quad +\gamma(M((x_{n-1},y_{n-1})),(x_n,y_n))) \\
&= \alpha(max\{d(x_n,x_{n+1}),d(x_{n-1},x_n)\})+\beta(max\{d(x_n,x_{n+1}),d(x_{n-1},x_n)\}) \\
&\quad +\gamma\big(d(x_n,F(x_n,y_n))\frac{2+d(x_{n-1},F(x_{n-1},y_{n-1})+d(y_{n-1},F(y_{n-1},x_{n-1})}{2+d(x_{n-1},x_n)+d(y_n,y_{n-1})}\big) \\
&= \alpha(max\{d(x_n,x_{n+1}),d(x_{n-1},x_n)\})+\beta(max\{d(x_n,x_{n+1}),d(x_{n-1},x_n)\}) \\
&\quad +\gamma(d(x_n,x_{n+1})).
\end{aligned} \tag{3.2}
$$

Similarly, we also obtain

$$\begin{aligned} d(y_n,y_{n+1}) &= d(F(y_{n-1},x_{n-1}),F(y_n,x_n)) \\ &\le \alpha(max\{\frac{d(y_{n-1},F(y_{n-1},x_{n-1}))d(y_n,F(y_n,x_n))}{d(y_{n-1},y_n)},d(y_{n-1},y_n)\}) \\ &\quad +\beta(max\{\frac{d(y_{n-1},F(y_{n-1},x_{n-1}))d(y_n,F(y_n,x_n))}{d(y_{n-1},y_n)},d(y_{n-1},y_n)\}) \\ &\quad +\gamma(d(y_n,F(y_n,x_n))\frac{2+d(y_{n-1},F(y_{n-1},x_{n-1})+d(x_{n-1},F(x_{n-1},y_{n-1})}{2+d(y_{n-1},y_n)+d(x_n,x_{n-1})} \\ &= \alpha(max\{d(y_n,y_{n+1}),d(y_{n-1},y_n)\})+\beta(max\{d(y_n,y_{n+1}),d(y_{n-1},y_n)\}) \\ &\quad +\gamma(d(y_n,y_{n+1})). \end{aligned} \tag{3.3}$$

Suppose that $max\{d(x_n,x_{n+1}),d(x_{n-1},x_n)\} = d(x_n,x_{n+1})$ for some $n \ge 1$. Then the inequality turns into

$$d(x_n,x_{n+1}) \le \alpha(d(x_n,x_{n+1}))+\beta(d(x_n,x_{n+1}))+\gamma(d(x_n,x_{n+1})),$$

which is a contradiction. Thus $max\{d(x_n,x_{n+1}),d(x_{n-1},x_n)\} = d(x_{n-1},x_n)$ for some $n \ge 1$. Hence, the inequality yields

$$d(x_n,x_{n+1}) \le \alpha d(x_{n-1},x_n)+\beta d(x_{n-1},x_n)+\gamma d(x_n,x_{n+1}), \tag{3.4}$$

$$d(y_n,y_{n+1}) \le \alpha d(y_{n-1},y_n)+\beta d(y_{n-1},y_n)+\gamma d(y_n,y_{n+1}), \tag{3.5}$$

which imply that

$$(1-\gamma)d(x_n,x_{n+1}) \le (\alpha+\beta)d(x_{n-1},x_n), \tag{3.6}$$

$$(1-\gamma)d(y_n,y_{n+1}) \le (\alpha+\beta)d(y_{n-1},y_n). \tag{3.7}$$

From (3.6) and (3.7), we have

$$d_n \le \frac{(\alpha+\beta)}{(1-\gamma)}d_{n-1}. \tag{3.8}$$

Let $d_n = d(x_n,x_{n+1})+d(y_n,y_{n+1})$. Consequently, if we set $\lambda = \frac{(\alpha+\beta)}{(1-\gamma)}$, then we have

$$d_n \le \lambda d_{n-1} \le \cdots \le \lambda^n d_0. \tag{3.9}$$

If $d_0 = 0$, then (x_0,y_0) is a coupled fixed point of F.

Suppose that $d_0 \geq 0$. Then, for each $r \in N$, we obtain by the repeated application of triangle inequality that

$$\begin{aligned}
&d(x_n, x_{n+r}) + d(y_n, y_{n+r}) \\
&\leq [d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+r-1}, x_{n+r})] \\
&\quad + [d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots + d(y_{n+r-1}, y_{n+r})] \\
&= [d(x_n, x_{n+1}) + d(y_n, y_{n+1})] + [d(x_{n+1}, x_{n+2}) \\
&\quad + d(y_{n+1}, y_{n+2})] + \ldots + [d(x_{n+r-1}, x_{n+r}) + d(y_{n+r-1}, y_{n+r})] \\
&\leq d_n + d_{n+1} + \ldots + d_{n+r-1} \\
&\leq \frac{\lambda^n(1-\lambda^r)}{1-\lambda} d_0.
\end{aligned} \tag{3.10}$$

Let $0 \ll c$ be given. Choose a natural number M such that $\frac{\lambda^n(1-\lambda^r)}{1-\lambda} d_0 \ll c$ for all $m > M$. Thus $d(x_n, x_{n+r}) + d(y_n, y_{n+r}) \ll c$. Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Since X is a complete metric space, $\exists\, x, y \in X$ such that $\lim_{n\to\infty} x_n = x, \lim_{n\to\infty} y_n = y$. We now show if the assumption (1) holds, then (x, y) is coupled fixed point of F. As, we have

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n, y_n) = F(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) = F(x, y)$$

and

$$y = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} F(y_n, x_n) = F(\lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n) = F(y, x).$$

Therefore, (x, y) is coupled fixed point of F.

Suppose that the condition 2(a) and 2(b) of the theorem holds. The sequence $\{x_n\} \to x$, $\{y_n\} \to y$

$$\begin{aligned}
d(F(x,y), F(x_n, y_n)) \leq\ & \alpha(max\{\frac{d(x, F(x,y))d(x_n, F(x_n, y_n))}{d(x, x_n)}, d(x, x_n)\}) \\
& + \beta(max\{\frac{d(x, F(x,y))d(x_n, F(x_n, y_n))}{d(x, x_n)}, d(x, x_n)\}). \\
& + \gamma(d(x, F(x,y))\frac{2 + d(x_n, F(x_n, y_n)) + d(y_n, F(y_n, x_n))}{2 + d(x, x_n) + d(y, y_n)}).
\end{aligned}$$

Letting $n \to \infty$, we have $d(F(x,y), x) \leq 0$. This implies that $F(x,y) = x$. Similarly, we can show that $F(y,x) = y$. This completes the theorem.

Theorem 3.2. *Let the hypotheses of Theorem 3.1 hold. We obtain the uniqueness of the coupled fixed point of F.*

Proof. Suppose (x,y) and (x',y') are coupled fixed points of F, that is, $F(x,y)=x, F(y,x)=y, F(x',y')=x'$ and $F(y',x')=y'$. We shall prove that $x=x', y=y'$. Consider the following two cases:

Case 1. If (x,y) and (x',y') are comparable, we have

$$\begin{aligned} d(x,x') &= d(F(x,y),F(x',y')) \\ &\leq \alpha(max\{\frac{d(x,F(x,y))d(x',F(x',y'))}{d(x,x')},d(x,x')\}) \\ &\quad +\beta(max\{\frac{d(x,F(x,y))d(x',F(x',y'))}{d(x,x')},d(x,x')\}) \\ &\quad +\gamma\big(d(x,F(x,y))\frac{2+d(x',F(x',y'))+d(y',F(y',x'))}{2+d(x,x')+d(y,y')}\big), \end{aligned}$$

which gives $d(x,x')\leq 0$, $(\alpha+\beta+\gamma)<1$ (a contradiction). Thus $x=x'$. Similarly, we have $d(y,y')=d(F(y,x),F(y',x'))\leq 0$. Hence, $y=y'$. Therefore, (x,y) is a unique coupled fixed point of F.

Case 2. Suppose (x,y) and (x',y') are not comparable.

In view of the assumption, there exist $(z,t)\in X\times X$ comparable with both of them.

We define sequences $\{z_n\},\{t_n\}$ as follows $z_0=z$, $t_0=t$, $z_{n+1}=F(z_n,t_n)$ and $t_{n+1}=F(t_n,z_n)$. Since (z,t) is comparable with (x,y), we may assume that $(x,y)\geq(z,t)=(z_0,t_0)$. By using the mathematical induction, it is easy to prove that

$$(x,y)\geq(z_n,t_n), \quad \forall n. \tag{3.11}$$

From (3.1) and (3.11), we have

$$\begin{aligned} d(F(x,y),F(z_n,t_n)) &\leq \alpha(max\{\frac{d(x,F(x,y))d(z_n,F(z_n,t_n))}{d(x,z_n)},d(x,z_n)\}) \\ &\quad +\beta(max\{\frac{d(x,F(x,y))d(z_n,F(z_n,t_n))}{d(x,z_n)},d(x,z_n)\}) \\ &\quad +\gamma(d(x,F(x,y))\frac{2+d(z_n,F(z_n,t_n))+d(t_n,F(t_n,z_n))}{2+d(x,z_n)+d(y,t_n)}), \end{aligned}$$

or

$$d(x,z_{n+1})\leq(\alpha+\beta)d(x,z_n). \tag{3.12}$$

Similarly, we also have

$$d(t_{n+1},y)\leq(\alpha+\beta)d(t_n,y). \tag{3.13}$$

Using (3.12) and (3.13), we get

$$\begin{aligned} d(x,z_{n+1})+d(t_{n+1},y) &\leq (\alpha+\beta)[d(x,z_n)+d(t_n,y)] \\ &\leq (\alpha+\beta)^2[d(x,z_{n-1})+d(t_{n-1},y)] \\ &\vdots \\ &\leq (\alpha+\beta)^{n+1}[d(x,z_0)+d(t_0,y)] \to 0 \;\; as \;\; n\to\infty. \end{aligned}$$

Thus, we have

$$\lim_{n\to\infty} d(x,z_{n+1}) = \lim_{n\to\infty} d(t_{n+1},y) = 0. \tag{3.14}$$

Similarly, we find that

$$\lim_{n\to\infty} d(x',z_{n+1}) = \lim_{n\to\infty} d(t_{n+1},y') = 0. \tag{3.15}$$

From (3.14) and (3.15), we obtain $x = x'$ and $y = y'$.

Theorem 3.3. *Let $(X,\leq)$ be a partially ordered set and suppose that there exist a cone metric d in X such that cone metric space (X,d) is complete. Let $F: X\times X\to X$ be a mapping having the mixed monotone property on X such that*

$$\begin{aligned} d(F(x,y),F(u,v)) \leq \alpha(max\{&\frac{d(x,F(x,y))d(u,F(u,v))}{d(x,u)},d(x,u)\}) \\ &+\beta(max\{\frac{d(x,F(x,y))d(u,F(u,v))}{d(x,u)},d(x,u)\}), \end{aligned} \tag{3.16}$$

$\forall\; x,y,u,v\in X$ with $x\geq u, y\leq v$ and there exist non-negative real numbers $\alpha,\beta\in[0,1)$ with $\alpha+\beta<1$. Suppose either

1) F is continuous or

2) X has the following properties,

(a) if a non-decreasing sequence$\{x_n\}$ in X converges to some point $x\in X$, then $x_n\leq x$, $\forall n$,

(b) if a non-increasing sequence $\{y_n\}$ in X converges to some point $y\in X$, then $y_n\geq y$, $\forall n$.

Then F has a coupled fixed point.

Proof. Choose $x_0, y_0 \in X$ and set $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$. Repeating this process, set $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$. Then by (3.16), we have

$$\begin{aligned}
d(x_n, x_{n+1}) &= d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\
&\leq \alpha(max\{\frac{d(x_{n-1}, F(x_{n-1}, y_{n-1}))d(x_n, F(x_n, y_n))}{d(x_{n-1}, x_n)}, d(x_{n-1}, x_n)\})) \\
&\quad + \beta(max\{\frac{d(x_{n-1}, F(x_{n-1}, y_{n-1}))d(x_n, F(x_n, y_n))}{d(x_{n-1}, x_n)}, d(x_{n-1}, x_n)\}) \\
&= \alpha(max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\}) + \beta(max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\}).
\end{aligned} \tag{3.17}$$

From (3.16), we have

$$\begin{aligned}
d(y_n, y_{n+1}) &= d(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\
&\leq \alpha(max\{\frac{d(y_{n-1}, F(y_{n-1}, x_{n-1}))d(y_n, F(y_n, x_n))}{d(y_{n-1}, y_n)}, d(y_{n-1}, y_n)\}) \\
&\quad + \beta(max\{\frac{d(y_{n-1}, F(y_{n-1}, x_{n-1}))d(y_n, F(y_n, x_n))}{d(y_{n-1}, y_n)}, d(y_{n-1}, y_n)\}) \\
&= \alpha(max\{d(y_n, y_{n+1}), d(y_{n-1}, y_n)\}) + \beta(max\{d(y_n, y_{n+1}), d(y_{n-1}, y_n)\}).
\end{aligned} \tag{3.18}$$

Suppose that $max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} = d(x_{n-1}, x_n)$ for some $n \geq 1$. Then the inequality yields $d(x_n, x_{n+1}) \leq \alpha(d(x_n, x_{n+1})) + \beta(d(x_n, x_{n+1}))$, which is a contradiction. Thus

$$max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} = d(x_n, x_{n+1}).$$

Therefore, the inequality yields

$$d(x_n, x_{n+1}) \leq \alpha d(x_{n-1}, x_n) + \beta d(x_{n-1}, x_n), \tag{3.19}$$

$$d(y_n, y_{n+1}) \leq \alpha d(y_{n-1}, y_n) + \beta d(y_{n-1}, y_n), \tag{3.20}$$

which imply that

$$d(x_n, x_{n+1}) \leq (\alpha + \beta) d(x_{n-1}, x_n), \tag{3.21}$$

$$d(y_n, y_{n+1}) \leq (\alpha + \beta) d(y_{n-1}, y_n). \tag{3.22}$$

Using (3.21) and (3.22), we have

$$d_n \leq (\alpha + \beta) d_{n-1}. \tag{3.23}$$

Let $d_n = d(x_n, x_{n+1}) + d(y_n, y_{n+1})$. Consequently, if we set $\lambda = (\alpha + \beta)$, then we have

$$d_n \leq \lambda d_{n-1} \leq \ldots \leq \lambda^n d_0. \tag{3.24}$$

If $d_0 = 0$, then (x_0, y_0) is a coupled fixed point of F. Suppose that $d_0 \geq 0$. Then, for each $r \in N$, we obtain by d the repeated application of triangle inequality that

$$\begin{aligned} & d(x_n, x_{n+r}) + d(y_n, y_{n+r}) \\ & \leq [d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+r-1}, x_{n+r})] \\ & \quad + [d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots + d(y_{n+r-1}, y_{n+r})] \\ & = [d(x_n, x_{n+1}) + d(y_n, y_{n+1})] + [d(x_{n+1}, x_{n+2}) \\ & \quad + d(y_{n+1}, y_{n+2})] + \ldots + [d(x_{n+r-1}, x_{n+r}) + d(y_{n+r-1}, y_{n+r})] \\ & \leq d_n + d_{n+1} + \ldots + d_{n+r-1} \\ & \leq \frac{\lambda^n (1 - \lambda^r)}{1 - \lambda} d_0. \end{aligned} \tag{3.25}$$

Let $0 \ll c$ be given. Choose a natural number M such that $\frac{\lambda^n(1-\lambda^r)}{1-\lambda} d_0 \ll c$ for all $m > M$. Thus $d(x_n, x_{n+r}) + d(y_n, y_{n+r}) \ll c$. Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Since X is a complete metric space, $\exists\, x, y \in X$ such that $\lim_{n\to\infty} x_n = x, \lim_{n\to\infty} y_n = y$. We now show that if assumption (1) holds, then (x, y) is coupled fixed point of F. As, we have

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n, y_n) = F(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) = F(x, y),$$

and

$$y = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} F(y_n, x_n) = F(\lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n) = F(y, x).$$

Therefore, (x, y) is coupled fixed point of F.

Now suppose that the condition 2(a) and 2(b) of the theorem holds. The sequence $\{x_n\} \to x$, $\{y_n\} \to y$

$$\begin{aligned} d(F(x,y), F(x_n, y_n)) \leq\ & \alpha(max\{\frac{d(x, F(x,y)) d(x_n, F(x_n, y_n))}{d(x, x_n)}, d(x, x_n)\}) \\ & + \beta(max\{\frac{d(x, F(x,y)) d(x_n, F(x_n, y_n))}{d(x, x_n)}, d(x, x_n)\}). \end{aligned}$$

Letting $n \to \infty$, we have $d(F(x,y), x) \leq 0$. This implies that $F(x,y) = x$. Similarly, we can show that $F(y,x) = y$. This completes the theorem.

Theorem 3.4. *Let the hypotheses of Theorem 3.3 hold. We obtain the uniqueness of the coupled fixed point of F.*

Proof. Suppose (x,y) and (x',y') are coupled fixed points of F, that is, $F(x,y)=x, F(y,x)=y, F(x',y')=x'$ and $F(y',x')=y'$. Next, we prove $x=x', y=y'$. Consider the following two cases:

Case 1. If (x,y) and (x',y') are compareable, we have

$$\begin{aligned} d(x,x') &= d(F(x,y),F(x',y')) \\ &\leq \alpha(max\{\frac{d(x,F(x,y))d(x',F(x',y'))}{d(x,x')},d(x,x')\}) \\ &\quad +\beta(max\{\frac{d(x,F(x,y))d(x',F(x',y'))}{d(x,x')},d(x,x')\}), \end{aligned}$$

which gives $d(x,x') \leq 0$, $(\alpha+\beta)<1$ (a contradiction). Thus $x=x'$. Similarly, we have $d(y,y')=d(F(y,x),F(y',x')) \leq 0$. Hence, $y=y'$. Therefore, (x,y) is a unique coupled fixed point of F.

Case 2. Suppose (x,y) and (x',y') are not compareable.

From the assumption, we see that there exist $(z,t) \in X \times X$ comparable with both of them.

We define sequences $\{z_n\},\{t_n\}$ as follows $z_0=z$, $t_0=t$, $z_{n+1}=F(z_n,t_n)$ and $t_{n+1}=F(t_n,z_n)$. Since (z,t) is comparable with (x,y), we may assume that $(x,y) \geq (z,t)=(z_0,t_0)$. By using the mathematical induction, it is easy to prove that

$$(x,y) \geq (z_n,t_n), \quad \forall n. \tag{3.26}$$

From (3.16) and (3.26), we have

$$\begin{aligned} d(F(x,y),F(z_n,t_n)) &\leq \alpha(max\{\frac{d(x,F(x,y))d(z_n,F(z_n,t_n))}{d(x,z_n)},d(x,z_n)\}) \\ &\quad +\beta(max\{\frac{d(x,F(x,y))d(z_n,F(z_n,t_n))}{d(x,z_n)},d(x,z_n)\}) \end{aligned}$$

or

$$d(x,z_{n+1}) \leq (\alpha+\beta)d(x,z_n). \tag{3.27}$$

Similarly, we also have

$$d(t_{n+1},y) \leq (\alpha+\beta)d(t_n,y). \tag{3.28}$$

From (3.27) and (3.28), we get

$$\begin{aligned} d(x,z_{n+1})+d(t_{n+1},y) &\leq (\alpha+\beta)[d(x,z_n)+d(t_n,y)] \\ &\leq (\alpha+\beta)^2[d(x,z_{n-1})+d(t_{n-1},y)] \\ &\vdots \\ &\leq (\alpha+\beta)^{n+1}[d(x,z_0)+d(t_0,y)] \to 0 \ \ as \ \ n\to\infty. \end{aligned}$$

Thus,

$$\lim_{n\to\infty} d(x,z_{n+1}) = \lim_{n\to\infty} d(t_{n+1},y) = 0. \tag{3.29}$$

Similarly, we find that

$$\lim_{n\to\infty} d(x',z_{n+1}) = \lim_{n\to\infty} d(t_{n+1},y') = 0. \tag{3.30}$$

From (3.29) and (3.30), we obtain $x = x'$ and $y = y'$.

Example 3.5. *Let $E = R^2$, the Euclidean plane and $P = \{(x,y) \in R^2 : x,y \geq 0\}$ a normal cone in P. Let $X = \{(x,0) \in R^2 : 0 \leq x \leq 1\} \cup \{(0,x) \in R^2 : 0 \leq x \leq 1\}$. The mapping $d : X \times X \to E$ is defined by*

$$\begin{aligned} d((x,0),(y,0)) &= (\frac{5}{3}|x-y|,|x-y|), \\ d((0,x),(0,y)) &= (|x-y|,\frac{2}{3}|x-y|), \\ d((x,0),(0,y)) &= d((0,y),(x,0)) = (\frac{5}{3}x+y,x+\frac{2}{3}y). \end{aligned}$$

Then (X,d) is complete cone metric space.

Consider the mapping $F : X \times X \to X$ defined by

$$\begin{aligned} F((x,0),(0,x)) &= (\frac{x}{4},0), \\ F((0,x),(x,0)) &= (0,\frac{x}{2}). \end{aligned}$$

X satisfies properties (i) and (ii) in Theorem 3.3. Clearly F is continous and has the mixed monotone property. Also there are $x_0 = 0; y_0$ =0 in X such that $x_0 = 0 \leq F(0,0) = F(x_0,y_0)$ and $y_0 = 0 \geq F(0,0) = F(y_0,x_0)$. We claim that (3.16) holds for each $x \geq u$, $y \leq v$.

Case 1.

$$\begin{aligned}
d(F((x,0),(0,x)),F((0,y),(y,0)) &= d((\frac{x}{4},0),(0,\frac{y}{2})) \\
&= (\frac{5}{3}(\frac{x}{4})+\frac{y}{2},\frac{x}{4}+\frac{2}{6}y) \\
&\leq (\frac{5}{12}x+y,\frac{x}{2}+\frac{2}{3}y) \\
&\leq \frac{2}{3}(\frac{5}{3}x+y,x+\frac{2}{3}y) \\
&= \alpha(max\{\frac{d(x,F(x,y))d(u,F(u,v))}{d(x,u)},d(x,u)\}) \\
&\quad +\beta(max\{\frac{d(x,F(x,y))d(u,F(u,v))}{d(x,u)},d(x,u)\}).
\end{aligned}$$

Case 2.

$$\begin{aligned}
d(F((0,x),(x,0),F((0,y),(y,0)) &= d((0,\frac{x}{2}),(0,\frac{y}{2})) \\
&= (|\frac{x}{2}-\frac{y}{2}|,\frac{2}{3}|\frac{x}{2}-\frac{y}{2}|) \\
&\leq \frac{2}{3}(|x-y|,\frac{2}{3}|x-y|) \\
&= \alpha(max\{\frac{d(x,F(x,y))d(u,F(u,v))}{d(x,u)},d(x,u)\}) \\
&\quad +\beta(max\{\frac{d(x,F(x,y))d(u,F(u,v))}{d(x,u)},d(x,u)\}).
\end{aligned}$$

Case 3.

$$\begin{aligned}
d(F((x,0),(0,x)),F((y,0),(0,y)) &= d((\frac{x}{4},0),(\frac{y}{4},0)) \\
&= (\frac{5}{3}|\frac{x}{4}-\frac{y}{4}|,|\frac{x}{4}-\frac{y}{4}|) \\
&\leq \frac{2}{3}(\frac{5}{3}|x-y|,|x-y|) \\
&= \alpha(max\{\frac{d(x,F(x,y))d(u,F(u,v))}{d(x,u)},d(x,u)\}) \\
&\quad +\beta(max\{\frac{d(x,F(x,y))d(u,F(u,v))}{d(x,u)},d(x,u)\}).
\end{aligned}$$

Case 4.

$$
\begin{aligned}
d(F((0,x),(x,0),F((y,0),(0,y)) &= d((0,\frac{x}{2}),(\frac{y}{4},0)) \\
&= (|\frac{x}{2}-\frac{y}{4}|,\frac{2}{3}|\frac{x}{2}-\frac{y}{4}|) \\
&\leq \frac{2}{3}(\frac{5}{3}x+y,x+\frac{2}{3}y) \\
&= \alpha(max\{\frac{d(x,F(x,y))d(u,F(u,v))}{d(x,u)},d(x,u)\}) \\
&\quad +\beta(max\{\frac{d(x,F(x,y))d(u,F(u,v))}{d(x,u)},d(x,u)\}).
\end{aligned}
$$

We deduce that all the hypotheses of Theorem 3.3 are satisfied $\alpha+\beta<1$*, where* α,β *are such that* $\alpha=\beta=\frac{1}{3}$*. Then it is obvious that* $(0,0)$ *is the coupled fixed point of* F*.*

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