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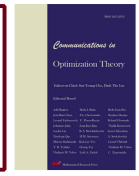

GENERAL SET-VALUED VECTOR VARIATIONAL INEQUALITY PROBLEMS

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Abstract. In this paper, we establish the existence results for general set-valued vector variational inequalities. As an application, we also discuss some coincidence point results.

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1. Introduction

Let $\mathscr{X}$ and $\mathscr{Y}$ be two arbitrary sets and $f : \mathscr{X} \to \mathscr{Y}$, $T : \mathscr{X} \rightrightarrows \mathscr{Y}$ be two given mappings. We say that a point $x \in X$ is a coincidence point of f and T if $f(x) \in T(x)$. Coincidence theory is, in most settings a generalization of fixed point theory, the study of a point x with $x \in T(x)$ indeed a fixed point is the special case obtained from the coincidence point by letting $\mathscr{X} = \mathscr{Y}$ and taking f to be the identity mapping. The theory variational inequality problem is very powerful techniques for studying problems arising in mechanics optimization, transportation, economics equilibrium, control theory, contact problems in elasticity and other branches of mathematics; see [2, 3, 12, 13, 15]. Let $\mathscr{X}$ and $\mathscr{Y}$ be two real Banach spaces. A nonempty set P of $\mathscr{X}$ is called convex cone, if $\lambda P \subseteq P$ for all $\lambda \geq 0$ and $P + P = P$. A cone P is called pointed cone, if P is a

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cone and $P\cap(-P)=\{0\}$ where 0 denotes the zero vector. Also a cone P is called proper if it is properly contained in $\mathscr{X}$. Let K be a nonempty subset of $\mathscr{X}$. We will denote by 2^K the set of all nonempty subsets of K, $cl_{\mathscr{X}}(K)$ the closure of K in $\mathscr{X}$. Let $T:K\to 2^{L(\mathscr{X},\mathscr{Y})}$ be a multivalued mapping where $L(\mathscr{X},\mathscr{Y})$ be the set of bounded linear function from $\mathscr{X}$ to $\mathscr{Y}$, the graph of T denoted by $\mathscr{G}(T)$ is the set $\{(x,y)\in\mathscr{X}\times\mathscr{Y}: x\in\mathscr{X}, y\in T(x)\}$. T is said to be upper semi continuous on $\mathscr{X}$ if for each $x\in\mathscr{X}$ and each open set U in $\mathscr{Y}$ containing $T(x)$ then exists an open neighbourhood V of x in $\mathscr{X}$ such that $T(y)\subseteq U$ for each $y\in\mathscr{Y}$. T is said to be upper hemi continuous at x if for each $y\in\mathscr{X},\lambda\in[0,1]$, the multivalued mapping $\lambda\to T(\lambda y+(1-\lambda)x)$ is upper semi continuous at 0^+. Let $C:K\to 2^{\mathscr{Y}}$ be a multivalued mapping such that for each $x\in\mathscr{X}, C(x)$ is a closed convex moving cone with $intC(x)\neq\emptyset$ where $intC(x)$ denotes the interior of $C(x)$. Let $f:K\to K$ be a single valued mapping and $T:K\rightrightarrows 2^{L(\mathscr{X},\mathscr{Y})}$ be a multivalued mapping. We consider the *general set valued vector variational inequality problems* for finding $x\in K, f(x)\in K, u\in T(x)$ such that

$$\langle u, f(y)-f(x)\rangle \notin -intC(x),\ \forall f(y)\in K. \tag{1}$$

Let us denote by $S_w(T,f,K)$ *respectively* $S(T,f,K)$ the set of solution of (1). We consider the following problem for finding an $x\in K$ such that

$$\forall y\in K, \exists u\in T(x): \langle u, f(y)-f(x)\rangle \notin -intC(x), \tag{2}$$

$$\exists u\in T(x), \forall y\in K: \langle u, f(y)-f(x)\rangle \notin -intC(x), \tag{3}$$

$$\forall y\in K, \exists v\in T(y): \langle v, f(y)-f(x)\rangle \notin -intC(x). \tag{4}$$

In this pape, inspired by the works [1, 4, 5, 9, 11, 14, 17, 18, 21, 22, 23, 24, 25], we give some existence results for the solutions of the problems (2)-(4) and also discuss the coincidence point results.

2. Preliminaries

Let $\mathscr{X}_1$ and $\mathscr{X}_2$ be the Hausdorff topological vector spaces and $T:\mathscr{X}_1\rightrightarrows\mathscr{X}_2$ be a set valued mapping with nonempty set values. T is said to be upper semicontinuous if for every $x_0\in\mathscr{X}_1$

and for every open set N containing $T(x_0)$ there exists a neighbourhood M of x_0 such that $T(M) \subseteq N$.

Lemma 2.1. *If T is compact valued then T is upper semicontinuous if and only if for every net $\{x_i\} \subseteq \mathscr{X}_1$ such that $x_i \to x_0 \in \mathscr{X}_1$ and for every $z_i \in T(x_i)$ there exists $z_0 \in T(x_0)$ and a subnet $\{z_{i_j}\}$ of $\{z_i\}$ such that $z_{i_j} \to z_0$.*

Let $\mathscr{X}$ and $\mathscr{Y}$ be two Banach spaces. $T : \mathscr{X} \to L(\mathscr{X}, \mathscr{Y})$ is called weak to norm-sequentially continuous at $x \in \mathscr{X}$ if every sequence $\{x_n\}$ that converges weakly to x, we know that $T(x_n)$ converges to $T(x)$ in the topology of the norm of $\mathscr{Y}$. An operator $T : \mathscr{X} \rightrightarrows L(\mathscr{X}, \mathscr{Y})$ is said to be weak to weak* upper semicontinuous if for every $x_0 \in \mathscr{X}$ and for every open set $N \subseteq \mathscr{Y}$ in the weak* topology of $\mathscr{Y}$ containing $T(x_0)$ there exists a neighbourhood M of x_0 in the weak topology of $\mathscr{X}$ such that $T(M) \subseteq N$.

Lemma 2.2. [10] *If $P \subset Q \subset \mathscr{X}$ where Q is weakly compact and P is weakly sequentially closed then P is weakly compact.*

Lemma 2.3. [20] *Consider a bounded net $\{(x_i, x_i^*)\}_{i \in I} \subset \mathscr{X} \times \mathscr{X}^*$ and assume that one of the following conditions is fulfilled:*

(i) $x_i \to x$ i.e., the net $\{x_i\}$ converges to x in the weak topology of $\mathscr{X}$ and $x_i^* \to x^*$ i.e., the net $\{x_i^*\}$ converges to x^* in the topology of norm of $\mathscr{X}^*$;

(ii) $x_i \to x$ i.e., the net $\{x_i\}$ converges to x in the weak topology of norm of $\mathscr{X}$ and $x_i^* \to x^*$ i.e., the net $\{x_i^*\}$ converges to x^* in the weak* topology of $\mathscr{X}^*$.

Then

$$\langle x_i^*, x_i \rangle \to \langle x^*, x \rangle.$$

Now we present the notion of KKM-mapping. Let $\mathscr{X}$ be a real Banach space and $D \subseteq \mathscr{X}$. Recall the convex hull of the set D is defined as the set

$$co(D) = \{\sum_{i=1}^{n} \lambda_i x_i : x_i \in D, \sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0, \ \forall i \in \{1, 2, \cdots, n\}, n \in \mathbb{N}\}.$$

Definition 2.4. [7] Let $\mathscr{X}$ be a Hausdorff topological real linear spaces and $M \subseteq \mathscr{X}$. The set valued mapping $G : M \rightrightarrows \mathscr{X}$ is called a KKM-mapping if for every finite number of elements

$x_1, x_2, \cdots, x_n \in M$ one has

$$co\{x_1, x_2, \cdots, x_n\} \subseteq \bigcup_{i=1}^{n} G(x_i).$$

Lemma 2.5. [7] *Let $\mathscr{X}$ be a Hausdorff topological real linear spaces and $M \subseteq \mathscr{X}$ and $G : M \rightrightarrows \mathscr{X}$ be a KKM-mapping. If $G(x)$ is closed for every $x \in M$ and there exists $x_0 \in M$ such that $G(x_0)$ is compact, then*

$$\bigcap_{x \in M} G(x) \neq \emptyset.$$

Lemma 2.6. [20] *Let $\mathscr{X}$ be a Banach spaces, $M \subseteq \mathscr{X}$ a nonempty set and $G : M \rightrightarrows \mathscr{X}$ be a KKM-mapping. If $G(x)$ is weakly sequentially closed for every $x \in M$ and there exists $x_0 \in M$ such that $G(x_0)$ is weakly compact, then*

$$\bigcap_{x \in M} G(x) \neq \emptyset.$$

Let $\mathscr{X}$ be a real linear space. For $x, y \in \mathscr{X}$ let us denote by $[x, y] = \{z = (1-t)x + ty : t \in [0, 1]\}$ the closed line segment with the end points x *respectively* y. The open line segment with the endpoints x *respectively* y is defined by $(x, y) = [x, y] \setminus \{x, y\} = \{z = (1-t)x + ty : t \in (0, 1)\}$.

Definition 2.7. Let $\mathscr{X}$ and $\mathscr{Y}$ be two real linear spaces. We say that the operator $T : D \subseteq \mathscr{X} \to \mathscr{Y}$ is a quasi linear type if for every $x, y \in D$ and every $z \in [x, y] \cap D$, one has

$$T(z) \in [T(x), T(y)].$$

Proposition 2.8. [20] Let $\mathscr{X}, \mathscr{Y}, \mathscr{Z}$ be real linear spaces, $D \subseteq \mathscr{X}, C \subseteq \mathscr{Y}$ and $A : D \to \mathscr{Y}, B : C \to \mathscr{Z}, A(D) \subseteq C$ be two operator of quasi linear type. Then $B \circ A : D \to \mathscr{Z}$ is of quasi linear type.

Lemma 2.9. [19] *Let $\mathscr{X}$ be a topological real linear space and $\mathscr{Y}$ be a Hausdorff topological real linear space. Let $D \subseteq \mathscr{X}$ be a convex and $A : D \to \mathscr{Y}$ an operator continuous on line segments and of quasi linear type. Then for every $x, y \in D$ one has*

$$A([x, y]) = [A(x), A(y)].$$

Theorem 2.10. [19] *Let $\mathscr{X}$ and $\mathscr{Y}$ be two real linear spaces, let $D \subseteq \mathscr{X}$ be a convex and $T : D \to \mathscr{Y}$ be an operator of quasi linear type. Then for every $n \in \mathbb{N}$ every $x_1, x_2, \cdots, x_n \in D$ and every $x \in co\{x_1, x_2, \cdots, x_n\}$, we have*

$$T(x) \in co\{T(x_1), T(x_2), \cdots, T(x_n)\}.$$

Let $\mathscr{X}$ be a real Banach spaces and $\mathscr{Y}$ be a topological vector spaces and let $T : D \subseteq \mathscr{X} \to L(\mathscr{X}, \mathscr{Y})$ and $f : D \to \mathscr{X}$ be a given operator. We say that (T, f) is f-pseudomonotone if for all $x, y \in D, u \in T(x), v \in T(y)$,

$$\langle u, f(y) - f(x) \rangle \notin -intC(x) \ \Rightarrow \langle v, f(y) - f(x) \rangle \notin -intC(x),$$

and the pair (T, F) is called weakly f-pseudomonotone in K if for all $x, y \in K$

$$\exists u \in T(x), \langle u, f(y) - f(x) \rangle \notin -intC(x) \ \Rightarrow \exists v \in T(y), \langle v, f(y) - f(x) \rangle \notin -intC(x).$$

Theorem 2.11. [6] *Let $\mathscr{Y}$ be a topological vector spaces with a closed, convex, pointed cone C such that $intC \neq \emptyset$ then for all $x, y, z \in \mathscr{Y}$ we have*

(i) $x - y \in -intC$ *and* $x \notin -intC \Rightarrow y \notin -intC$;
(ii) $x + y \in -C$ *and* $x + z \notin -intC \Rightarrow z - y \notin -intC$;
(iii) $x + z - y \notin -intC$ *and* $-y \in -C \Rightarrow x + z \notin -intC$;
(iv) $x + y \notin -intC$ *and* $y - z \in -C \Rightarrow x + z \notin -intC$.

3. Existence results

In this section, we discuss the existence results of the general set valued vector variational inequality problems.

Theorem 3.1. *Let K be a nonempty weakly compact convex subset of $\mathscr{X}$ and a set valued mapping $T : K \to L(\mathscr{X}, \mathscr{Y})$ be the generalized upper hemicontinuous in K with nonempty compact values. Further, let $f : K \to \mathscr{X}$ be an operator of quasi linear type, convex and pair (T, f) is weakly f-pseudo monotone for each $x \in K$. Assume that $C : K \to 2^{\mathscr{Y}}$ is a mapping such that $x \in K, C(x)$ is a proper closed convex moving cone with $intC(x) \neq \emptyset$. Assume that mapping*

$x \to \mathscr{Y} \setminus (-intC(x))$ for $x \in K$ is a weakly closed mapping that is its graph is closed in $\mathscr{X} \times \mathscr{Y}$ with weak topology of $\mathscr{X}$ and $\mathscr{Y}$. Assume that the following conditions are fulfilled.

(a) *f is weak to weak-sequentially continuous on K, T is weak to norm upper semi continuous on K and $T(x)$ is compact for every $x \in K$,*

(b) *f is weak to norm-sequentially continuous on K, T is weak to weak* upper semi continuous on K and $T(x)$ is weak* compact for every $x \in K$,*

(c) *there is a nonempty weakly compact subset D of K and a subset D_0 of a weakly compact convex subset of K such that for all $x \in K \setminus D$ there exists $z \in D_0, u \in T(z)$,*

$$\langle u, f(y) - f(x) \rangle \notin -intC(x),\ \forall y \in D_0.$$

Then $S_\omega(T, f, K) \neq \emptyset$. If, in addition, T is f-pseudomonotone, then $M(T, f, K) \neq \emptyset$.

Proof. To prove the theorem, we first define the mapping $G : K \rightrightarrows K$

$$G(y) = \{x \in K : \exists u \in T(x) \text{ such that } \langle u, f(y) - f(x) \rangle \notin -intC(x)\},$$

which satisfies the assumptions of Ky Fan Lemma. Let $y_1, y_2, \cdots, y_n \in K$ and $y \in co\{y_1, y_2, \cdots, y_n\}$. Suppose that

$$y \notin \bigcup_{i=1}^{n} G(y_i).$$

Then for all $u \in T(y)$ we have

$$\langle u, f(y_1) - f(y) \rangle \in -intC(y),\ \langle u, f(y_2) - f(y) \rangle \in -intC(y),$$

$$\cdots \langle u, f(y_n) - f(y) \rangle \in -intC(y).$$

Since f is quasi linear type and $y \in co\{y_1, y_2, \cdots, y_n\}$, we find from Theorem 2.10 that

$$f(y) \in co\{f(y_1), f(y_2), \cdots, f(y_n)\}.$$

Hence there exists $\lambda_i \geq 0, i = \overline{1,n}$ where $\sum_{i=1}^{n} \lambda_i = 1$ such that

$$f(y) = \sum_{i=1}^{n} \lambda_i f(y_i).$$

For every fixed $u \in T(y)$, we have

$$0 = \sum_{i=1}^{n} \lambda_i \langle u, f(y_i) - f(y) \rangle = \sum_{i=1}^{n} \langle u, \lambda_i (f(y_i) - f(y)) \rangle = -intC(y),$$

which is a contradiction. Hence, G is a KKM mapping. We show next that $G(y)$ is weakly compact for all $y \in K$. Obviously $G(y) \neq \emptyset$. Since for all $y \in K$ we have $y \in G(y)$. For $y \in K$ consider a sequence $\{x_n\} \subseteq G(y)$ that converges weakly to $x \in K$. Hence there exists $u_n \in T(x_n)$ such that

$$\langle u_n, f(y) - f(x_n)\rangle \notin -intC(x_n).$$

Assume that (a) hold. From Lemma 2.1, we obtain the sequence $\{u_n\}$ contains a subsequence $\{u_{n_k}\}$ that converges to $v \in T(x)$ in the norm topology $\mathscr{Y}$. Since f is weak to weak sequentially continuous, we obtain that $f(x_{n_k})$ converges to $f(x), k \to \infty$ in the weak topology of $\mathscr{X}$. From Lemma 2.3 (i), we have

$$\langle u_{n_k}, f(y) - f(x_{n_k})\rangle \to \langle v, f(y) - f(x)\rangle, k \to \infty.$$

Hence

$$\langle u_{n_k}, f(y) - f(x_{n_k})\rangle \notin -intC(x_{n_k}).$$

This implies that

$$\langle v, f(y) - f(x)\rangle \notin -intC(x).$$

Assume that (b) holds. From Lemma 2.1 we obtain that the sequence $\{u_n\}$ contain a subsequence $\{u_{n_k}\}$ that converges to $v \in T(x)$ in the weak* topology of $\mathscr{Y}$. Since f is weak to norm-sequentially continuous, we obtain that $f(x_{n_k}) \to f(x), k \to \infty$. From Lemma 2.3 (ii), we have

$$\langle u_{n_k}, f(y) - f(x_{n_k})\rangle \to \langle v, f(y) - f(x)\rangle, k \to \infty.$$

Hence, we have

$$\langle u_{n_k}, f(y) - f(x_{n_k})\rangle \notin -intC(x_{n_k}),\ \forall k \in \mathbb{N}.$$

It follows that

$$\langle v, f(y) - f(x)\rangle \notin -intC(x).$$

Hence $x \in G(y)$, which means that $G(y)$ is weakly sequentially closed for all $y \in K$. But $G(y) \subseteq K$ and K is weakly compact thus from Lemma 2.2, $G(y)$ is weakly compact for all $y \in K$, implies that is weakly closed as well. Hence G is a KKM mapping that satisfies the assumptions of Ky Fan Lemma, consequentially

$$\bigcap_{y\in K} G(y) \neq \emptyset.$$

In other words there exists $x \in K$ such that for every $y \in K$ there exists $u \in T(x)$ satisfying

$$\langle u, f(y) - f(x)\rangle \notin -intC(x).$$

This show that $S_\omega(T, f, K) \neq \emptyset$.

Now, we assume that T is f-pseudomonotone. Let $x \in S_\omega(T, f, K)$, then for all $y \in K$ there exists $u \in T(x)$ such that

$$\langle u, f(y) - f(x)\rangle \notin -intC(x),$$

and from f-pseudomonotonicity of T, we have

$$\langle v, f(y) - f(x)\rangle \notin -intC(x),\ \forall v \in T(y).$$

Hence $x \in M(T, f, K)$.

Corollary 3.2. *Let K be a nonempty weakly compact convex subset of $\mathscr{X}$ and $T : K \to L(\mathscr{X}, \mathscr{Y})$ be a generalized upper hemicontinuous in K with nonempty compact values. Let $f : K \to X$ be a convex and pair (T, f) is weakly f-pseudo monotone for each $x \in K$. Further assume that f is of quasi linear type operator. Assume that $C : K \to 2^{\mathscr{Y}}$ is mapping such that $x \in K, C(x)$ is a proper closed convex moving cone with $intC(x) \neq \emptyset$. Assume that mapping $x \to \mathscr{Y} \setminus (-intC(x))$ for $x \in K$ is a weakly closed mapping that is its graph is closed in $\mathscr{X} \times \mathscr{Y}$ with weak topology of $\mathscr{X}$ and $\mathscr{Y}$. Then the following conditions are holds:*

(a) *f is weak to weak-sequentially continuous on K, T is weak to norm -sequentially continuous on K,*
(b) *f is weak to norm-sequentially continuous on K, T is weak to weak* continuous on K,*
(c) *there is a nonempty weakly compact subset D of K and a subset D_0 of a weakly compact convex subset of K such that for all $x \in K \setminus D$ then there exists $z \in D_0, u \in T(z)$*

$$\langle u, f(y) - f(x)\rangle \notin -intC(x),\ \forall y \in D_0.$$

Then $S_\omega(T,f,K) \neq \emptyset$. If, in addition, T is f-pseudomonotone, then $M(T,f,K) \neq \emptyset$.

Theorem 3.3. *Let $\mathscr{X}$ be a reflexive Banach space and let $K \subset \mathscr{X}$ be a nonempty weakly sequentially closed convex subset. Assume that $C : K \to 2^{\mathscr{Y}}$ is mapping such that $x \in K, C(x)$ is a proper closed convex moving cone with $intC(x) \neq \emptyset$. Assume that mapping $x \to \mathscr{Y} \setminus (-intC(x))$ for $x \in K$ is a weakly closed mapping that is its graph is closed in $\mathscr{X} \times \mathscr{Y}$ with weak topology of $\mathscr{X}$ and $\mathscr{Y}$. Consider a set valued mapping $T : K \to L(\mathscr{X}, \mathscr{Y})$ and $f : K \to \mathscr{X}$ is a quasi linear type. Assume that there exists $y_0 \in K$ such that*

$$\lim_{\|x\| \to \infty} \inf \inf_{u \in T(x)} \langle u, f(x) - f(y_0) \rangle \in -intC(x).$$

Assume that T is generalized upper hemi continuous in K with nonempty compact values and f is convex and pair (T, f) is weakly f-pseudomonotone for each $x \in K$ then there is a nonempty weakly compact subset D of K and a subset of K such that for all $x \in K \setminus D$ there exists $z \in D_0, u \in T(z)$

$$\langle u, f(y) - f(x) \rangle \notin -intC(x), \ \forall y \in D_0.$$

Moreover, assume that one of the following conditions are fulfilled:

(a) *f is weak to weak-sequentially continuous on K, T is weak to norm -upper semicontinuous on K and $T(x)$ is compact for every $x \in K$,*

(b) *f is weak to norm-sequentially continuous on K, T is weak to weak* upper semi continuous on K and $T(x)$ is weak* compact for every $x \in K$.*

Then $S_\omega(T,f,K) \neq \emptyset$. If, in addition, T is f-pseudomonotone, then $M(T,f,K) \neq \emptyset$.

Proof. Let us define the mapping $G : K \rightrightarrows K$ as in the proof of Theorem 3.1. From the proof of Theorem 3.1, $G(y)$ is weakly sequentially closed for all $y \in K$. We show that $G(y_0)$ is weakly compact. The rest of the proof is similar to the proof of Theorem 3.1 and is here omitted. We prove that $G(y_0)$ is bounded. Indeed suppose to contrary, we obtain that there exists $\{x_k\} \subseteq G(y_0)$ such that

$$\inf_{u \in T(x_k)} \langle u, f(x_k) - f(y_0) \rangle \notin -intC(x_k).$$

Hence

$$\liminf_{\|x_K\| \to \infty} \inf_{u \in T(x_k)} \langle u, f(x_k) - f(y_0) \rangle \notin -intC(x_k),$$

which contradicts the assumptions of the theorem. Thus we have $G(y_0)$ is bounded and weakly sequentially closed. But then there exists $N > 0$ such that $G(y_0) \subseteq B_N$ where B_N denotes the closed ball centered in 0 with radius N. Since $\mathscr{X}$ is reflexive it is known that B_N is weakly compact. From Lemma 2.2, we conclude that $G(y_0)$ is weakly compact. According to Lemma 2.6, we have

$$\bigcap_{x\in K} G(x) \neq \emptyset.$$

Corollary 3.4. *Let $\mathscr{X}$ be a reflexive Banach space and let $K \subset \mathscr{X}$ be a nonempty weakly sequentially closed convex subset. Assume that $C : K \to 2^{\mathscr{Y}}$ is mapping such that $x \in K, C(x)$ is a proper closed convex moving cone with $intC(x) \neq \emptyset$. Assume that the set valued mapping $x : K \to \mathscr{Y} \setminus (-intC(x))$ for $x \in K$ is a weakly closed mapping that is its graph is closed in $\mathscr{X} \times \mathscr{Y}$ with weak topology of $\mathscr{X}$ and $\mathscr{Y}$. Let a set valued mapping $T : K \to L(\mathscr{X}, \mathscr{Y})$ be a generalized upper hemi continuous in K with nonempty compact values and $f : K \to \mathscr{X}$ be a quasi linear type. Let f be a convex and pair (T, f) is weakly f-pseudomonotone for each $x \in K$. Assume that there exists $y_0 \in K$ such that*

$$\liminf_{\|x\|\to\infty} \inf_{x\in K} \langle T(x), f(x) - f(y_0)\rangle \in -intC(x).$$

Then there is a nonempty weakly compact subset D of K and a subset of K such that for all $x \in K \setminus D$ there exists $z \in D_0, u \in T(z)$

$$\langle u, f(y) - f(x)\rangle \notin -intC(x),\ \forall y \in D_0.$$

Moreover, assume that one of the following conditions are fulfilled:

(a) *f is weak to weak-sequentially continuous on K, T is weak to norm -sequentially continuous on K,*

(b) *f is weak to norm-sequentially continuous on K, T is weak to weak* sequentially continuous on K.*

Then $S_\omega(T, f, K) \neq \emptyset$. If, in addition, T is f-pseudomonotone, then $M(T, f, K) \neq \emptyset$.

The following concepts were introduced by Ky Fan [7]. Let $\mathscr{X}$ and $\mathscr{Y}$ be arbitrary sets. A function $h : \mathscr{X} \times \mathscr{X} \to R$ is said to be convex like on $\mathscr{X}$ if for any $u_1, u_2 \in \mathscr{X}$ and $t \in (0,1)$, there exists $u_0 \in \mathscr{X}$ such that for all $y \in \mathscr{Y}$ one has

$$h(u_0,y) \leq th(u_1,y) + (1-t)h(u_2,y).$$

Similarly h is said to be concave like on $\mathscr{Y}$ if for any $v_1, v_2 \in \mathscr{Y}$ and $t \in (0,1)$ there exists $v_0 \in \mathscr{Y}$ such that for all $x \in \mathscr{X}$ one has

$$h(x,v_0) \geq th(x,v_1) + (1-t)h(x,v_2).$$

Theorem 3.5. [8] *Let $\mathscr{X}$ be a compact space, $\mathscr{Y}$ a set and $h : \mathscr{X} \times \mathscr{Y} \to R$ a function that is concave like on $\mathscr{Y}$, convex like on $\mathscr{X}$ and for each $y \in \mathscr{Y}$ the function $x \to h(x,y)$ is lower semi continuous on $\mathscr{X}$. Then*

$$\sup_{y\in\mathscr{Y}} \min_{x\in\mathscr{X}} h(x,y) = \min_{x\in\mathscr{X}} \sup_{y\in\mathscr{Y}} h(x,y).$$

Theorem 3.6. Let K be a nonempty weakly compact convex subset of $\mathscr{X}$. Assume that $C : K \to 2^{\mathscr{Y}}$ is mapping such that $x \in K, C(x)$ is a proper closed convex moving cone with $intC(x) \neq \emptyset$. Assume that mapping $x \to \mathscr{Y} \setminus (-intC(x))$ for $x \in K$ is a weakly closed mapping that is its graph is closed in $\mathscr{X} \times \mathscr{Y}$ with weak topology of $\mathscr{X}$ and $\mathscr{Y}$. Let a set valued mapping $T : K \to L(\mathscr{X},\mathscr{Y})$ be a generalized upper hemicontinuous in K with nonempty compact values and $f : K \to \mathscr{X}$ be a quasi linear type. Let f be a convex and pair (T,f) is weakly f-pseudo monotone for each $x \in K$. Then there is a nonempty weakly compact subset D of K and a subset D_0 of a weakly compact convex subset of K such that for all $x \in K \setminus D$, there exists $z \in D_0, u \in T(z)$

$$\langle u, f(y) - f(x) \rangle \notin -intC(x), \ \forall y \in D_0.$$

Assume that one of the following conditions are fulfilled:

(a) f is weak to weak-sequentially continuous on K, T is weak to norm -upper semicontinuous on K and $T(x)$ is compact for every $x \in K$,

(b) f is weak to norm-sequentially continuous on K, T is weak to weak* upper semi continuous on K and $T(x)$ is weak* compact for every $x \in K$.

Then $S_\omega(T,f,K) \neq \emptyset$.

Proof. From Theorem 3.1, whenever (a) and (b) holds, we have $S_\omega(T,f,K) \neq \emptyset$. Let $x \in S_\omega(T,f,K)$. We now prove that $x \in S_\omega(T,f,K)$. Indeed by supposing the contrary, we obtain that for every $u \in T(x)$ there exists $y \in K$ such that

$$\langle u, f(y) - f(x) \rangle \in -intC(x).$$

Hence

$$\min_{y \in K} \langle u, f(y) - f(x) \rangle \in -intC(x).$$

Since $T(x)$ is compact (in the norm topology or in the weak* topology), we obtain that

$$\max_{u \in T(x)} \min_{y \in K} \langle u, f(y) - f(x) \rangle \in -intC(x).$$

Consider the function $h : K \times T(x) \to Y$

$$h(y,u) = \langle u, f(y) - f(x) \rangle \notin -intC(x).$$

We show that h satisfies the assumptions of Fan's minimax Theorem. Since $T(x)$ is convex, we have that $h(y,\cdot) : T(x) \to Y$ is concave for every $y \in K$. Hence h is concave like on $T(x)$. Since f is of quasi linear type. From Lemma 2.9, for every $y_1, y_2 \in K$ we have

$$f([y_1,y_2]) = [f(y_1), f(y_2)].$$

Hence for every $t \in (0,1)$ we have

$$tf(y_1) + (1-t)f(y_2) = f(y_0) \text{ for some } y_0 \in [y_1,y_2].$$

Thus for every $y_1, y_2 \in K$ and $t \in (0,1)$ there exists $y_0 \in [y_1,y_2]$ such that

$$\langle u, f(y_0) - f(x) \rangle = \langle u, tf(y_1) + (1-t)f(y_2) - f(x) \rangle \notin -intC(x), \ \forall u \in T(x).$$

That is,

$$h(u,y_0) = th(u,y_1) + (1-t)h(u,y_2) \notin -intC(x).$$

Hence h is convex like on K. Obviously $h(\cdot,u)$ is weakly sequentially continuous on K for all $u \in T(x)$. Hence from Theorem 7.1.2 from [16], $h(\cdot,u)$ is weakly lower semi continuous from Theorem 3.5 we have

$$\max_{u \in T(x)} \min_{y \in K} h(y,u) = \min_{y \in K} \max_{u \in T(x)} h(y,u).$$

$$\max_{u\in T(x)}\min_{y\in K}\langle u, f(y)-f(x)\rangle = \max_{u\in T(x)}\min_{y\in K} h(y,u) \in -intC(x).$$

On the other hand, from $x \in S_\omega(T, f, K)$, we have that for every $y \in K$

$$\max_{u\in T(x)}\langle u, f(y)-f(x)\rangle = \max_{u\in T(x)} h(y,u) \notin -intC(x).$$

which leads to

$$\min_{y\in K}\max_{u\in T(x)} h(y,u) \notin -intC(x).$$

This a contradiction.

3. Applications

In this section, we apply the existence results of solutions for the *general set valued vector variational inequality problems*. From (2)-(4) that we have obtain in previous section, to established some coincidence point results involving operators of quasi linear type. We obtain Kakutani's fixed point Theorem. Every where in the sequel $\mathscr{X}$ denotes a real Hilbert space identified with its dual. The range of the set valued operator $F : K \subseteq \mathscr{X} \rightrightarrows \mathscr{X}$ is the set

$$R(F) = \bigcup_{x\in K} F(x).$$

Theorem 4.1. *Let K be a nonempty weakly compact convex subset of a topological vector space $\mathscr{X}$ and consider $\{C(x) : x \in K\}$ be a family of nonempty convex compact pointed solid cone with $-intC(x) \neq \emptyset$. Consider the weak to weak upper semi continuous set valued mapping $F : K \rightrightarrows \mathscr{X}$ with weakly compact and convex valued. Let $f : K \to \mathscr{X}$ be a weak to norm continuous operator which is of quasi linear type. Assume that $R(F) \subseteq f(K)$. Then there exists $x \in K$ such that $f(x) \in F(x)$.*

Proof. Consider the set valued mapping $T : K \rightrightarrows L(\mathscr{X}, \mathscr{Y})$, $T(x) = f(x) - F(x)$, which is nonempty convex and weakly compact valued. We show that T is weak to weak upper semicontinuous . From Lemma 2.1, it is enough to show that for every weak convergent net $\{x_i\} \subseteq K$ that is $x_i \to x^0 \in K$ and for every net $z_i \in T(x_i)$ there exists $z^0 \in T(x^0)$ and a subnet $\{z_{i_j}\} \subseteq \{z_i\}$ such that $z_{i_j} \to z^0$ that is $\{z_{i_j}\}$ converges to z^0 in the weak topology of $\mathscr{X}$. Let $\{x_i\} \subseteq K, x_i \to x^0 \in K$ and $z_i \in T(x_i)$. Then $z_i = f(x_i) - y_i$, where $y_i \in F(x_i)$. Since F is weakly compact valued and weak to weak upper semi continuous. From Lemma 2.1, there exists $y^0 \in F(x^0)$ and a sub net $\{y_{i_j}\} \subseteq \{y_i\}$ such that $y_{i_j} \to y^0$. But $z_{i_j} \to f(x^0) - y^0 \in T(x^0)$, hence T is weak to weak upper semi continuous. From Theorem 3.6 (b), $S_\omega(T, f, K) \neq \emptyset$. Let $x \in S_\omega(T, f, K)$, that is there exists $u \in T(x)$ such that

$$\langle u, f(y) - f(x) \rangle \notin -intC(x), \ \forall y \in K.$$

But $u = f(x) - v$ for some $v \in F(x)$. Hence for all $y \in K$,

$$\langle f(x) - v, f(y) - f(x) \rangle \notin -intC(x), \ \forall y \in K.$$

Since $R(F) \subseteq f(K)$, let $y \in K$ such that $f(y) = v$. Then we obtain

$$\langle f(x) - v, v - f(x) \rangle \notin -intC(x)$$

or equivalently

$$-\|v - f(x)\|^2 \notin -intC(x),$$

which leads to $f(x) = v \in F(x)$.

Theorem 4.2. *Let K be a nonempty weakly compact convex set of a topological vector space $\mathscr{X}$ and $\{C(x) : x \in \mathscr{X}\}$ be a family of nonempty convex compact point solid cone with $intC(x) \neq \emptyset$. Consider the weak to weak upper semi continuous set valued map $F : K \rightrightarrows \mathscr{X}$ with weakly compact convex values. Let $f : K \to \mathscr{X}$ be a weak to weak sequentially continuous operator which is quasi linear type. Assume that $R(F) \subseteq f(K)$ and the map $f - F$ is weak to norm upper semi continuous. Then there exists $x \in K$ such that*

$$f(x) \in F(x).$$

Proof. Consider the map $T : K \rightrightarrows L(\mathscr{X}, \mathscr{Y})$, $T(x) = f(x) - F(x)$. From Theorem 3.6 (a), similar to the proof of Theorem 4.1, we obtain that there exists $x \in K$ such that

$$f(x) \in F(x).$$

In the virtue of weak to weak sequentially continuity of the map $id_K : K \to K, id_K(x) = x$ as a corollary. We have the following fixed point results.

Corollary 4.3. *Let $K \subseteq \mathscr{X}$ be a nonempty weakly compact convex set of a topological vector space $\mathscr{X}$ and $\{C(x) : x \in \mathscr{X}\}$ be a family of nonempty convex compact point solid cone with $intC(x) \neq \emptyset$. Consider a set valued map $F : K \rightrightarrows \mathscr{X}$ with weakly compact convex values. Assume that $id_K - F$ is weak to norm upper semi continuous. Then F has a fixed point.*

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