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EXPANDING THE APPLICABILITY OF THE GENERALIZED NEWTON METHOD FOR GENERALIZED EQUATIONS

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Abstract. In this paper, we consider a generalized Newton method for solving the generalized equation $0 \in F(x^*) + T(x^*)$, where F is Fréchet differentable and T is set valued and maximal monotone. Using the center Lipschitz conditions, we prove the convergence of the method with the following advantages: tighter error estimates on the distances involved and the information on the location of the solution is at least as precise. These advantages were obtained under the same computational cost.

Keywords. Generalized Newton method; Generalized equation; Generalized Lipschitz condition; Semi-local convergence.

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1. Introduction

Let H be a Hilbert space and let $T : H \rightrightarrows H$ be a set valued maximal monotone operator. Let $F : H \longrightarrow H$ be a Fréchet differentiable function. In this paper, we are interested in the

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approximately solving the generalized equation: Find $x^* \in H$ such that

$$0 \in F(x^*) + T(x^*). \tag{1.1}$$

Importance of studying problem (1.1) and its applications in the physical and engineering sciences and many other areas can be found in [1]- [31]. Generalized Newton method for (1.1) is defined iteratively for $n = 1, 2, 3, \ldots$ by

$$0 \in F(x_n) + F'(x_n)(x_{n+1} - x_n) + T(x_{n+1}). \tag{1.2}$$

Using the classical Lipschitz condition, Uko [24, 25] established the convergence of (1.2). Recently, the convergence (1.2) has been studied by many other authors under various conditions; see Robinson [17], Josephy [12] and [18, 26] for more details.

The convergence of generalized Newton method (1.2) was shown in [27, 31] using the Lipschitz continuity conditions on F'. However, there are problems where the Lipschitz continuity of F' does not hold (see also the numerical examples). Motivated by this constrains, we present a convergence analysis of the generalized Newton method (1.2) using the generalized continuity on F'. Our results are weaker even if we specialize the conditions on F' to the condition given in [21]. This way we expand the applicability of generalized Newton method (1.2).

The rest of the paper is organized as follows. In Section 2, we give the mathematical preliminaries. In Section 3, the convergence of generalized Newton method (1.2) is obtained.

2. Preliminaries

Let $x \in H$ and $r > 0$. We use $U(x,r)$ and $\overline{U(x,r)}$ to denote the open and closed metric ball, respectively, center at at x with radius r i.e.,

$$U(x,r) := \{y \in H : \|x - y\| < r\} \text{ and } \overline{U(x,r)} := \{y \in H : \|x - y\| \leq r\}.$$

Recall that a bounded linear operator $G : H \to H$ is called a positive operator if G is self conjugate and $\langle Gx, x\rangle \geq 0$, for each $x \in H$ (cf. [19], p.313). The following lemma about properties of positive operators is taken from [27].

Lemma 2.1. *Let G be a positive operator. Then the following conditions hold:*

- $\|G^2\| = \|G\|^2$
- *If G^{-1} exists, then G^{-1} is also a positive operator and*

$$\langle Gx, x\rangle \geq \frac{\|x\|^2}{\|G^{-1}\|}, \text{ for each } x \in H. \tag{2.1}$$

Let $T : H \rightrightarrows H$ be a set valued operator. The domain $domT$ of T is defined as $domT := \{x \in H : Tx \neq \emptyset\}$. Next, let us recall notions of montonicity for set-valued operators (see [30]).

Definition 2.2. *Let $T : H \rightrightarrows H$ be a set valued operator. T is said to be*

(a) *monotone if*

$$\langle u - v, y - x\rangle \geq 0, \text{ for each } u \in T(y) \text{ and } v \in T(x) \tag{2.2}$$

- *maximal monotone if it is monotone and the following implications hold:*

$$\langle u - v, x - y\rangle \geq 0, \text{ for each } y \in domT \text{ and } v \in T(y) \Rightarrow x \in domT \text{ and } u \in T(x). \tag{2.3}$$

Let $G : H \longrightarrow H$ be a bounded linear operator. Then $\widehat{G} := \frac{1}{2}(G + G^*)$, where G^* is the adjoint of G. Hereafter, we assume that $T : H \rightrightarrows H$ is a set valued maximal monotone operator and $F : H \longrightarrow H$ is a Fréchet differentiable operator.

3. Semi-local convergence

Definition 3.1. Let $R > 0$ and $\bar{x} \in H$ be such that $\widehat{F'(\bar{x})}^{-1}$ exists. Then, operator $\|\widehat{F'(\bar{x})}^{-1}\|F$ is said to satisfy the center-Lipschitz condition with $L-$ average at $\bar{x} \in U(\bar{x}, R)$, if

$$\|\widehat{F'(\bar{x})}^{-1}\|\|F'(x) - F'(\bar{x})\| \leq \int_0^{\|x-\bar{x}\|} L_0(u)du,$$

for each $\bar{x} \in U(\bar{x}, R)$, where L_0 is a non-negative non-decreasing integrable function on the interval $[0, R]$ satisfying $\int_0^{R_0} L_0(u)du = 1$, for some $R_0 \geq 0$.

Definition 3.2. Let $R > 0$ and $\bar{x} \in H$ be such that $\widehat{F'(\bar{x})}^{-1}$ exists. Then, operator $\|\widehat{F'(\bar{x})}^{-1}\|F'$ is said to satisfy the center-Lipschitz condition in the inscribed sphere with $(L_0, L)-$ average at $\bar{x}$ *on* $U(\bar{x}, \overline{R})$, $\overline{R} = \min\{R_0, R\}$, if

$$\|\widehat{F'(\bar{x})}^{-1}\|\|F'(y) - F'(x)\| \leq \int_{\|x-\bar{x}\|}^{\|x-\bar{x}\|+\|y-x\|} L(u)du,$$

for each $x, y \in U(\bar{x}, R_0)$, with $\|x-\bar{x}\|+\|y-x\| < \overline{R}$ and L is a non-negative non-decreasing integrable function on the interval $[0, \overline{R}]$.

Definition 3.3. Let $R > 0$ and $\bar{x} \in H$ be such that $\widehat{F'(\bar{x})}^{-1}$ exists. Then, operator $\|\widehat{F'(\bar{x})}^{-1}\|F'$ is said to satisfy the center-Lipschitz condition in the inscribed sphere with L_1- average at $\bar{x}$ *on* $U(\bar{x}, \overline{R})$, if

$$\|\widehat{F'(\bar{x})}^{-1}\|\|F'(y)-F'(x)\| \leq \int_{\|x-\bar{x}\|}^{\|x-\bar{x}\|+\|y-x\|} L_1(u)du$$

for each $x, y \in U(\bar{x}, R)$, with $\|x-\bar{x}\|+\|y-x\| < \overline{R}$ and L_1 is a non-negative non-decreasing integrable function on the interval $[0, R]$.

The convergence analysis of generalized Newton's method (1.2) was based on Definition 3.3 [31]. However, we have that

$$L_0(u) \leq L_1(u) \tag{3.1}$$

and

$$L(u) \leq L_1(u) \tag{3.2}$$

for each $u \in [0, \overline{R}]$, since $\overline{R} \leq R$.

Therefore if the functions L_0 and L are used instead of L_1 a more precise convergence analysis can be obtained. This is the main objective in the present paper. Notice that in practice the computation of function L_1 requires the computation of functions L_0 and L as special cases. Let $\beta > 0$. Define scalar majorizing function h by

$$h_1(t) = \beta - t + \int_0^t L_1(u)(t-u)du, \text{ for each } t \in [0, R] \tag{3.3}$$

and majorizing sequences $\{t_n\}$ by $t_0 = 0$, $t_{n+1} = t_n - h_1'(t_n)^{-1}h_1(t_n)$.
Sequence $\{t_n\}$ can also be written as $t_0 = 0,\ \ t_1 = \beta$,

$$t_{n+1} = t_n + \frac{\int_0^1 \int_{t_{n-1}}^{t_{n-1}+\theta(t_n-t_{n-1})} L_1(u)(t_n-t_{n-1})d\theta}{1-\int_0^{t_n} L_1(u)du}. \tag{3.4}$$

Sequence $\{t_n\}$ was used in [31]. However, in our study, we use the sequence $\{s_n\}$ defined by $s_0 = 0,\ \ s_1 = \beta,$

$$s_{n+1} = s_n + \frac{\int_0^1 \int_{s_{n-1}}^{s_{n-1}+\theta(s_n - s_{n-1})} L(u)du(s_n - s_{n-1})d\theta}{1 - \int_0^{s_n} L_0(u)du}. \tag{3.5}$$

We need an auxillary result where we compare sequence $\{t_n\}$ to $\{s_n\}$. Let $R_1 > 0$ and $b > 0$ be such that

$$\int_0^{R_1} L_1(u)du = 1 \ \text{ and } \ b = \int_0^{R_1} L_1(u)udu. \tag{3.6}$$

Lemma 3.4. *The following items hold.*

(a) *The function h_1 is monotonically decreasing on $[0, R_1]$ and monotonically increasing on $[R_1, R]$. Suppose that*

$$\beta \le b. \tag{3.7}$$

Then, h has a unique zero, respectively in $[0, R_1]$ and $[R_1, R]$, which are denoted by r_1 and r_2 which satisfy

$$\beta < r_1 < \frac{R_1}{b}\beta < R_1 < r_2 < R. \tag{3.8}$$

Moreover, if $\beta \le b$ and $r_1 = r_2$ if $\beta = b$.

(b) *Under hypothesis (3.7) sequences $\{t_n\}$ and $\{s_n\}$ are non-decreasing, converge to r_1 and $s^* = \lim_{n\to\infty} s_n$, respectively so that*

$$s_n \le t_n \le r_1 \tag{3.9}$$

$$0 \le s_{n+1} - s_n \le t_{n+1} - t_n \tag{3.10}$$

and

$$s^* \le r_1. \tag{3.11}$$

Proof. The proof of part (a) and that sequence $\{t_n\}$ non-decreasingly converges to r_1 can be found in [27, 31]. Using a simple inductive arguement, (3.1), (3.2), (3.4), (3.5) estimates (3.9) and (3.10) are obtained. Hence, sequence $\{s_n\}$ is non-decreasing and bounded above by r_1 and as such it converges to s^* which satisfies (3.11). □

We also need the following Banach-type perturbation lemma.

Lemma 3.5. *Let $r < R_0$. Let also $\bar{x} \in H$ be such that $\widehat{F'(\bar{x})}$ is a positive operator and $\widehat{F'(\bar{x})}^{-1}$ exists. Suppose that $\|\widehat{F'(\bar{x})}\|F'$ satisfies the center-Lipschitz condition with L_0- average at $\bar{x} \in U(\bar{x}, r)$. Then, for each $x \in U(\bar{x}, r)$, $\widehat{F'(x)}$ is a positive operator, $\widehat{F'(x)}^{-1}$ exists and*

$$\|\widehat{F'(x)}^{-1}\| \leq \frac{\|\widehat{F'(\bar{x})}^{-1}\|}{1 - \int_0^{\|x-\bar{x}\|} L_0(u)du}. \tag{3.12}$$

Proof. Simply use L_0 instead of L_1 in the proof of corresponding result in [27, 31].

Remark 3.6. The estimate corresponding to (3.12) is

$$\|\widehat{F'(x)}^{-1}\| \leq \frac{\|\widehat{F'(\bar{x})}^{-1}\|}{1 - \int_0^{\|x-\bar{x}\|} L_1(u)du}. \tag{3.13}$$

In view of (3.1), (3.12) and (3.13), estimate (3.12) is more precise than (3.13), if $L_0(u) < L_1(u)$.

Next, we present the semi-local convergence analysis of the Newton's method (1.2).

Theorem 3.7. *Suppose that there exists $x_0 \in D$ such that $\widehat{F'(x_0)}^{-1}$ exists, (3.7) holds, $\|\widehat{F'(x_0)}^{-1}\|F'$ satisfies the center-Lipschitz condition in the inscribed sphere with L_0- average at x_0 on $U(x_0, s^*)$, the center-Lipschitz condition in the inscribed sphere with $(L_0, L)-$ average at $\bar{x}$ on $U(x_0, s^*)$, and $F'(x_0)$ is a positive operator. Then, sequence $\{x_n\}$ generated with initial point x_0 by Newton's method (1.2) provided that*

$$\|x_1 - x_0\| \leq \beta \tag{3.14}$$

is well defined in $\overline{U}(x_0, s^*)$, remains in $\overline{U}(x_0, s^*)$ for each $n = 0, 1, 2, \cdots$ and converges to a solution x^* of (1.1) in $\overline{U}(x_0, s^*)$. Moreover, the following estimates hold

$$\|x_{n+1} - x_n\| \leq s^* - s_n, \text{ for each } n = 0, 1, 2, \cdots \tag{3.15}$$

Proof. We shall show using mathematical induction that sequence $\{x_n\}$ is well defined and satisfies

$$\|x_{k+1} - x_k\| \leq s_{k+1} - s_k, \text{ for each } k = 0, 1, 2, \cdots \tag{3.16}$$

Estimate (3.16) holds for $k=0$ by (3.14). Let us suppose that (3.16) holds for $n=0,1,2,\cdots,k-1$. We shall show that (3.16) holds for $n=k$. Notice that

$$\|x_k-x_0\| \leq \|x_k-x_{k+1}\|+\cdots+\|x_1-x_0\| \leq s_k-s_{k-1}+\cdots+s_1-s_0=s_k-s_0<s^*. \tag{3.17}$$

By (3.17) and Lemma 3.5, $F'(x_k)$ is a positive operator, $\widehat{F'(x_k)}^{-1}$ exists and

$$\|\widehat{F'(x_k)}^{-1}\| \leq \frac{\|\widehat{F'(x_0)}^{-1}\|}{1-\int_0^{\|x_k-x_0\|} L_0(u)du}. \tag{3.18}$$

By Lemma 2.1, we have that

$$\frac{\|x\|^2}{\|\widehat{F'(x_k)}^{-1}\|} \leq \langle F'(x_k)x,x\rangle = \langle F'(x_k)x,x\rangle \text{ for each } x\in H, \tag{3.19}$$

By Remark 2.5 [27, 31], we have

$$0\in F(x_k)+F'(x_k)(x_{k+1}-x_k)+T(x_{k+1}). \tag{3.20}$$

We have by hypotheses that

$$0\in F(x_{k-1})+F'(x_{k-1})(x_k-x_{k-1})+T(x_k). \tag{3.21}$$

Using (3.20), (3.21) and T being a maximal monotone operator, we get that $\langle -F(x_{k-1})-F'(x_{k-1})(x_k-x_{k-1})+F(x_k)+F'(x_k)(x_{k+1}-x_k), x_k-x_{k+1}\rangle \geq 0$. It follows that

$$\langle F(x_k)-F(x_{k-1})-F'(x_{k-1})(x_k-x_{k-1}),x_k-x_{k+1}\rangle \geq \langle F'(x_k)(x_k-x_{k+1}),x_k-x_{k+1}\rangle. \tag{3.22}$$

By (3.19), we have that

$$\frac{\|x_k-x_{k+1}\|^2}{\|\widehat{F'(x_k)}^{-1}\|} \leq \langle \widehat{F'(x_k)}(x_k-x_{k+1}),x_k-x_{k+1}\rangle = \langle F'(x_k)(x_k-x_{k+1}),x_k-x_{k+1}\rangle,$$

which leads together with (3.22) to

$$
\begin{aligned}
\|x_k - x_{k+1}\| &\leq \|\widehat{F'(x_k)}^{-1}\| \|F(x_k) - F(x_{k-1}) - F'(x_{k-1})(x_k - x_{k-1})\| \\
&\leq \|\widehat{F'(x_k)}^{-1}\| \| \int_0^1 (F'(x_{k-1} + \theta(x_k - x_{k-1})) - F'(x_{k-1})(x_k - x_{k-1}) d\theta\| \\
&\leq \frac{\|\widehat{F'(x_0)}^{-1}\|}{1 - \int_0^{\|x_k - x_0\|} L_0(u)du} \| \int_0^1 (F'(x_{k-1} + \theta(x_k - x_{k-1})) - F'(x_{k-1})\| \\
&\quad \times \|(x_k - x_{k-1}) d\theta\| \\
&\leq \frac{1}{1 - \int_0^{s_k - s_0} L_0(u)du} \int_0^1 \int_{\|x_{k-1} - x_0\|}^{\|x_{k-1} - x_0\| + \theta\|x_k - x_{k-1}\|} L(u)du \|x_k - x_{k-1}\| d\theta \\
&\leq \frac{1}{1 - \int_0^{s_k} L_0(u)du} \int_0^1 \int_{s_{k-1}}^{s_{k-1} + \theta(s_k - s_{k-1})} L(u)du (s_k - s_{k-1}) d\theta \\
&= s_{k+1} - s_k. \qquad (3.23)
\end{aligned}
$$

It follows from (3.23) that sequence $\{x_k\}$ is complete in a Hilbert space H and as such it converges to some $x^* \in \overline{U}(x_0, s^*)$ (since $\overline{U}(x^*, s^*)$ is a closed set). Moreover, since T is a maximal monotone and $F \in C^1$, we deduce that x^* solves (1.1). Furthermore, (3.15) follows from (3.16) by using standard majorization techniques [1,4,5].

Concerning the uniqueness of the solution x^* around the point x_0, we have the following result.

Proposition 3.8. *Suppose that the hypotheses of Theorem 3.7 hold except the center-Lipschitz (L_0, L) condition for $x_1 \in H$ such that*

$$0 \in F(x_0) + F'(x_0)(x_1 - x_0) + T(x_1).$$

Then, there exists a unique solution x^* of (1.1) in $\overline{U}(x_0, s^*)$.

Proof. Simply replace function L_1 by L_0 in the proof of Theorem 4.1 in [31].

Remark 3.9.

(a) It is worth noticing that function L_1 is not needed in the proof of Theorem 4.1 in [31], since it can be replaced by L_0 which is more precise than L_1(see (3.1)) in all stages of the proof. Moreover, in view of (3.11), we obtain a better information on the uniqueness of x^*.

(b) The results obtained in this paper can be improved even further, if in Definition 3.2 we consider instead the center Lipschitz condition in the inscribed sphere with $(L_0, K)-$ average at $\bar{x}$ on $U(\bar{x}, \overline{R} - \|x_1 - x_0\|)$ (or $\overline{U}(\bar{x}, \overline{R} - \beta)$) given by

$$\|\widehat{F'(\bar{x})}^{-1}\| \|F'(y) - F'(x)\| \leq \int_{\|x-\bar{x}\|}^{\|x-\bar{x}\|+\|y-x\|} K(u)du,$$

for each $x, y \in U(\bar{x}, \overline{R} - \|x_1 - x_0\|)$ (or $\overline{U}(\bar{x}, \overline{R} - \beta)$). Notice that

$$K(u) \leq L(u), \text{ for each } u \in U(0, \overline{R}). \tag{3.24}$$

Then, by simply noticing that the iterates $\{x_n\}$ remain in $U(x_0, \overline{R} - \|x_1 - x_0\|)$ which is more precise location than $U(x_0, \overline{R})$, function K can replace L in all the preceding results.

Moreover, define sequence $\{\alpha_n\}$ by $\alpha_0 = 0$, $\alpha_1 = \beta$,

$$\alpha_{n+1} = \alpha_n + \frac{\int_0^1 \int_{\alpha_{n-1}}^{\alpha_{n-1}+\theta(\alpha_n - \alpha_{n-1})} K(u)du(\alpha_n - \alpha_{n-1})d\theta}{1 - \int_0^{\alpha_n} L_0(u)du}. \tag{3.25}$$

Then, in view of (3.5), (3.24) and (3.25), we have that

$$\alpha_n \leq s_n, \tag{3.26}$$

$$0 \leq \alpha_{n+1} - \alpha_n \leq s_{n+1} - s_n \tag{3.27}$$

and

$$\alpha^* = \lim_{n\to\infty} \alpha_n \leq s^* \tag{3.28}$$

hold under the hypotheses (3.7).

(c) We have extended the applicability of Newton's method under hypotheses (3.7). At this point we are wondering, if sequences $\{\alpha_n\}$ and $\{s_n\}$ converge under a hypotheses weaker than (3.7). It turns out that this is indeed the case, when L_0, L, L_1 and K are constant functions.

It follows from (3.4), (3.5) and (3.25) that sequences $\{t_n\}$, $\{s_n\}$ and $\{\alpha_n\}$ reduce respectively to

$$t_0 = 0,\ \ t_1 = \beta,\ \ t_{n+1} = t_n + \frac{L(t_n - t_{n-1})^2}{2(1 - Lt_n)}, \tag{3.29}$$

$$s_0 = 0,\ \ s_1 = \beta,\ \ s_{n+1} = s_n + \frac{L(s_n - s_{n-1})^2}{2(1 - L_0 s_n)}, \tag{3.30}$$

$$\alpha_0 = 0,\ \ \alpha_1 = \beta,\ \ \alpha_{n+1} = \alpha_n + \frac{K(\alpha_n - \alpha_{n-1})^2}{2(1 - L_0 \alpha_n)}. \tag{3.31}$$

Sequence $\{t_n\}$, converges provided that the Kantorovich condition [13, 14, 23]

$$h_k = L\beta \leq \frac{1}{2} \tag{3.32}$$

is satisfied, whereas $\{s_n\}$ and $\{\alpha_n\}$ converge [6], if

$$h_1 = L_1\beta \leq \frac{1}{2} \tag{3.33}$$

and

$$h_2 = L_2\beta \leq \frac{1}{2} \tag{3.34}$$

are satisfied, respectively, where $L_1 = \frac{1}{8}(L + 4L_0 + \sqrt{L^2 + 8L_0 L})$ and $L_2 = \frac{1}{8}(L + 4L_0 + \sqrt{L^2 + 8L_0 K})$. Notice that

$$h_k \leq \frac{1}{2} \Rightarrow h_1 \leq \frac{1}{2} \Rightarrow h_2 \leq \frac{1}{2} \tag{3.35}$$

but not necessarily vice versa, unless, if $L_0 = L = K$.

Examples, where $L_0 < K < L < L_1$ can be found in [6]. Hence, the convergence domain is also extended in this case. Similar advantages are obtained in the case of the Smale's alpha theory [21, 22] of Wang's $\gamma-$ condition [28], [31]. However, we leave the details to the motivated readers.

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