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PERMANENCE OF A NONLINEAR COMPETITION MODEL WITH DELAY

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Abstract. Sufficient conditions are obtained for the permanence of the following nonlinear competition model

$$\frac{dN_1(t)}{dt} = r_1(t)N_1(t)\left[\frac{K_1(t)+\alpha_1(t)N_2^{\beta_{12}}(t-\tau_2(t))}{1+N_2^{\beta_{12}}(t-\tau_2(t))} - N_1^{\beta_{11}}(t-\sigma_1(t))\right],$$

$$\frac{dN_2(t)}{dt} = r_2(t)N_2(t)\left[\frac{K_2(t)+\alpha_2(t)N_1^{\beta_{21}}(t-\tau_1(t))}{1+N_1^{\beta_{21}}(t-\tau_1(t))} - N_2^{\beta_{22}}(t-\sigma_2(t))\right],$$

where r_i, K_i, α_i, τ_i and $\sigma_i, i=1,2$ are continuous functions bounded above and below by positive constants, $K_i > \alpha_i, i=1,2$, $\beta_{ij}, i,j=1,2$ are all positive constants.

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1. Introduction

Throughout this paper, for a continuous function $g(t)$, we set

$$g^l = \inf_{t\in R} g(t), \; g^u = \sup_{t\in R} g(t).$$

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The aim of this paper is to investigate the persistent property of the following nonlinear competition model

$$\begin{aligned} \frac{dN_1(t)}{dt} &= r_1(t)N_1(t)\left[\frac{K_1(t)+\alpha_1(t)N_2^{\beta_{12}}(t-\tau_2(t))}{1+N_2^{\beta_{12}}(t-\tau_2(t))}-N_1^{\beta_{11}}(t-\sigma_1(t))\right], \\ \frac{dN_2(t)}{dt} &= r_2(t)N_2(t)\left[\frac{K_2(t)+\alpha_2(t)N_1^{\beta_{21}}(t-\tau_1(t))}{1+N_1^{\beta_{21}}(t-\tau_1(t))}-N_2^{\beta_{22}}(t-\sigma_2(t))\right]. \end{aligned} \tag{1.1}$$

We assume that the coefficients of system (1.1) satisfies:

(A) r_i, K_i, α_i, τ_i and $\sigma_i, i=1,2$ are continuous functions bounded above and below by positive constants. $K_i > \alpha_i, i=1,2$. $\beta_{ij}, i,j=1,2$ are all positive constants.

Let $\tau = \sup_t\{\tau_i(t), \sigma_i(t), i=1,2\}$, we consider (1.1) together with the following initial conditions

$$N_i(s) = \varphi_i(s) \geq 0, s\in[-\tau,0],\ \varphi_i(0)>0. \tag{1.2}$$

It is not difficult to see that solutions of (1.1)-(1.2) are well defined for all $t \geq 0$ and satisfy

$$N_i(t) > 0 \quad \text{for } t \geq 0, i=1,2.$$

The essential assumption in (A) is $K_i > \alpha_i, i=1,2$, we mention here that such an assumption implies that the relationship between two species is competition. Indeed, from $K_1 > \alpha_1$ and the first equation of system (1.1), we have

$$\begin{aligned} &\frac{dN_1(t)}{dt} \\ =\ & r_1(t)N_1(t)\left[\frac{K_1(t)+\alpha_1(t)N_2^{\beta_{12}}(t-\tau_2(t))}{1+N_2^{\beta_{12}}(t-\tau_2(t))}-N_1^{\beta_{11}}(t-\sigma_1(t))\right] \\ =\ & r_1(t)N_1(t)\left[\frac{K_1(t)+K_1(t)N_2^{\beta_{12}}(t-\tau_2(t))}{1+N_2^{\beta_{12}}(t-\tau_2(t))}-N_1^{\beta_{11}}(t-\sigma_1(t))\right. \\ &\qquad \left. -\frac{(K_1(t)-\alpha_1(t))N_2^{\beta_{12}}(t-\tau_2(t))}{1+N_2^{\beta_{12}}(t-\tau_2(t))}\right] \end{aligned}$$

$$= r_1(t)N_1(t)\Bigg[K_1(t) - N_1^{\beta_{11}}(t-\sigma_1(t)) - \frac{(K_1(t)-\alpha_1(t))N_2^{\beta_{12}}(t-\tau_2(t))}{1+N_2^{\beta_{12}}(t-\tau_2(t))}\Bigg]. \tag{1.3}$$

Similarly, the second equation of system (1.1) can be rewrite as follows:

$$\frac{dN_2(t)}{dt} = r_2(t)N_2(t)\Bigg[K_2(t) - N_2^{\beta_{22}}(t-\sigma_2(t)) - \frac{(K_2(t)-\alpha_2(t))N_1^{\beta_{21}}(t-\tau_1(t))}{1+N_1^{\beta_{21}}(t-\tau_1(t))}\Bigg]. \tag{1.4}$$

From (1.3) and (1.4), one could easily see that the first species has negative effect on the second species, and the second species has negative effect on the first species, that is, under the assumption $K_i > \alpha_i, i = 1,2$, the relationship between two species is competition.

During the past decades, many scholars focused their attention to the study of the dynamic behaviors of the cooperative system, see [1]-[24], Li [1] studied the following two species mutualism model

$$\begin{aligned} \frac{dN_1(t)}{dt} &= r_1(t)N_1(t)\left[\frac{K_1(t)+\alpha_1(t)N_2(t-\tau_2(t))}{1+N_2(t-\tau_2(t))} - N_1(t-\sigma_1(t))\right], \\ \frac{dN_2(t)}{dt} &= r_2(t)N_2(t)\left[\frac{K_2(t)+\alpha_2(t)N_1(t-\tau_1(t))}{1+N_1(t-\tau_1(t))} - N_2(t-\sigma_2(t))\right]. \end{aligned} \tag{1.4}$$

Under the assumption r_i, K_i, α_i and $\tau_i, \sigma_i, i = 1,2$ are continuous periodic functions with common period ω. $\alpha_i > K_i, i = 1,2$. The author investigated the existence, uniqueness and stability property of the positive periodic solution of system (1.4). Huang[2] argued that the general non-autonomous case is more appropriate, and she investigated the persistent property of a n-species mutualism model with delay, which is the generalization of the system (1.4).

As for as system (1.4) is concerned, one interesting issue is proposed: What would happen if $K_i > \alpha_i, i = 1,2$? Is it possible for the system admits the similar dynamic behaviors as that of the case $\alpha_i > K_i, i = 1,2$?

On the other hand, during the past decades, many scholars argued that nonlinear population model is more appropriate then the Logistic type model, and they investigated the extinction, persistent, and stability property of the nonlinear population models ([22]-[30]). For example, Chen and Shi [27] investigated the following nonlinear model:

$$\begin{cases} \dot{x}_i = x_i[b_i(t) - \sum_{k=1}^{n} a_{ik}(t) x_k^{\alpha_{ik}} - \sum_{k=1}^{n} c_{ik}(t) x_i^{\alpha_{ii}} x_k^{\alpha_{ik}} - \sum_{k=1}^{m} d_{ik}(t) y_k^{\beta_{ik}}], \\ \dot{y}_j = y_j[-r_j(t) + \sum_{k=1}^{n} e_{jk}(t) x_k^{\delta_{jk}} - \sum_{k=1}^{m} f_{jk}(t) y_j^{\eta_{jj}} y_k^{\eta_{jk}} - \sum_{k=1}^{m} g_{jk}(t) y_k^{\eta_{jk}}], \end{cases} \tag{1.6}$$

where $i = 1,2,...,n$, $j = 1,2,...,m$, $x_i(t)$ denotes the density of prey species X_i at time t, $y_j(t)$ denotes the density of predator species Y_j at time t. They obtained a set of sufficient conditions which ensure the existence of a unique globally attractive almost periodic solution. Chen [25] investigated the permanence and extinction property of following general nonautonomous n-species Gilpin-Ayala competition system

$$\dot{x}_i(t) = x_i(t)\left[b_i(t) - \sum_{j=1}^{n} a_{ij}(t)(x_j(t))^{\alpha_{ij}}\right],\ i = 1,2,...,n, \tag{1.6}$$

where $b_i(t), 1 \le i \le n$ and $a_{ij}(t), i, j = 1,2,...,n$ are continuous for $c \le t < +\infty$, α_{ij} are positive constants. The results of Chen[28] is then generalized to the delayed case in [27].

The success of [22]-[30] motivated us to proposed the system (1.1). The aim of this paper is, by further developing the analysis technique of [2, 25], to obtain a set of sufficient conditions to ensure the permanence of the system (1.1). More precisely, we will prove the following result.

Theorem 1.1. *Under the assumption (A), system (1.1) is permanent, that is, there exist positive constants $m_i, M_i, i = 1,2$ which are independent of the solutions of system (1.1), such that for any positive solution $(x_1(t), x_2(t))^T$ of system (1.1) with initial condition (1.2), one has:*

$$m_i \le \liminf_{t\to+\infty} x_i(t) \le \limsup_{t\to+\infty} x_i(t) \le M_i, i = 1,2.$$

2. Proof of the main results

Now let's state several lemmas which will be useful in the proving of main result.

Lemma 2.1. [25] *If $a > 0, b > 0$ and $\dot{x} \ge x(b - ax^{\alpha})$, where α is a positive constant, when $t \ge 0$ and $x(0) > 0$, we have*

$$\liminf_{t\to+\infty} x(t) \ge \left(\frac{b}{a}\right)^{1/\alpha}.$$

If $a>0, b>0$ and $\dot{x}\le x(b-ax^{\alpha})$, where α is a positive constant, when $t\ge 0$ and $x(0)>0$, we have

$$\limsup_{t\to+\infty} x(t)\le\left(\frac{b}{a}\right)^{1/\alpha}.$$

Now we are in the position of proving the main result of this paper.

Proof of Theorem 1.1. Set

$$\tau=\sup_t\{\tau_i(t),\sigma_i(t),i=1,2\}.$$

Let $(N_1(t),N_2(t))$ be any positive solution of system (1.1) with initial condition (1.2). From $K_1(t)>\alpha_1(t)$, we know that the first equation of (1.1) could be rewrite as (1.3), and it follows from (1.3) that

$$\begin{aligned}\frac{dN_1(t)}{dt} &= r_1(t)N_1(t)\Bigg[K_1(t)-N_1^{\beta_{11}}(t-\sigma_1(t))\\ &\qquad -\frac{(K_1(t)-\alpha_1(t))N_2^{\beta_{12}}(t-\tau_2(t))}{1+N_2^{\beta_{12}}(t-\tau_2(t))}\Bigg] \qquad (2.1)\\ &\le K_1^u r_1^u N_1(t).\end{aligned}$$

Integrating both sides of (2.1) from $t-\sigma_1(t)$ to t leads to

$$\ln\frac{N_1(t)}{N_1(t-\sigma_1(t))}\le\int_{t-\sigma_1(t)}^{t} r_1^u K_1^u ds\le r_1^u K_1^u\tau,$$

and so

$$N_1(t-\sigma_1(t))\ge N_1(t)\exp\{-r_1^u K_1^u\tau\}. \qquad (2.2)$$

Substituting (2.2) into (1.3), it follows that

$$\begin{aligned}\frac{dN_1(t)}{dt} &\le r_1(t)N_1(t)\left[K_1(t)-N_1^{\beta_{11}}(t-\sigma_1(t))\right]\\ &\le N_1(t)\left[r_1^u K_1^u - r_1^l\left(N_1(t)\exp\{-r_1^u K_1^u\tau\}\right)^{\beta_{11}}\right] \qquad (2.3)\\ &= N_1(t)\left[r_1^u K_1^u - r_1^l N_1^{\beta_{11}}(t)\exp\{-\beta_{11} r_1^u K_1^u\tau\}\right].\end{aligned}$$

Thus, as a direct corollary of Lemma 2.1, according to (2.3), one has

$$\begin{aligned}\limsup_{t\to+\infty} N_1(t) &\le \left(\frac{r_1^u K_1^u}{r_1^l}\exp\{\beta_{11} r_1^u K_1^u \tau\}\right)^{\frac{1}{\beta_{11}}} \\ &= \left(\frac{r_1^u K_1^u}{r_1^l}\right)^{\frac{1}{\beta_{11}}} \exp\{r_1^u K_1^u \tau\} \overset{\text{def}}{=} M_1.\end{aligned} \tag{2.4}$$

By using (1.4), similarly to the analysis of (2.1)-(2.4), we can obtain

$$\limsup_{t\to+\infty} N_2(t) \le \left(\frac{r_2^u K_2^u}{r_2^l}\right)^{\frac{1}{\beta_{22}}} \exp\{r_2^u K_2^u \tau\} \overset{\text{def}}{=} M_2. \tag{2.5}$$

For any small positive constant $\varepsilon > 0$, from (2.4)-(2.5) it follows that there exists a $T_1 > 0$ such that for all $t > T_1$ and $i = 1,2$,

$$N_i(t) \;<\; M_i + \varepsilon. \tag{2.6}$$

For $t \ge T_1 + \tau$, from (2.6) and (1.3), we have

$$\begin{aligned}&\frac{dN_1(t)}{dt} \\ =\;& r_1(t)N_1(t)\Bigg[K_1(t) - N_1^{\beta_{11}}(t-\sigma_1(t)) \\ &\qquad - \frac{(K_1(t)-\alpha_1(t))N_2^{\beta_{12}}(t-\tau_2(t))}{1+N_2^{\beta_{12}}(t-\tau_2(t))}\Bigg] \\ \ge\;& r_1(t)N_1(t)\Bigg[K_1(t) - N_1^{\beta_{11}}(t-\sigma_1(t)) \\ &\qquad - \frac{(K_1(t)-\alpha_1(t))N_2^{\beta_{12}}(t-\tau_2(t))}{N_2^{\beta_{12}}(t-\tau_2(t))}\Bigg] \\ =\;& r_1(t)N_1(t)\Big[K_1(t) - N_1^{\beta_{11}}(t-\sigma_1(t)) - (K_1(t)-\alpha_1(t))\Big] \\ \ge\;& N_1(t)\left[r_1^l \alpha_1^l - r_1^u (M_1+\varepsilon)^{\beta_{11}}\right].\end{aligned} \tag{2.7}$$

Noting that

$$\begin{aligned} r_1^l\alpha_1^l - r_1^u\big(M_1+\varepsilon\big)^{\beta_{11}} &\le r_1^u\Big(\alpha_1^l - \big(M_1+\varepsilon\big)^{\beta_{11}}\Big) \\ &\le r_1^u\Big(\alpha_1^l - \big(M_1\big)^{\beta_{11}}\Big) \\ &\le r_1^u\Big(\alpha_1^l - \frac{r_1^u K_1^u}{r_1^l}\exp\{\beta_{11} r_1^u K_1^u \tau\}\Big) \\ &\le r_1^u\Big(\alpha_1^l - K_1^u\Big) \le 0. \end{aligned}$$

Integrating both sides of (2.7) from $t-\sigma_1(t)$ to t leads to

$$\begin{aligned} \ln\frac{N_1(t)}{N_1(t-\sigma_1(t))} &\ge \int_{t-\sigma_1(t)}^{t}\Big[r_1^l\alpha_1^l - r_1^u\big(M_1+\varepsilon\big)^{\beta_{11}}\Big]ds \\ &\ge \Big[r_1^l\alpha_1^l - r_1^u\big(M_1+\varepsilon\big)^{\beta_{11}}\Big]\tau, \end{aligned}$$

and so

$$N_1(t-\sigma_1(t)) \le N_1(t)\exp\Big\{-\Big[r_1^l\alpha_1^l - r_1^u\big(M_1+\varepsilon\big)^{\beta_{11}}\Big]\tau\Big\}. \tag{2.8}$$

Substituting (2.8) into (1.3), similarly to the analysis of (2.7), for $t \ge T_1+\tau$, it follows that

$$\begin{aligned} &\frac{dN_1(t)}{dt} \\ \ge\ & r_1(t)N_1(t)\Bigg[\alpha_1(t) - N_1^{\beta_{11}}(t-\sigma_1(t))\Bigg] \\ \ge\ & N_1(t)\Big[r_1^l\alpha_1^l - r_1^u N_1^{\beta_{11}}(t-\sigma_1(t))\Big] \\ \ge\ & N_1(t)\Big[r_1^l\alpha_1^l - r_1^u N_1^{\beta_{11}}(t)\exp\Big\{-\Big[r_1^l\alpha_1^l - r_1^u\big(M_1+\varepsilon\big)^{\beta_{11}}\Big]\beta_{11}\tau\Big\}\Big], \end{aligned} \tag{2.9}$$

thus, as a direct corollary of Lemma 2.1, according to (2.9), one has

$$\begin{aligned} \liminf_{t\to+\infty} N_1(t) &\ge \left(\frac{r_1^l\alpha_1^l}{r_1^u}\exp\Big\{\Big[r_1^l\alpha_1^l - r_1^u\big(M_1+\varepsilon\big)^{\beta_{11}}\Big]\beta_{11}\tau\Big\}\right)^{\frac{1}{\beta_{11}}} \\ &= \left(\frac{r_1^l\alpha_1^l}{r_1^u}\right)^{\frac{1}{\beta_{11}}}\exp\Big\{\Big[r_1^l\alpha_1^l - r_1^u\big(M_1+\varepsilon\big)^{\beta_{11}}\Big]\tau\Big\}. \end{aligned} \tag{2.10}$$

Setting $\varepsilon \to 0$, it follows that

$$\liminf_{t\to+\infty} N_1(t) \ge \frac{1}{2}\left(\frac{r_1^l\alpha_1^l}{r_1^u}\right)^{\frac{1}{\beta_{11}}}\exp\Big\{\Big[r_1^l\alpha_1^l - r_1^u\big(M_1\big)^{\beta_{11}}\Big]\tau\Big\} \overset{\text{def}}{=} m_1. \tag{2.11}$$

Similarly to the analysis of (2.7)-(2.11), by applying (2.6), from (1.4), we can also have

$$\liminf_{t\to+\infty} N_2(t) \geq \frac{1}{2}\left(\frac{r_2^l\alpha_2^l}{r_2^u}\right)^{\frac{1}{\beta_{22}}} \exp\left\{\left[r_2^l\alpha_2^l - r_2^u(M_2)^{\beta_{22}}\right]\tau\right\} \overset{\text{def}}{=} m_2. \tag{2.12}$$

(2.4)-(2.5), (2.11)-(2.12) show that under the assumptions of Theorem 1.1, system (1.1) is permanent. This ends the proof of Theorem 1.1.

3. Numeric simulations

This section we will give an example to show the feasibility of the Theorem 1.1.

Example 3.1.

$$\begin{aligned} \frac{dN_1(t)}{dt} &= N_1(t)\left[\frac{4+(1+\frac{1}{2}\cos(t))N_2^{\frac{1}{2}}(t)}{1+N_2^{\frac{1}{2}}(t)} - N_1^2(t)\right], \\ \frac{dN_2(t)}{dt} &= N_2(t)\left[\frac{3+(1+\frac{1}{10}\sin(t))N_1^3(t)}{1+N_1^3(t)} - N_2^{\frac{1}{2}}(t)\right]. \end{aligned} \tag{3.1}$$

Corresponding to system (1.1), one has

$$r_1(t) = r_2(t) = 1, \alpha_1(t) = 1+\tfrac{1}{2}\cos(t),\ \beta_{12} = \tfrac{1}{2},\ \beta_{11} = 2;$$

$$\alpha_2(t) = 1+\tfrac{1}{10}\sin(t),\ K_1(t) = 4,\ K_2(t) = 3,\ \beta_{21} = 3,\ \beta_{22} = \tfrac{1}{2}.$$

Obviously, $\alpha_i(t) > K_i(t), i = 1,2$, hence, the conditions of Theorem 1.1 holds, it follows from Theorem 1.1 that system (3.1) is permanent. Fig. 1 and 2 also support this assertion.

4. Discussion

Li[1] proposed a delay model of mutualism (i.e., system (1.4)). Under the assumption $\alpha_i > K_i, i = 1,2$, he showed that the system admits at least one positive periodic solution. However, the author did not investigated the case $\alpha_i < K_i$. In this paper, we first generalize the system (1.4) to the nonlinear case, then under the assumption $K_i > \alpha_i$, by using the theory of differential inequality, and applying the analysis technique of Chen[23], we show that the system is also permanent.

Our result shows that delay and nonlinear term only infect the upper and lower bound of the

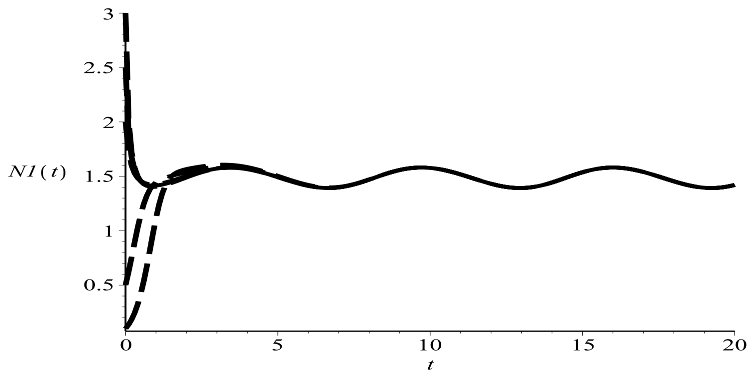

FIGURE 1. Dynamic behavior of the first species in system (3.1) with the initial conditions $(N_1(0), N_2(0)) = (0.1, 0.1)$, $(2,2)$, $(3,3)$, $(2.5,2.5)$ and $(0.5,0.5)$, respectively.

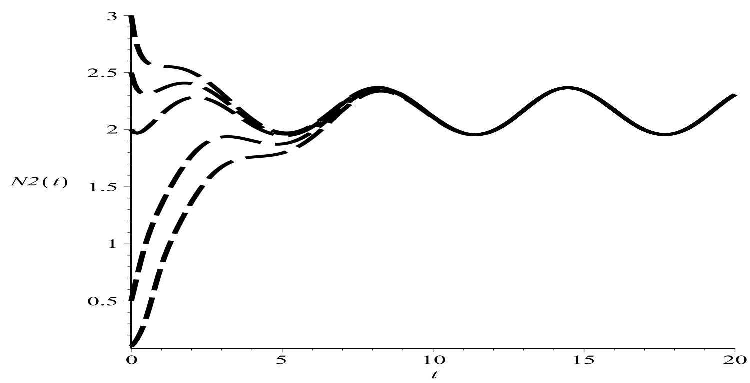

FIGURE 2. Dynamic behavior of the second species in system (3.1) with the initial conditions $(N_1(0), N_2(0)) = (0.1, 0.1)$, $(2,2)$, $(3,3)$, $(2.5,2.5)$ and $(0.5,0.5)$, respectively.

solution, and has no influence on the persistent property of the system. Whether delay could induce the bifurcation or not is still unknown, we leave this for future investigation.

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